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Transmission system for waves with nonlinear weights and delay

Aissa Benseghir*, Hamid Benseridi, Mourad Dilmi

Applied Mathematical Laboratory (LaMa), Faculty of Sciences, University of Setif 1- SETIF, 19000, Algeria

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Abstract

In this paper we consider a transmission problem for one dimensional waves with nonlinear weights on the frictional damping and time delay. We prove first, the existence and the uniqueness of the solution using the semigroup theory. Second, we chow the exponential stability of the solution by introducing a suitable Lyaponov functional.

Keywords: transmission system, delay term, nonlinear weights, exponential stability
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1 Introduction

In this paper, we consider global existence and decay properties of solutions for a transmission problem for waves with nonlinear weights and delay. We consider the following system form

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t) + \mu_1(t)u_t(x,t) & \\ +\mu_2(t)u_t(x,t-\tau) = 0, & (x,t) \in \Omega \times]0,+\infty[, \\ & \\ v_{tt}(x,t) - bv_{xx}(x,t) = 0, & (x,t) \in]L_1,L_2[\times]0,+\infty[, \end{cases} \tag{1.1}$$

where $0 < L_1 < L_2 < L_3$, $\Omega =]0,L_1[\cup]L_2,L_3[$, a, b, are positive constants, $\mu_1(t)$ and $\mu_2(t)$ are nonlinear weights acting on the frictional damping $\tau > 0$ is the delay. System (1.1) is subjected to the following boundary conditions, and transmission conditions:

$$\begin{cases} u(0,t) = u(L_3,t) = 0, & \\ u(L_i,t) = v(L_i,t), & i = 1,2 \\ au_x(L_i,t) = bv_x(L_i,t), & i = 1,2 \end{cases} \tag{1.2}$$

and the initial conditions:

$$\begin{cases} u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t-\tau) = f_0(x,t-\tau), & x \in \Omega,\ t \in [0,\tau], \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & x \in]L_1,L_2[. \end{cases} \tag{1.3}$$

*Corresponding author
Email addresses: aissa.benseghir@univ-setif.dz (Aissa Benseghir), hamid.benseridi@univ-setif.dz (Hamid Benseridi), Mourad.dilmi@univ-setif.dz (Mourad Dilmi)

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We are interested in proving the exponential stability for the problem (1.1)-(1.3) under the assumption

$$\frac{a}{b} < \frac{L_1 + L_3 - L_2}{2(L_2 - L_1)}. \tag{1.4}$$

Transmission problems are closely related to the design of material components, attracting considerable attention in recent years, e.g., in the analysis if damping mechanisms in the metallurgical industry or smart material technology, see [2]. From the mathematical point of view a transmission problem for wave propagation consists on a hyperbolic equation for the corresponding elliptic operator has discontinuous coefficients.

Time delay is the property of a physical system by which to an applied force is delayed in its effect, and the central question is that the delays source can destabilize a system that is asymptotically stable in the absence of delay, see [5, 8, 16]. Another type of works have been done on similar problems but have focused on the asymptotic solution of different transmission problems in a thin domain. For example, the authors in [4] have proved the asymptotic behavior of an interface problem in a thin domain. The asymptotic analysis of a frictionless contact between two elastic bodies with a dissipative term in a dynamic regime was studied in [9].

The first contribution in literature for transmission problem with time delay was given by A. Benseghir in [3]. More precisely, in [3] the transmission problem

$$\begin{cases} u_{tt}(x,t) - a u_{xx}(x,t) + \mu_1 u_t(x,t) & \\ +\mu_2 u_t(x, t-\tau) = 0, & (x,t) \in \Omega \times]0, +\infty[, \\ v_{tt}(x,t) - b v_{xx}(x,t) = 0, & (x,t) \in]L_1, L_2[\times]0, +\infty[, \end{cases} \tag{1.5}$$

with constants μ_1, μ_2 and time delay $\tau > 0$ was studied. Under appropriate assumption on the weights of the two feedbacks ($\mu_1 < \mu_2$), it was proved the well-possessedness of the system and, under condition (1.4), it was established an exponential decay result.
The result in [3] were improved by S. Zitouni et al. [15]. There, the authors considered the system with time-varying delay $\tau(t)$ of the form

$$\begin{cases} u_{tt}(x,t) - a u_{xx}(x,t) + \mu_1 u_t(x,t) & \\ +\mu_2 u_t(x, t-\tau(t)) = 0, & (x,t) \in \Omega \times]0, +\infty[, \\ v_{tt}(x,t) - b v_{xx}(x,t) = 0, & (x,t) \in]L_1, L_2[\times]0, +\infty[, \end{cases} \tag{1.6}$$

In [13] the authors examined a system of wave equation with a linear boundary damping term with a delay:

$$\begin{cases} u_{tt} - \Delta u = 0, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \Gamma_0,\ t > 0, \\ \dfrac{\partial u}{\partial \nu}(x,t) = \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau) & x \in \Gamma_1,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u_t(x,0) = u_1(x) & x \in \Omega, \\ u_t(x, t-\tau) = g_0(x, t-\tau) & x \in \Omega, \tau \in]0,1[, \end{cases} \tag{1.7}$$

and proved under the assumption

$$\mu_2 < \mu_1, \tag{1.8}$$

that the solution is exponentially stable. On the contrary, if (1.8) does not hold, they found a sequence of delays for which the corresponding solution of (1.7) will be unstable. We also recall the result by Xu *et al.* [16], where the authors proved the same result as in [13] for the one space dimension by adopting the spectral analysis approach.

The aim of this paper is to study the well-possessedness and asymptotic stability of system (1.1)-(1.3) under proper conditions on nonlinear weights $\mu_1(t), \mu_2(t)$, on the contrary in [3] where the author considered that the weights μ_1, μ_2 are positive constants, we prove global existence and an estimate for the decay rate of the energy. The paper is organized as follows. In Section 2 we provide notations that will be used later. In Section 3 we state and prove the global existence result. In Section 4, we prove the exponential decay of the energy when time goes to infinity.

2 Notation and preliminaries

In this section, we present some material in the proof of our main result. We assume
(A_1) $\mu_1 : \mathbb{R}_+ \longrightarrow]0,\infty[$ is a non-increasing function of class $C^1(\mathbb{R}_+)$ satisfying

$$\left|\frac{\mu_1'(t)}{\mu_2(t)}\right| \leq M_1, \quad \forall t \geq 0, \tag{2.1}$$

(A_2) $\mu_2 : \mathbb{R}_+ \longrightarrow]0,\infty[$ is a non-increasing function of class $C^1(\mathbb{R}_+)$, which is not necessarily positive or monotone, such that

$$|\mu_2(t)| \leq \beta\mu_1(t), \tag{2.2}$$

$$|\mu_2'(t)| \leq M_2\mu_1(t), \tag{2.3}$$

for some $0 < \beta < 1$ and $M_2 > 0$

3 Global existence and energy decay

In this section, we prove the local existence and the uniqueness of the solution of system (1.1)-(1.3) by using the semi-group theory. So let us introduce the following new variable [13]

$$y(x,\rho,t) = u_t(x,t-\tau\rho). \tag{3.1}$$

Then, we get

$$\tau y_t(x,\rho,t) + y_\rho(x,\rho,t) = 0, \qquad \text{in } \Omega \times]0,1[\times]0,+\infty[. \tag{3.2}$$

Therefore, problem (1.1) is equivalent to

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t) + \mu_1(t)u_t(x,t) + \mu_2(t)y(x,1,t) = 0, & (x,t) \in \Omega \times]0,+\infty[, \\ v_{tt}(x,t) - bv_{xx}(x,t) = 0, & (x,t) \in]L_1,L_2[\times]0,+\infty[, \\ \tau y_t(x,\rho,t) + y_\rho(x,\rho,t) = 0, & \text{in } \Omega \times]0,1[\times]0,+\infty[, \end{cases} \tag{3.3}$$

which together with (1.3) can be rewritten as:

$$\begin{cases} U' = \mathcal{A}U, \\ U(0) = (u_0,u_1,v_0,v_1,f_0(.,-.\tau))^{\mathrm{T}}, \end{cases} \tag{3.4}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ \varphi \\ v \\ \psi \\ y \end{pmatrix} = \begin{pmatrix} \varphi \\ au_{xx} - \mu_1(t)\varphi - \mu_2(t)y(.,1) \\ \psi \\ bv_{xx} \\ -\frac{1}{\tau}y_\rho \end{pmatrix} \tag{3.5}$$

with the domain

$$D(\mathcal{A}) = \left\{ \begin{array}{ll} (u,\varphi,v,\psi,y)^{\mathrm{T}} \in \mathcal{H}; & y(.,0) = \varphi \text{ on } \Omega \\ u(L_i,t) = v(L_i,t), & i = 1,2 \\ au_x(L_i,t) = bv_x(L_i,t), & i = 1,2 \end{array} \right\},$$

where

$$\mathcal{H} = H^2(\Omega) \cap H^1(\Omega) \times H^1(\Omega) \times H^2(]L_1,L_2[) \cap H^1(]L_1,L_2[) \times H^1(]L_1,L_2[) \times L^2(0,1,H^1(\Omega)).$$

Now the energy space is defined by

$$\mathcal{K} = H^1(\Omega) \times L^2(\Omega) \times H^1(]L_1,L_2[) \times L^2(]L_1,L_2[) \times L^2((\Omega) \times]0,1[).$$

Let

$$U = (u,\varphi,v,\psi,y)^{\mathrm{T}}, \quad \bar{U} = (\bar{u},\bar{\varphi},\bar{v},\bar{\psi},\bar{y})^{\mathrm{T}}.$$

Let ξ be a non-increasing function of class $C^1(\mathbb{R}_+)$ such that

$$\zeta(t) = \overline{\zeta}\mu_1(t), \tag{3.6}$$

where

$$\tau\beta < \overline{\zeta} < \tau(2-\beta). \tag{3.7}$$

We define the inner product in $\mathcal{K}$ as follows:

$$(U,\bar{U})_{\mathcal{K}} = \int_\Omega \{\varphi\bar{\varphi} + a u_x \bar{u}_x\} dx + \int_{L_1}^{L_2} \{\psi\bar{\psi} + b v_x \bar{v_x}\} dx + \frac{\zeta(t)}{2}\int_\Omega\int_0^1 y(x,\rho)\bar{y}(x,\rho) d\rho dx.$$

The existence and uniqueness result is stated as follows;

Theorem 3.1. Suppose that (A_1) and (A_2) hold, for any $U_0 \in \mathcal{K}$ there exists a unique solution $U \in C([0,+\infty[,\mathcal{K})$ of problem (3.4). Moreover, if $U_0 \in D(\mathcal{A})$, then

$$U \in C([0,+\infty[, D(\mathcal{A})) \cap C^1([0,+\infty[,\mathcal{K}).$$

Proof . In order to prove the result stated in Theorem 3.1, we use the semigroup theory, that is, we show that the operator $\mathcal{A}$ generates a C_0-semigroup in $\mathcal{K}$. In this step, we concern ourselves to prove that the operator $\mathcal{A}$ is dissipative. Indeed, for $U = (u,\varphi,v,\psi,y)^{\mathrm{T}} \in D(\mathcal{A})$, where $\varphi(L_2) = \psi(L_2)$, using (3.6) and (3.7), we have

$$\begin{aligned}(\mathcal{A}U,U)_{\mathcal{K}} &= a\int_\Omega u_{xx}\varphi dx + b\int_{L_1}^{L_2} v_{xx}\psi dx - \mu_1(t)\int_\Omega \varphi^2 dx \\ &- \mu_2(t)\int_\Omega y(.,1)\varphi dx - \frac{\zeta(t)}{\tau}\int_\Omega\int_0^1 y(x,\rho)y_\rho(x,\rho) d\rho dx \\ &+ a\int_\Omega u_x\varphi_x dx + b\int_{L_1}^{L_2} v_x\psi_x dx.\end{aligned} \tag{3.8}$$

Looking now at the last term of the right-hand side of (3.8), we have

$$\begin{aligned}\zeta(t)\int_\Omega\int_0^1 y(x,\rho)y_\rho(x,\rho) d\rho dx &= \zeta(t)\int_\Omega \frac{1}{2}\frac{\partial}{\partial\rho} y^2(x,\rho) d\rho dx \\ &= \frac{\zeta(t)}{2}\int_\Omega (y^2(x,1) - y^2(x,0)) dx.\end{aligned} \tag{3.9}$$

Performing an integration by parts in (3.8), keeping in mind the fact that $y(x,0,t) = \varphi(x,t)$ and using (3.9), we have from (3.8)

$$\begin{aligned}(\mathcal{A}U,U)_{\mathcal{K}} &= a[u_x\varphi]_{\partial\Omega} + b[v_x\psi]_{L^1}^{L_2} \\ &- \mu_1(t)\int_\Omega \varphi^2 dx + \frac{\zeta(t)}{2\tau}\int_\Omega \varphi^2 dx - \mu_2(t)\int_\Omega y(.,1)\varphi dx - \frac{\zeta(t)}{2\tau}\int_\Omega y^2(x,1) dx.\end{aligned} \tag{3.10}$$

Using Young's inequality, (1.2) and the equality $\varphi(L_2) = \psi(L_2)$, we obtain from (3.10), that

$$(\mathcal{A}U,U)_{\mathcal{K}} \le -\mu_1(t)\left(1 - \frac{\overline{\zeta}}{2\tau} - \frac{\beta}{2}\right)\int_\Omega \varphi^2 dx - \mu_1(t)\left(\frac{\overline{\zeta}}{2\tau} - \frac{\beta}{2}\right)\int_\Omega y^2(x,1) dx. \tag{3.11}$$

Consequently, using (3.7), then we deduce that $(\mathcal{A}U,U)_{\mathcal{K}} \le 0$. Thus, the operator $\mathcal{A}$ is dissipative.

Now to show that the operator $\mathcal{A}$ is maximal monotone, it is sufficient to show that the operator $\lambda I - \mathcal{A}$ is surjective for a fixed $\lambda > 0$. Indeed, given $(f_1, f_2, g_1, g_2, h)^{\mathrm{T}} \in \mathcal{K}$, we seek $U = (u,\varphi,v,\psi,y)^{\mathrm{T}} \in D(\mathcal{A})$ solution of

$$\begin{pmatrix} \lambda u - \varphi \\ \lambda\varphi - a u_{xx} + \mu_1(t)y(.,0) + \mu_2(t)y(.,1) \\ \lambda v - \psi \\ \lambda\psi - b v_{xx} \\ \lambda y + \frac{1}{\tau}y_\rho \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ g_1 \\ g_2 \\ h \end{pmatrix}. \tag{3.12}$$

Suppose we have find (u, v) with the appropriate regularity, then

$$\begin{aligned} \varphi &= \lambda u - f_1, \\ \psi &= \lambda v - g_1. \end{aligned} \tag{3.13}$$

It is clear that $\varphi \in H^1(\Omega)$ and $\psi \in H^1(L_1, L_2)$, furthermore, by (3.12), we can find y as $y(x,0) = \varphi(x)$, $x \in \Omega$, using the approach as in Nicaise & Pignotti [13], we obtain, by using the equation in (3.12)

$$y(x,\rho) = \varphi(x)e^{-\lambda\rho\tau} + \tau e^{-\lambda\rho\tau}\int_0^\rho h(x,\sigma)e^{\lambda\sigma\tau}d\sigma.$$

From (3.13), we obtain

$$y(x,\rho) = \lambda u(x)e^{-\lambda\rho\tau} - f_1(x)e^{-\lambda\rho\tau} + \tau e^{-\lambda\rho\tau}\int_0^\rho h(x,\sigma)e^{\lambda\sigma\tau}d\sigma.$$

By using (3.12) and (3.13), the functions u, v satisfying the following equations:

$$\begin{aligned} \lambda^2 u - a u_{xx} + \mu_1(t)\varphi + \mu_2(t)y(x,1) &= f_2 + \lambda f_1, \\ \lambda^2 v - b v_{xx} &= g_2 + \lambda g_1. \end{aligned} \tag{3.14}$$

Since

$$\begin{aligned} y(x,1) &= \varphi(x)e^{-\lambda\tau} + \tau e^{-\lambda\tau}\int_0^1 h(x,\sigma)e^{\lambda\tau}d\sigma, \\ &= \lambda u e^{-\lambda\tau} + y_0(x), \end{aligned}$$

for $x \in \Omega$, we have

$$y_0(x) = -f_1(x) + \tau e^{-\lambda\tau}\int_0^1 h(x,\sigma)e^{\lambda\tau}d\sigma. \tag{3.15}$$

The problem (3.14) can be reformulated as

$$\begin{aligned} &\int_\Omega (\lambda^2 u - a u_{xx} + \mu_1(t)\lambda u + \mu_2(t)\lambda u e^{-\lambda\tau})\omega_1 dx \\ &= \int_\Omega (f_2 + \lambda f_1 - \mu_2(t)\lambda y_0(x))\omega_1 dx, \qquad \forall \omega_1 \in H^1(\Omega). \\ &\int_{L_1}^{L_2} (\lambda^2 v - b v_{xx})\omega_2 dx \\ &= \int_{L_1}^{L_2} (g_2 + \lambda g_1)\omega_2 dx, \qquad \forall \omega_2 \in H^1(]L_1, L_2[). \end{aligned} \tag{3.16}$$

Integrating the first equation in (3.16) by parts, we obtain

$$\begin{aligned} &\int_\Omega (\lambda^2 u - a u_{xx} + \mu_1(t) u + \mu_2(t)\lambda u e^{-\lambda\tau})\omega_1 dx \\ &= \int_\Omega \lambda^2 u\omega_1 dx - a\int_\Omega u_{xx}\omega_1 dx + \mu_1(t)\int_\Omega \lambda u dx + \mu_2(t)\int_\Omega \lambda u e^{-\lambda\tau}\omega_1 dx \\ &= \int_\Omega \lambda^2 u\omega_1 dx + a\int_\Omega u_x(\omega_1)_x dx - [a u_x\omega_1]_{\partial\Omega} + \mu_1(t)\int_\Omega \lambda u dx + \mu_2(t)\int_\Omega \lambda u e^{-\lambda\tau}\omega_1 dx \\ &= \int_\Omega (\lambda^2 + \mu_1(t)\lambda + \mu_2(t)\lambda e^{-\lambda\tau})u\omega_1 dx + a\int_\Omega u_x(\omega_1)_x dx - [a u_x\omega_1]_{\partial\Omega}. \end{aligned} \tag{3.17}$$

Integrating the second equation in (3.16) by parts, we obtain

$$\int_{L_1}^{L_2} (\lambda^2 v - b v_{xx})\omega_2 dx = \int_{L_1}^{L_2} \lambda^2 v\ \omega_2 dx + b\int_{L_1}^{L_2} v_x(\omega_2)_x dx - [b v_x\omega_2]_{L_1}^{L_2}. \tag{3.18}$$

Using (3.17) and (3.18), the problem (3.16) is equivalent to the problem

$$\Phi((u,v),(\omega_1,\omega_2)) = l(\omega_1,\omega_2), \tag{3.19}$$

where the bilinear form $\Phi : (H^1(\Omega))^2 \times (H^1(]L_1,L_2[)^2) \to \mathbb{R}$ and the linear form $l : H^1(\Omega)\times H^1(]L_1,L_2[) \to \mathbb{R}$ are defined by

$$\begin{aligned}\Phi((u,v),(\omega_1,\omega_2)) = & \int_\Omega (\lambda^2 + \mu_1(t)\lambda + \mu_2(t)\lambda e^{-\lambda\tau})u\omega_1 dx + a\int_\Omega u_x(\omega_1)_x dx - [a u_x \omega_1]_{\partial\Omega} \\ & + \int_{L_1}^{L_2} \lambda^2 v\ \omega_2 dx + b\int_{L_1}^{L_2} v_x(\omega_2)_x dx - [b v_x \omega_2]_{L_1}^{L_2},\end{aligned}$$

and

$$l(\omega_1,\omega_2) = \int_\Omega (f_2 + \lambda f_1 - \mu_2(t)\lambda y_0(x))\omega_1 dx + \int_{L_1}^{L_2}(g_2+\lambda g_1)\omega_2 dx.$$

It is clear that Φ is continuous and coercive, and l is continuous. So applying the Lax-Milgram theorem, we deduce that for all $(\omega_1,\omega_2) \in H^1(\Omega)\times H^1(]L_1,L_2[)$, problem (3.19) admits a unique solution $(u,v) \in H^1(\Omega)\times (H^1(]L_1,L_2[)$. It follows from (3.17) and (3.18) that $(u,v) \in (H^2(\Omega)\cap H^1(\Omega)\times H^2(]L_1,L_2[)\cap H^1(]L_1,L_2[))$. Therefore, the operator $(\lambda I - \mathcal{A})$ is dissipative for any $\lambda > 0$. Then the result in Theorem 3.1 follows from the Hille-Yoshida theorem. □

4 Exponential decay of the solution

In this section we investigate the asymptotic of the system (1.1)-(1.3). For any regular solution of (1.1)-(1.3), we define the energy as:

$$E_1(t) = \frac{1}{2}\int_\Omega u_t^2(x,t)dx + \frac{a}{2}\int_\Omega u_x^2(x,t)dx, \tag{4.1}$$

and

$$E_2(t) = \frac{1}{2}\int_{L_1}^{L_2} v_t^2(x,t)dx + \frac{b}{2}\int_{L_1}^{L_2} v_x^2(x,t)dx. \tag{4.2}$$

The total energy is defined as:

$$E(t) = E_1(t) + E_2(t) + \frac{\zeta(t)}{2}\int_\Omega\int_0^1 y^2(x,\rho,t)d\rho dx, \tag{4.3}$$

where ζ defined in (3.6).

Our decay result reads as follows:

Theorem 4.1. Let (u,v) be the solution of (1.1)-(1.3). Assume that (2.1)-(2.3) and

$$\frac{a}{b} < \frac{L_3 + L_1 - L_2}{2(L_2 - L_1)} \tag{4.4}$$

hold. Then there exist two positive constants C and d such that

$$E(t) \le Ce^{-dt}, \qquad \forall t \ge 0. \tag{4.5}$$

The proof of Theorem 4.1 will be done through some lemmas:

Lemma 4.2. Let (u,v,y) be the solution of (3.3), (1.3). Then the energy functional defined by (4.3) satisfies

$$\frac{dE(t)}{dt} \le -\mu_1(t)\left(1 - \frac{\overline{\zeta}}{2\tau} - \frac{\beta}{2}\right)\int_\Omega y^2(x,0,t)dx - \mu_1(t)\left(\frac{\overline{\zeta}}{2\tau} - \frac{\beta}{2}\right)\int_\Omega y^2(x,1,t)dx. \tag{4.6}$$

Proof . We have from (4.3) that

$$\frac{dE_1(t)}{dt} = \int_\Omega u_{tt}(x,t)u_t(x,t)dx + a\int_\Omega u_{xt}(x,t)u_x(x,t)dx. \tag{4.7}$$

Using system (3.3), and integrating by parts, we obtain

$$\frac{dE_1(t)}{dt} = a[u_x u_t]_{\partial\Omega} - \mu_1(t)\int_\Omega u_t^2\,(x,t) - \mu_2(t)\int_\Omega u_t(x,t)y\,(x,1,t))dx. \tag{4.8}$$

On the other hand, we have

$$\frac{dE_2(t)}{dt} = b[v_x v_t]_{L_1}^{L_2}. \tag{4.9}$$

Using the fact that

$$\begin{aligned}\frac{d}{dt}\frac{\zeta(t)}{2}\int_\Omega\int_0^1 y^2(x,\rho,t)d\rho dx &= \zeta(t)\int_\Omega\int_0^1 y(x,\rho,t)y_t(x,\rho,t)d\rho dx\\ &+\frac{\zeta'(t)}{2}\int_\Omega\int_0^1 y^2(x,\rho,t)d\rho dx,\\ &= -\frac{\zeta(t)}{\tau}\int_\Omega\int_0^1 y_\rho(x,\rho,t)y(x,\rho,t)d\rho dx\\ &+\frac{\zeta'(t)}{2}\int_\Omega\int_0^1 y^2(x,\rho,t)d\rho dx\\ &= -\frac{\zeta(t)}{2\tau}\int_\Omega\int_0^1 \frac{d}{d\rho}y^2(x,\rho,t)d\rho dx\\ &+\frac{\zeta'(t)}{2}\int_\Omega\int_0^1 y^2(x,\rho,t)d\rho dx\\ &= -\frac{\zeta(t)}{2\tau}\int_\Omega (y^2(x,1,t)-y^2(x,0,t))dx\\ &+\frac{\zeta'(t)}{2}\int_\Omega\int_0^1 y^2(x,\rho,t)d\rho dx.\end{aligned} \tag{4.10}$$

From (4.8), (4.9), (4.10) and using the conditions (1.2), we know that

$$\begin{aligned}E'(t) = \frac{\zeta(t)}{2\tau}\int_\Omega y^2(x,0,t)dx - \frac{\zeta(t)}{2\tau}\int_\Omega y^2(x,1,t)dx + \frac{\zeta'(t)}{2}\int_\Omega\int_0^1 y^2(x,\rho,t)d\rho dx\\ -\mu_1(t)\int_\Omega u_t^2\,(x,t) - \mu_2(t)\int_\Omega u_t(x,t)y\,(x,1,t))dx,\end{aligned} \tag{4.11}$$

Due to the Young's inequality, we have

$$\mu_2(t)\int_\Omega u_t(x,t)y\,(x,1,t))dx \le \frac{|\mu_2(t)|}{2}\int_\Omega u_t^2(x,t)dx + \frac{|\mu_2(t)|}{2}\int_\Omega y^2(x,1,t)dx. \tag{4.12}$$

□ Inserting (4.12) in (4.11), we obtain

$$\begin{aligned}E'(t) &\le \left(-\mu_1(t) - \frac{\zeta(t)}{2\tau} - \frac{|\mu_2(t)|}{2}\right)\int_\Omega u_t^2(x,t)dx\\ &-\left(\frac{\zeta(t)}{2\tau} - \frac{|\mu_2(t)|}{2}\right)\int_\Omega y^2(x,1,t)dx\\ &+\frac{\zeta'(t)}{2}\int_\Omega\int_0^1 y^2(x,\rho,t)d\rho dx\\ &\le -\mu_1(t)\left(1 - \frac{\overline{\zeta}}{2\tau} - \frac{\beta}{2}\right)\int_\Omega u_t^2(x,t)dx\\ &-\mu_1(t)\left(\frac{\overline{\zeta}}{2\tau} - \frac{\beta}{2}\right)\\ &\le 0.\end{aligned} \tag{4.13}$$

Hence, the proof is complete.
Following [1], we define the functional

$$I(t)=\int_\Omega\int_{t-\tau}^{t} e^{s-t}u_t^2(x,s)dsdx,$$

and we have the following lemma.

Lemma 4.3. Let (u,v) be the solution of (1.1)-(1.3). Then we have

$$\frac{dI(t)}{dt}\leq\int_\Omega u_t^2(x,t)dx-e^{-\tau}\int_\Omega u_t^2(x,t-\tau)dx-e^{-\tau}\int_\Omega\int_{t-\tau}^{t} u_t^2(x,s)dsdx. \tag{4.14}$$

The proof of Lemma 4.3 is straightforward, we omit the details.

Now, we define the functional $\mathcal{D}(t)$ as follows:

$$\mathcal{D}(t)=\int_\Omega uu_tdx+\int_{L_1}^{L_2} vv_tdx. \tag{4.15}$$

Thus, we have the following estimate.

Lemma 4.4. For any $\varepsilon_1>0$ and C_p is the Poincaré's constant, the functional $\mathcal{D}(t)$ satisfies the following estimate:

$$\begin{aligned}\frac{d}{dt}\mathcal{D}(t)\leq&\left(1+\frac{1}{2\varepsilon_1}\right)\int_\Omega u_t^2dx-\left(a-\mu_1^2(0)C_p\varepsilon_1\right)\int_\Omega u_x^2dx-b\int_{L_1}^{L_2} v_x^2dx\\&+\int_{L_1}^{L_2} v_t^2dx+\frac{\beta^2}{2\varepsilon_1}\int_\Omega y^2(x,1,t)dx.\end{aligned} \tag{4.16}$$

Proof . Taking the derivative of $\mathcal{D}(t)$ with respect to t, we find

$$\begin{aligned}\frac{d}{dt}\mathcal{D}(t)=&\int_\Omega u_t^2dx-a\int_\Omega u_x^2dx-\mu_1(t)\int_\Omega uu_tdx+\int_{L_1}^{L_2} v_t^2dx-b\int_{L_1}^{L_2} v_x^2dx\\&-\mu_2(t)\int_\Omega u(x,t)y(x,1,t)dx+[au_xu]_{\partial\Omega}+[bv_xv]_{L_1}^{L_2}.\end{aligned} \tag{4.17}$$

From hypothesis (A_1) and (A_2), we have

$$\begin{aligned}\frac{d}{dt}\mathcal{D}(t)\leq&\int_\Omega u_t^2dx-a\int_\Omega u_x^2dx+\mu_1(0)\left|\int_\Omega uu_tdx\right|+\beta\mu_1(0)\left|\int_\Omega u(x,t)y(x,1,t)dx\right|\\&+\int_{L_1}^{L_2} v_t^2dx-b\int_{L_1}^{L_2} v_x^2dx+[au_xu]_{\partial\Omega}+[bv_xv]_{L_1}^{L_2}.\end{aligned} \tag{4.18}$$

Using the boundary conditions (1.2), we have

$$\begin{aligned}[au_xu]_{\partial\Omega}+[bv_xv]_{L_1}^{L_2}=&\,au_x(L_1,t)u(L_1,t)-au_x(L_2,t)u(L_2,t)\\&+bv_x(L_2,t)v(L_2,t)-bv_x(L_1,t)v(L_1,t)=0.\end{aligned}$$

Now, by Young's inequality and Poincaré's inequality we conclude the lemma. □

Now, inspired by [10], we introduce the functional

$$q(x)=\begin{cases}x-\dfrac{L_1}{2}, & x\in[0,L_1],\\ x-\dfrac{L_2+L_3}{2}, & x\in[L_2,L_3],\\ \dfrac{L_2-L_3-L_1}{2(L_2-L_1)}(x-L_1)+\dfrac{L_1}{2}, & x\in[L_1,L_2].\end{cases} \tag{4.19}$$

It is easy to see that $q(x)$ is bounded, i.e., $|q(x)| \leq M$, where

$$M = \max\left\{\frac{L_1}{2}, \frac{L_3 - L_2}{2}\right\}.$$

Next, we define the following functionals

$$\mathcal{F}_1(t) = -\int_\Omega q(x) u_x u_t dx,$$

and

$$\mathcal{F}_2(t) = -\int_{L_1}^{L_2} q(x) v_x v_t dx.$$

Then, we have the following estimates:

Lemma 4.5. For any $\varepsilon_2 > 0$, we have the estimates:

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_1(t) \leq & \left(\frac{1}{2} + \frac{1}{2\varepsilon_2}\right)\int_\Omega u_t^2 dx + \left(\frac{a}{2} + M^2 \mu_1(0)^2 \varepsilon_2\right)\int_\Omega u_x^2 dx + \frac{\beta^2}{2\varepsilon_2}\int_\Omega y^2\left(x,1,t\right)dx \\ & - \frac{a}{4}\left[(L_3 - L_2)u_x^2(L_2,t) + L_1 u_x^2(L_1,t)\right] \\ & - \frac{1}{4}\left[(L_3 - L_2)u_t^2(L_2,t) + L_1 u_t^2(L_1,t)\right],\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_2(t) \leq & \frac{L_2 - L_3 - L_1}{4\left(L_2 - L_1\right)}\left(\int_{L_1}^{L_2} v_t^2 dx + \int_{L_1}^{L_2} b v_x^2 dx\right) \\ & + \frac{b}{4}\left((L_3 - L_2)v_x^2(L_2,t) + L_1 v_x^2(L_1,t)\right) \\ & + \frac{1}{4}\left((L_3 - L_2)v_t^2(L_2,t) + L_1 v_t^2(L_1,t)\right).\end{aligned} \tag{4.21}$$

Proof . Taking the derivative of $\mathcal{F}_1(t)$ with respect to t and using equation (3.3), we get

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_1(t) &= -\int_\Omega q(x) u_{tx} u_t dx - \int_\Omega q(x) u_x u_{tt} dx, \\ &= -\int_\Omega q(x) u_{tx} u_t dx \\ &- \int_\Omega q(x) u_x \left(a u_{xx}\left(x,t\right) - \mu_1(t) u_t(x,t) - \mu_2(t) y\left(x,1,t\right)\right) dx, \\ &= -\int_\Omega q(x) u_{tx} u_t dx - a\int_\Omega q(x) u_x u_{xx}(x,t) dx \\ &+ \mu_1(t)\int_\Omega u_t(x,tdx + \mu_2\int_\Omega y(x,1,t)dx.\end{aligned}$$

Using integration by parts, we find

$$\int_\Omega q(x) u_{tx} u_t dx = -\frac{1}{2}\int_\Omega q'(x) u_t^2 dx + \frac{1}{2}\left[q(x) u_t^2\right]_{\partial\Omega}. \tag{4.22}$$

On the other hand, we have

$$\int_\Omega a q(x) u_{xx} u_x dx = -\frac{1}{2}\int_\Omega a q'(x) u_x^2 dx + \frac{1}{2}\left[a q(x) u_x^2\right]_{\partial\Omega}. \tag{4.23}$$

Inserting (4.22) and (4.23) into (4.22), we find

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_1(t) = & \frac{1}{2}\int_\Omega q'(x) u_t^2 dx + \frac{1}{2}\int_\Omega a q'(x) u_x^2 dx - \frac{1}{2}\left[q(x) u_t^2\right]_{\partial\Omega} - \frac{1}{2}\left[a q(x) u_x^2\right]_{\partial\Omega} \\ & + \int_\Omega q(x) u_x\left(\mu_1(t) u_t(x,t) + \mu_2(t) y\left(x,1,t\right)\right) dx.\end{aligned} \tag{4.24}$$

By (A_1) and (A_2), we have

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_1(t) &\leq \frac{1}{2}\int_\Omega q'(x)u_t^2dx + \frac{1}{2}\int_\Omega aq'(x)u_x^2dx - \frac{1}{2}\left[q(x)u_t^2\right]_{\partial\Omega} - \frac{1}{2}\left[aq(x)u_x^2\right]_{\partial\Omega} \\ &+ \mu_1(0)\left|\int_\Omega q(x)u_xu_t(x,t)dx\right| + \beta\mu_1(0)\left|\int_\Omega q(x)u_xy(x,1,t)dx\right|, \\ &\leq \frac{1}{2}\int_\Omega u_t^2dx + \frac{1}{2}\int_\Omega au_x^2dx - \frac{1}{2}\left[q(x)u_t^2\right]_{\partial\Omega} - \frac{1}{2}\left[aq(x)u_x^2\right]_{\partial\Omega} \\ &+ \mu_1(0)M\left|\int_\Omega u_xu_t(x,t)dx\right| + \beta\mu_1(0)M\left|\int_\Omega u_xy(x,1,t)dx\right|.\end{aligned} \tag{4.25}$$

By using the boundary conditions, we have

$$\begin{aligned}\frac{1}{2}\left[q(x)u_t^2\right]_{\partial\Omega} &= \frac{1}{4}\left[(L_3-L_2)u_t^2(L_2,t) + L_1u_t^2(L_1,t)\right], \\ -\frac{a}{2}\left[q(x)u_t^2\right]_{\partial\Omega} &\leq \frac{a}{4}\left[(L_3-L_2)u_x^2(L_2,t) + L_1u_x^2(L_1,t)\right].\end{aligned}$$

Inserting the above two equalities into (4.25) and by Young's inequality we obtain (4.20).

By the same method, taking the derivative of $\mathcal{F}_2(t)$ with respect to t, we get

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_2(t) &= -\int_{L_1}^{L_2} q(x)v_{tx}v_tdx - \int_{L_1}^{L_2} q(x)v_xv_{tt}, \\ &= \frac{1}{2}\int_{L_1}^{L_2} q'(x)v_t^2dx - \frac{1}{2}\left[q(x)v_t^2\right]_{L_1}^{L_2} + \frac{1}{2}\int_{L_1}^{L_2} bq'(x)v_x^2dx - \frac{b}{2}\left[q(x)v_x^2\right]_{L_1}^{L_2}, \\ &\leq \frac{L_2-L_3-L_1}{4(L_2-L_1)}\left(\int_{L_1}^{L_2} v_t^2dx + \int_{L_1}^{L_2} bv_x^2dx\right) \\ &+ \frac{b}{4}\left((L_3-L_2)v_x^2(L_2,t) + L_1v_x^2(L_1,t)\right) \\ &+ \frac{1}{4}\left((L_3-L_2)v_t^2(L_2,t) + L_1v_t^2(L_1,t)\right).\end{aligned}$$

which is exactly (4.21). □

Proof .[Proof Theorem 4.1] We define the Lyapunov functional $\mathcal{L}(t)$ as follows

$$\mathcal{L}(t) = NE(t) + I(t) + \gamma_2\mathcal{D}(t) + \gamma_3\mathcal{F}_1(t) + \gamma_4\mathcal{F}_2(t), \tag{4.26}$$

where N, γ_2, γ_3 and γ_4 are positive constants that will be fixed later. By the Lemma 4.2, there exists a positive constant K such that

$$E'(t) \leq -K\left[\int_\Omega u_t^2dx + \int_\Omega y^2(x,1,t)dx\right]. \tag{4.27}$$

Now, it is clear from the boundary conditions (1.2), that

$$a^2u_x^2(L_i,t) = b^2v_x^2(L_i,t), \qquad i = 1,2. \tag{4.28}$$

Taking the derivative of (4.26) with respect to t and making use of (4.6), (4.14), (4.16), (4.20),(4.21) and taking into account (4.28), we obtain

$$
\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) \leq & -\left\{KN - \left(1+\frac{1}{2\varepsilon_1}\right)\gamma_2 - \left(\frac{1}{2}+\frac{1}{2\varepsilon_2}\right)\gamma_3 + 1)\right\}\int_\Omega u_t^2 dx \\
& - \left((KN - \frac{\beta^2}{2\varepsilon_1}\gamma_2 - \frac{\beta^2}{2\varepsilon_2}\gamma_3 + e^{-\tau}\right)\int_\Omega y^2(x,1,t)dx \\
& - \left[\left(a - \mu_1^2(0)C_p\varepsilon_1\right)\gamma_2 - \left(\frac{a}{2} + M^2\mu_1^2(0)\varepsilon_2\right)\gamma_3\right]\int_\Omega u_x^2 dx \\
& - \left[\gamma_2 b - b\frac{L_2 - L_3 - L_1}{4(L_2 - L_1)}\gamma_4\right]\int_{L_1}^{L_2} v_x^2 dx \\
& + \left\{\frac{L_2 - L_3 - L_1}{4(L_2 - L_1)}\gamma_4 + \gamma_2\right\}\int_{L_1}^{L_2} v_t^2 dx - e^{-\tau}\int_\Omega\int_{t-\tau}^t u_t^2(x,s)dsdx \\
& - \left(\gamma_3 - \frac{a}{b}\gamma_4\right)\frac{a(L_3 - L_2)}{4}u_x^2(L_2,t) - \left(\gamma_3 - \frac{a}{b}\gamma_4\right)\frac{aL_1}{4}u_x^2(L_1,t) \\
& - (\gamma_3 - \gamma_4)\frac{L_1}{4}u_t^2(L_1,t)) - (\gamma_3 - \gamma_4)\frac{L_3 - L_2}{4}u_t^2(L_2,t)).
\end{aligned}
$$

At this point, we choose our constants in (4.29), carefully, such that all the coefficients in (4.29) will be negative. Indeed, under the assumption (4.4), we can always find γ_2, γ_3 and γ_4 such that

$$
\frac{L_2 - L_3 - L_1}{4(L_2 - L_1)}\gamma_4 + \gamma_2 < 0, \qquad \gamma_3 > \frac{a}{b}\gamma_4, \qquad \gamma_2 > \frac{\gamma_3}{2}. \tag{4.29}
$$

Once the above constants are fixed, we may choose ε_1 and ε_2 small enough such that

$$
\mu_1^2(0)C_p\varepsilon_1\gamma_2 + M\mu_1^2(0)\varepsilon_2\gamma_3 < a(\gamma_2 - \gamma_3/2).
$$

Finally, keeping in mind (3.6) and choosing N large enough such that the first and the second coefficients in (4.29) are negatives.

Consequently, from above, we deduce that there exist a positive constant η_1, such that (4.29) becomes

$$
\begin{aligned}
\frac{d\mathcal{L}(t)}{dt} \leq & -\eta_1\int_\Omega (u_t^2(x,t) + u_x^2(x,t) + u_t^2(x,t-\tau))dx \\
& - \eta_1\int_{L_1}^{L_2}(v_t^2(x,t) + v_x^2(x,t))dx - \eta_1\int_\Omega\int_{t-\tau}^t u_t^2(x,s)dsdx, \qquad \forall t \geq 0.
\end{aligned}
$$

Consequently, recalling (4.3), then, we deduce that there exist also $\eta_2 > 0$, such that

$$
\frac{d\mathcal{L}(t)}{dt} \leq -\eta_2 E(t), \qquad \forall t \geq 0. \tag{4.30}
$$

On the other hand, it is not hard to see that from (4.26) and for N large enough, there exist two positive constants β_1 and β_2 such that

$$
\beta_1\ E(t) \leq \mathcal{L}(t) \leq \beta_2 E(t), \qquad \forall t \geq 0. \tag{4.31}
$$

Combining (4.30) and (4.30), we deduce that there exists $\Lambda > 0$ for which the estimate

$$
\frac{d\mathcal{L}(t)}{dt} \leq -\Lambda\mathcal{L}(t), \qquad \forall t \geq 0, \tag{4.32}
$$

holds. Integrating (4.30) over $(0,t)$ and using (4.30) once again, then (4.5) holds. Then, the proof of the Theorem 4.1 is completed. □

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