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Inertial residual algorithm for fixed points of finite family of strictly pseudocontractive mappings in Banach spaces

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Abstract

In this paper, we introduce a new iteration method called inertial residual algorithm for finding a common fixed point of finite family of strictly pseudocontractive mappings in a real uniformly smooth Banach spaces. We also establish weak and strong convergence theorems for the scheme. Finally, we give numerical experiment to explain the proposed method. Our results generalize and improve many recent results in the literature.

Keywords: Nonexpansive, Strictly pseudocontractive mappings, Residual algorithm, fixed point, Banach spaces
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1 Introduction

Throughout this paper, X is assume to be a real Banach space with its dual X^*. The generalized duality map is a map $J_\varphi : X \to 2^{X^*}$ associated with a gauge function φ defined by

$$J_\varphi(x) = \{x^* \in X^* : \langle x, x^*\rangle = ||x||||x^*||, ||x^*|| = \varphi(||x||)\},$$

where $\varphi(t) = t^{p-1}$ for all $t \geq 0$ and $1 < p < \infty$. In particular, if $p = 2$, then,$J_\varphi = J_2$ is known as the normalized duality map written as J which is defined by

$$J(x) = \{x^* \in X^* : \langle x, x^*\rangle = ||x||^2, ||x^*|| = ||x||\}.$$

It is known (see [18, 19]) that if X is a real Hilbert space H, the normalized duality map reduces to identity map, i.e., $Jx = \{x\}$.

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Let $T : X \to X$ be a nonlinear map. The fixed point problem with respect to T is to find a point $x \in X$ such that

$$Tx = x. \tag{1.1}$$

We denote by $Fix(T)$ the set of all fixed points of T, i.e., $Fix(T) = \{x \in X : Tx = x\}$. A nonlinear map $T : X \to X$ said to be contraction, if there exists a real number $k \in (0, 1)$ such that

$$||Tx - Ty|| \leq k||x - y|| \qquad \forall x, y \in X. \tag{1.2}$$

However, if $k = 1$ in (1.2), then T is called nonexpansive. The map T is called pseudocontractive (see [6]), if for all $x, y \in X$ and $k > 0$, we have

$$||x - y|| \leq ||x - y + k[(I - T)x - (I - T)y]||. \tag{1.3}$$

As a result of Kato [26], (1.3) is equivalent to

$$\langle Tx - Ty, j(x - y)\rangle \leq ||x - y||^2, \tag{1.4}$$

for some $j(x - y) \in J(x - y)$. T is also called k - strictly pseudocontractive map (see [6]) if there exists $k > 0$ such that, for all $x, y \in X$ and for some $j(x - y) \in J(x - y)$,

$$\langle Tx - Ty, j(x - y)\rangle \leq ||x - y||^2 - k||x - y - (Tx - Ty)||^2. \tag{1.5}$$

Equivalently, if I is the identity operator, then, (1.5) becomes

$$\langle (I - T)x - (I - T)y, j(x - y)\rangle \geq k||(I - T)x - (I - T)y||^2. \tag{1.6}$$

If $X = H$, a real Hilbert space, then T is said to be k - strictly pseudocontractive (see [34]), if there exists $k \in [0, 1)$ such that for all $x, y \in X$ we have

$$||Tx - Ty||^2 \leq ||x - y||^2 + k||(I - T)x - (I - T)y||^2. \tag{1.7}$$

The map T is said to be L-Lipschitzian if there exists a constant $L > 0$ such that, for all $x, y \in X$,

$$||Tx - Ty|| \leq L||x - y||.$$

Remark 1.1. It is worthy mentioning that the class of strictly pseudocontractive mappings contains properly the class of nonexpansive mappings with $k = 0$ in (1.7) and every k - strictly pseudocontractive mapping is $\frac{1+k}{k}$ - Lipschitzian, see [12]. Furthermore, the class of strictly pseudocontractive mappings is a subclass of Lipschitz pseudocontractive mappings.

It was shown by Banach and Cacciopoli (see [7, 8, 9]) that if T satisfies (1.2), then it has a unique fixed point in X. They furthermore showed that if $x_0 \in X$ is arbitrarily chosen, the sequence $\{x_n\}$ defined by

$$x_{n+1} = Tx_n = T^{n+1}x_0 \qquad \forall n \geq 0, \tag{1.8}$$

converges strongly to the unique fixed point of T.
The iterative formula (1.8) was due to Picard [37], which is very useful in approximating fixed point of a map satisfying (1.2) (see [5]), and does not converge for general nonexpansive mappings (see [39]). Mann [31] introduced the Mann algorithm

$$x_{n+1} = (1 - \upsilon_n)x_n + \upsilon_n Tx_n, \qquad n \geq 0. \tag{1.9}$$

and showed that the sequence generated by (1.9) converges weakly to a fixed points of nonexpansive mappings, where $\{\upsilon_n\} \subset (0, 1)$ satisfying some conditions.
Halpern [23] introduced the following algorithm as a motivation for failure of (1.9) to converge strongly to fixed points of nonexpansive mappings.

$$x_{n+1} = \upsilon_n x_0 + (1 - \upsilon_n)Tx_n, \tag{1.10}$$

where $\{\upsilon_n\} \subset (0, 1)$ satisfying some conditions. Ishikawa [24] introduced the Ishikawa's iteration method

$$\begin{cases} y_n = \vartheta_n Tx_n + (1 - \vartheta_n)x_n, \\ x_{n+1} = (1 - \upsilon_n)x_n + \upsilon_n Ty_n, \ \forall n \geq 0, \end{cases} \tag{1.11}$$

where $\{\upsilon_n\}$, $\{\vartheta_n\} \subset (0,1)$ satisfy some conditions.

However, many authors have introduced different iterative algorithms to approximate fixed points of nonexpansive and strictly pseudocontractive mappings, see [14, 15, 16] and the references contained therein. There have been an interests in developing performance of algorithms (see [12, 21, 22, 28, 29, 33]), by use of inertial-type method, which was first proposed by Polyak in [36] as a measure to speeding up the convergence properties. The main character of inertial-type method is that the next iterate is defined by making use of the previous two iterates. Recently, the inertial methods have been studied by several authors (see [4, 10, 11, 25, 20, 30] and the references contained therein). Mainge [32] proposed the following inertial Mann algorithm by combining the Mann algorithm in [31] and inertial extrapolation.

$$\begin{cases} w_n = x_n + \vartheta_n(x_n - x_{n-1}), \\ x_{n+1} = (1-\upsilon_n)w_n + \upsilon_n T w_n, \quad n \geq 1. \end{cases} \tag{1.12}$$

Under some standard assumption on the control parameters, Mainge [32] proved weak convergence of the sequence generated by (1.12).
Furthermore, note that Mann iteration defined by (1.9) may fail to converge for Lipschizian pseudocontractive mapping in Hilbert space, (see [17]). However Browder and Petryshyn [6] proved a weak convergence theorems for k-srtictly pseudocontractive mappings, using(1.9) with constant sequence $\upsilon_n = \upsilon$, for all $n \geq 0$, in the setting of a real Hilbert space. Leemon [27] proposed the class of algorithms called residual algorithms for performing reinforcement learning with function approximation system. He proved that the direct and residual gradient algorithms are special cases of his proposed residual algorithms and showed that the residual algorithms combine the beneficial properties of both the direct and the residual gradient algorithms, such as speed, generalization and stability.

Recently, La Cruz [13] introduced a residual algorithm for approximating a fixed point of nonexpansive mapping, which is better and more computationally efficient than many existing methods, such as Mann algorithm (1.9), Halpern algorithm (1.10) and Ishikawa's algorithm (1.11) as follows;

Algorithm (1) (La Cruz [13])
Step 0: Choose $x_0 \in H$, $1 << \upsilon_{max} < \infty$, $\upsilon_0 \in (0, \upsilon_{max}]$, $\gamma, \delta \in (0,1)$ and positive sequence $\{\eta_n\}$ such that $\sum_{n=0}^{\infty} \eta_n < +\infty$.

Let $n := 0$ and $y_0 = x_0 - Tx_0$.
Step 1: If $||x_n - Tx_n|| = 0$, Stop.
Step 2: Compute x_{n+1}, y_{n+1} and υ_{n+1} as follows. Find m_n as the smallest nonnegative integer m such that;

$$||x_n - \delta^{m_n}\upsilon_n y_n - T(x_n - \delta^{m_n}\upsilon_n y_n)||^2 \leq (1 - \gamma\delta^{2m_n})||y_n||^2 + \eta_n. \tag{1.13}$$

$$Set \quad \theta_n = \delta^{m_n}, \quad x_{n+1} = x_n - \theta_n \upsilon_n y_n, \quad y_{n+1} = x_{n+1} - Tx_{n=1} \; and$$

$$\upsilon_{n+1} = \begin{cases} \frac{\theta_n \upsilon_n ||y_n||^2}{||y_n||^2 - \langle y_{n+1}, y_n\rangle}, & if \langle y_{n+1}, y_n\rangle \leq (1 - \frac{\upsilon_n}{\upsilon_{max}})||y_n||^2 \\ \upsilon_{max}, & Otherwise. \end{cases} \tag{1.14}$$

Step 3: Set $n = n+1$ and go back to step 1.
(A1) $||x_0|| < \infty$.
(A2) The set $\Omega_0 = \{x \in H : ||Fx||^2 \leq ||Fx_0||^2 + \eta\}$ is compact, where $F : H \to H$ is defined by $Fx = x - Tx$.
Under conditions (A1) and (A2), he proved weak convergence of the algorithm (1) in Hilbert space.

Inspired and motivated by the results of La Cruz [13] and Polyak [36], our intention in this paper is to introduce a new iteration method called inertial residual algorithm for finding a common fixed point of finite family of strictly pseudocontractive mappings in a real uniformly smooth Banach space. Our results improve and extend many recent results in the literature.

2 Preliminaries

A Banach space X is said to be smooth if for every $x_0 \in X$ with $||x_0|| = 1$, there exists a unique $x_0^* \in X^*$ such that

$$||x_0^*|| = 1 \; and \; \langle x_0, x_0^*\rangle = ||x_0||.$$

Geometrically, the smoothness condition means that a normed space's closed unit ball is smooth if the unit sphere has no corners or sharp bends. However the smoothness of a Banach space X is characterized by the function $\rho_X : [0,\infty) \to [0,\infty)$, called modulus of smoothness of X, defined by

$$\rho_X(t) = \sup\left\{\frac{||x+y|| + ||x-y||}{2} - 1 : ||x|| = 1, ||y|| = t\right\}.$$

Equivalently,

$$\rho_X(t) = \sup\left\{\frac{||x+ty|| + ||x-ty||}{2} - 1 : ||x|| = 1, ||y|| = 1\right\}.$$

X is said to be uniformly smooth if for any given $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$ with $||x|| = 1$ and $||y|| \leq \delta$, then

$$||x+y|| + ||x-y|| < 2 + \epsilon||y||.$$

In other words, it is said to be uniformly smooth if and only if

$$\lim_{t\to 0}\frac{\rho_X(t)}{t} = 0.$$

Remark 2.1. (see, [19])
(i) Every Hilbert space is uniformly smooth Banach space.
(ii) Every uniformly smooth Banach space is reflexive.
(iii) Furthermore, see, for example, [18] if X is smooth, then the duality map J is single valued and if X is uniformly smooth, then the norm on X is fréchet differentiable and J is norm-to-norm uniformly continuous on bounded subsets of X.

Definition 2.2. Let $T : X \to X$ be a map,

(i) T is said to be demiclosed at $y_0 \in X$, if for any sequence $\{x_n\}$ in X which converges weakly to $x_0 \in X$ and $Tx_n \to y_0$, it holds that $Tx_0 = y_0$.

(ii) T is said to be semicompact, if for any bounded sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x^* \in X$.

The following Lemmas will be needed in the proof of the main results.

Lemma 2.3. (see, [1]) Let X be a real Banach space and $J : X \to 2^{X^*}$ the duality mapping. Then, the following inequalities hold.

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j(x+y)\rangle \;\; \forall x, y \in X, \; \forall j(x+y) \in J(x+y).$$

Lemma 2.4. [40] Let X be a real Banach space with Fréchet differentiable norm. For $x \in X$, let β^* be a function define for $0 < t < \infty$ by

$$\beta^*(t) = \sup\left\{\left|\frac{||x+ty||^2 + ||x||^2}{t} - 2\langle y, j(x)\rangle\right| : ||y|| = 1\right\}. \tag{2.1}$$

Then, $\underset{t\to 0}{\beta}{}^*(t) = 0$ and

$$||x+h||^2 \leq ||x||^2 + 2\langle y, j(x)\rangle + ||h||\beta^*(||h||),$$

for all $h \in E - \{0\}$.

Remark 2.5. If $X = L_p$ $2 \leq p < \infty$, we know that

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j(x)\rangle + (p-1)||y||^2, \; \forall x, y \in X.$$

Then, β^* in (2.1) is estimated by $\beta^*(t) = (p-1)t$ for $t > 0$.
In our more general setting, throughout this work, we will assume that

$$\beta^*(t) \leq ct, \; t > 0 \; and \; for \; some \; c > 1, \tag{2.2}$$

where β^* is the function described by (2.1).

Lemma 2.6. (see [38] Lemma 1) Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers, such that

$$a_{n+1} \leq a_n + b_n,$$

with $\sum_{n=1}^{\infty} b_n < \infty$ for all $n \geq 1$, then $\lim\limits_{n\to\infty} a_n$ exists.

Lemma 2.7. (see, [41]) Let X be a real reflexive Banach space which satisfies Opial's property, K a nonempty, closed and convex subset of X and $T : K \to K$ a continuous pseudocontractive mapping. Then $I - T$ is demiclosed at 0.

Lemma 2.8. (see [35] Opial's property) A Banach space X is said to satisfy Opial's property if for every sequence $\{x_n\}$ in X such that $x_n \rightharpoonup x \in X$, then for any $y \in X$ such that $x \neq y$, we have

$$\liminf_{n\to\infty} ||x_n - x|| < \liminf_{n\to\infty} ||x_n - y||.$$

Lemma 2.9. Let $\{T_i\}_{i=1}^{p}$ be a k_i - strictly pseudocontractive mappings and let $k = max\{k_i\}$ and $\{\alpha_i\}_{i=1}^{p}$ be a finite real sequence in $(0, 1)$. Define $T_{\alpha_i}x := (1-\alpha_i)x+\alpha_i T_i x$ and $T := T_{\alpha_1} \circ T_{\alpha_2} \circ \ldots \circ T_{\alpha_p}$. Then, T_{α_i} for each $i = 1, 2, \ldots, p$ and T are strictly pseudocontractive mappings.

Proof . Consider the following computations;

$$\begin{aligned}
&\langle T_{\alpha_i}x - T_{\alpha_i}y, j(x-y)\rangle = \big\langle (1-\alpha_i)x + \alpha_i T_i x - \big((1-\alpha_i)y + \alpha_i T_i y\big), j(x-y)\big\rangle \\
&\quad = \quad \langle (1-\alpha_i)(x-y) + \alpha_i(T_i x - T_i y), j(x-y)\rangle \\
&\quad = \quad \langle (1-\alpha_i)(x-y), j(x-y)\rangle + \langle \alpha_i(T_i x - T_i y), j(x-y)\rangle \\
&\quad \leq \quad (1-\alpha_i)||x-y||^2 + \alpha_i\big(||x-y||^2 - k_i||(I-T_i)x - (I-T_i)y||^2\big) \\
&\quad = \quad ||x-y||^2 - \alpha_i k_i \Big|\Big| x - \frac{(T_{\alpha_i}x - x + \alpha_i x)}{\alpha_i} - \big(y - \frac{(T_{\alpha_i}y - y + \alpha_i y)}{\alpha_i}\big)\Big|\Big|^2 \\
&\quad = \quad ||x-y||^2 - \alpha_i k_i \Big|\Big| \frac{1}{\alpha_i}\big((x - T_{\alpha_i}x) - (y - T_{\alpha_i}y)\big)\Big|\Big|^2 \\
&\quad = \quad ||x-y||^2 - \frac{k_i}{\alpha_i}\big|\big|(I - T_{\alpha_i})x - (I - T_{\alpha_i})y\big|\big|^2.
\end{aligned}$$

Thus, T_{α_i} is $\frac{k_i}{\alpha_i}$ - strictly pseudocontractive mapping.
On the other hand

$$\begin{aligned}
&\langle Tx - Ty, j(x-y)\rangle = \big\langle T_{\alpha_1} \circ T_{\alpha_2} \circ \ldots \circ T_{\alpha_p}x - T_{\alpha_1} \circ T_{\alpha_2} \circ \ldots \circ T_{\alpha_p}y, j(x-y)\big\rangle \\
&\quad \leq \quad ||T_{\alpha_2}T_{\alpha_3} \circ \ldots \circ T_{\alpha_p}x - T_{\alpha_2} \circ T_{\alpha_3} \circ \ldots \circ T_{\alpha_p}y||^2 \\
&\qquad - \frac{k_1}{\alpha_1}||(I - T_{\alpha_1} \circ T_{\alpha_2} \circ \ldots \circ T_{\alpha_p})x - (I - T_{\alpha_1} \circ T_{\alpha_2} \circ \ldots \circ T_{\alpha_p})y||^2 \\
&\quad \leq \quad ||x-y||^2 - \frac{k_1}{\alpha_1}||(I-T)x - (I-T)y||^2.
\end{aligned}$$

This implies T is $\frac{k_1}{\alpha_1}$ - strictly pseudocontractive mapping. □

Lemma 2.10. Let X be a real Banach space with Fréchet differentiable norm and $T_i : X \to X$ be a k_i - strictly pseudocontractive mappings for $i = 1.2, 3, \ldots, p$. Let $\{\alpha_i\}_{i=1}^{p}$ be a finite real sequence in $(0, 1)$. Define $T_{\alpha_i}x := (1-\alpha_i)x + \alpha_i T_i x$ and $T = T_{\alpha_1} \circ T_{\alpha_2} \circ \ldots \circ T_{\alpha_p}$, as $\alpha_i \in (0, \mu]$, $\mu = min\{1, \frac{2k}{c}\}$, where $k = max\{k_i : i = 1, 2, \ldots, p\}$ and c is the constant appearing in (2.2), then

(i) T_{α_i} is nonexpansive.

(ii) $Fix(T_{\alpha_i}) = Fix(T_i)$.

(iii) T is nonexpansive.

Proof . (i) Let $x, y \in X$, then we have from Lemma 2.9 that

$$\begin{aligned}
||T_{\alpha_i}x - T_{\alpha_i}y||^2 &= ||(1-\alpha_i)x + \alpha_i T_i x - \big((1-\alpha_i)y + \alpha_i T_i y\big)||^2 \\
&= ||(x-y) - \alpha_i\big(x - y - (T_i x - T_i y)\big)||^2 \\
&\leq ||x-y||^2 - 2\alpha_i \langle x - y - (T_i x - T_i y), j(x-y)\rangle \\
&\quad + \alpha_i^2 ||x - y - (T_i x - T_i y)||\beta^*\big(||x - y - (T_i x - T_i y)||\big) \\
&\leq ||x-y||^2 - 2k_i\alpha_i||x - y - (T_i x - T_i y)||^2 + \alpha_i^2 c||x - y - (T_i x - T_i y||^2 \\
&= ||x-y||^2 - \alpha_i(2k_i - c\alpha_i)||x - y - (T_i x - T_i y)||^2 \\
&\leq ||x-y||^2.
\end{aligned}$$

Which implies that T_{α_i} is nonexpansive.

(ii) It is obvious that $x = T_{\alpha_i}x$ if and only if $x = T_i x$. Hence the result.
(iii) Let $x, y \in X$, then we have from (i)

$$\begin{aligned}
||Tx - Ty|| &= ||T_{\alpha_1} \circ T_{\alpha_2} \circ \ldots \circ T_{\alpha_p}x - T_{\alpha_1} \circ T_{\alpha_2} \circ \ldots \circ T_{\alpha_p}y|| \\
&\leq ||T_{\alpha_2} \circ T_{\alpha_3} \circ \ldots \circ T_{\alpha_p}x - T_{\alpha_2} \circ T_{\alpha_3} \circ \ldots \circ T_{\alpha_p}y|| \\
&\leq ||T_{\alpha_3} \circ T_{\alpha_4} \circ \ldots \circ T_{\alpha_p}x - T_{\alpha_3} \circ T_{\alpha_4} \circ \ldots \circ T_{\alpha_p}y|| \\
&\vdots \\
&\leq ||T_{\alpha_p}x - T_{\alpha_p}y|| \\
&\leq ||x-y||.
\end{aligned}$$

Which implies that T is nonexapnsive. $\square$

3 Main Results

In what follows, $T_i,\ i = 1, 2, 3, \ldots, p,\ p \in \mathbb{N}$ is a finite family of k_i - strictly pseudocontractive maps. Also, $\mu := min\{1, \frac{2k_1}{c}\}$, where c is the constant appearing in Lemma 2.4, $\alpha_i \in (0, \frac{\mu}{\sigma})$, $T_{\alpha_i}x := (1-\alpha_i)x + \alpha_i T_i x$, $T := T_{\alpha_1} \circ T_{\alpha_2} \circ \ldots \circ T_{\alpha_p}$, $M = diamX = \sup\limits_{x,y \in X} ||x-y||$ and m_* is the least positive integer for which $\sigma^m < \frac{2k_1}{c\alpha_1}$, for all $m \geq m_*$, $\sigma \in (0,1)$ fixed. Let $\{x_n\}$ be a sequence generated as follows:

Algorithm (2)
Step 0: Choose $x_0, x_1 \in X$, $\gamma, \sigma \in (0,1)$, positive sequence $\{\eta_n\}$ such that $\sum_{n=0}^{\infty} \eta_n < +\infty$ and

$$0 \leq \theta_n \leq \bar{\theta_n},\ \bar{\theta_n} = \begin{cases} min\{\frac{\delta_n}{||x_n - x_{n-1}||}, \frac{n-1}{n+\eta-1}\}, & if\ x_n \neq x_{n-1} \\ \frac{n-1}{n+\eta-1}, & Otherwise, \end{cases}$$

for some $\eta \geq 3$ and $\{\delta_n\}$ is a positive sequence such that $\sum_{n=0}^{\infty} \delta_n < \infty$. This idea was obtained from the recent inertial extrapolation step introduced in [2, 3].
Let $n := 1$, $w_1 = x_1 + \theta_1(x_1 - x_0)$ and $g_1 = w_1 - Tw_1$.
Step 1: If $||w_n - Tw_n|| = 0$, define $x_{n+k} := x_n\ \forall k \geq 1$ and stop. Otherwise,
Step 2: Find the least positive integer m_n such that for all $m \geq m_n$, the following inequality holds.

$$||w_n - \sigma^m g_n - T(w_n - \sigma^m g_n)||^2 \leq (1 - \gamma\sigma^{2m})||g_n||^2\ +\ \eta_n. \tag{3.1}$$

Step 3: Compute the step length λ_n using the formula

$$\lambda_n := \sigma^{max\{m_n,\, m_*\}}. \tag{3.2}$$

Step 4: Compute x_{n+1}, w_{n+1} and g_{n+1} using the following relations.

$$\begin{cases} x_{n+1} = w_n - \lambda_n g_n, \\ w_{n+1} = x_{n+1} + \theta_{n+1}(x_{n+1} - x_n), \\ g_{n+1} = w_{n+1} - Tw_{n+1}, \quad \forall n \geq 1, \end{cases} \tag{3.3}$$

Step 4: Set $n := n+1$ and go back to step 1.

Remark 3.1. We note that:

(i) If $\theta_n = 0$ and $w_n = Tw_n$, we get a fixed point of T. Then we define $x_{n+k} = x_n \ \forall k \geq 1$, so that our algorithm (2) generates an infinite sequence.

(ii) For each $n \geq 1$, we have $\theta_n||x_n - x_{n-1}|| \leq \delta_n$, together with $\sum_{n=0}^{\infty} \delta_n < \infty$ implies $\sum_{n=0}^{\infty} \theta_n||x_n - x_{n-1}|| < \infty$.

(iii) $\max\{m_n,\ m_*\} \in \mathbb{N}$ and $\max\{m_n,\ m_*\} \geq m_n$. So by the definition of m_n, the inequality (3.1) is satisfied with $m := \max\{m_n, m_*\}$.

Weak Convergence Theorem

Theorem 3.2. Let X be a real uniformly smooth Banach space and suppose that it satisfies Opial's property. Let $T_i : X \to X$, $i = 1, 2, 3, \ldots, p$ be a finite family of k_i - strictly pseudocontractive mappings with $\bigcap_{i=1}^{p} Fix(T_i) \neq \emptyset$. Define $T_{\alpha_i}x := (1-\alpha_i)x + \alpha_i T_i x$. Then, the sequence $\{x_n\}$ generated by the algorithm (2) converges weakly to $x^* \in Fix(T)$, provided the following conditions hold;

(C1) $\alpha_i \in (0, \frac{\mu}{\sigma}]$, $\mu = \min\left\{1, \frac{2k}{c}\right\}$, where $k = max\{k_i : i = 1, 2, \ldots, p\}$ and c is the constant in (2.2)

(C2) $\liminf\limits_{n\to\infty} \lambda_n\left(\frac{2k_1}{\alpha_1} - c\lambda_n\right) > 0.$

Proof . We divide the proof into the following steps:

Step (i): We show that there exists a smallest integer $m_n > 0$ such that the inequality (3.1) is true.

By contradiction. Suppose for every $m_n > 0$ such that $m \geq m_n$, we have

$$||w_n - \sigma^{m_n} g_n - T(w_n - \sigma^{m_n} g_n)||^2 > (1 - \gamma\sigma^{2m_n})||g_n||^2 \ + \ \eta_n. \tag{3.4}$$

Passing limit as $m_n \to \infty$ to both sides of (3.4), we obtain

$$||g_n||^2 > ||g_n||^2 + \eta_n,$$

which is impossible. Hence, the line-search rule defined by (3.1) is well defined.

Step (ii): Next we show that the $\lim\limits_{n\to\infty} \parallel x_n - p \parallel$ exists for any point $p \in Fix(T)$.

Let $p \in Fix(T)$, using (3.3) and Lemma 2.4, we get

$$\begin{aligned}
||w_n - p||^2 &= ||x_n + \theta_n(x_n - x_{n-1}) - p||^2 \\
&= ||(x_n - p) + \theta_n(x_n - x_{n-1})||^2 \\
&\leq ||x_n - p||^2 + 2\left\langle \theta_n(x_n - x_{n-1}), J(x_n - p)\right\rangle \\
&\quad + ||\theta_n(x_n - x_{n-1})||\beta^*\big(||\theta_n(x_n - x_{n-1})||\big) \\
&\leq ||x_n - p||^2 + 2\theta_n \left\langle x_n - x_{n-1}, J(x_n - p)\right\rangle \\
&\quad + c\theta_n^2||x_n - x_{n-1}||^2 \\
&\leq ||x_n - p||^2 + 2\theta_n\big| \left\langle x_n - x_{n-1}, J(x_n - p)\right\rangle \big| \\
&\quad + c\theta_n^2||x_n - x_{n-1}||^2 \\
&\leq ||x_n - p||^2 + 2\theta_n||x_n - x_{n-1}||||J(x_n - p)|| \\
&\quad + c\theta_n^2||x_n - x_{n-1}||^2 \\
&= ||x_n - p||^2 + 2\theta_n||x_n - x_{n-1}||||x_n - p|| \\
&\quad + c\theta_n^2||x_n - x_{n-1}||^2 \\
&\leq ||x_n - p||^2 + 2M\theta_n||x_n - x_{n-1}|| \\
&\quad + c\theta_n^2||x_n - x_{n-1}||^2.
\end{aligned} \tag{3.5}$$

Again using (3.5) and Lemma 2.4, we have

$$\begin{aligned}
||x_{n+1}-p||^2 &= ||w_n-\lambda_n g_n-p||^2\\
&= ||(w_n-p)-\lambda_n g_n||^2\\
&\le ||w_n-p||^2-2\lambda_n\langle g_n, J(w_n-p)\rangle\\
&\quad +||\lambda_n g_n||\beta^*\big(||\lambda_n g_n||\big)\\
&\le ||w_n-p||^2-2\lambda_n\langle w_n-Tw_n, J(w_n-p)\rangle\\
&\quad +c\lambda_n^2||w_n-Tw_n||^2\\
&\le ||w_n-p||^2-2\frac{k_1}{\alpha_1}\lambda_n||w_n-Tw_n||^2\\
&\quad +c\lambda_n^2||w_n-Tw_n||^2\\
&= ||w_n-p||^2-\lambda_n\left(\frac{2k_1}{\alpha_1}-c\lambda_n\right)||w_n-Tw_n||^2\\
&\le ||x_n-p||^2+2M\theta_n||x_n-x_{n-1}||\\
&\quad +c\theta_n^2||x_n-x_{n-1}||^2-\lambda_n\left(\frac{2k_1}{\alpha_1}-c\lambda_n\right)||w_n-Tw_n||^2\\
&= ||x_n-p||^2+V_n-\lambda_n\left(\frac{2k_1}{\alpha_1}-c\lambda_n\right)||w_n-Tw_n||^2 \qquad (3.6)
\end{aligned}$$

$$\le \quad ||x_n-p||^2+V_n \ by\ the\ definition\ of\ \lambda_n,$$

where $V_n=2M\theta_n||x_n-x_{n-1}||+c\theta_n^2||x_n-x_{n-1}||^2$. It follows from Remark 3.1(ii) that $\sum_{n=0}^{\infty}V_n<\infty$. Therefore, by Lemma 2.6, we have $\lim_{n\to\infty}||x_n-p||$ exists. This implies that the sequence $\{||x_n-p||\}$ is bounded. Consequently, the sequence $\{x_n\}$ is bounded.

Step (iii): We show that $\lim_{n\to\infty}\parallel x_n-Tx_n\parallel=0$.

From (3.6), we obtain

$$\lambda_n\left(\frac{2k_1}{\alpha_1}-c\lambda_n\right)||w_n-Tw_n||^2\le||x_n-p||^2-||x_{n+1}-p||^2+V_n.$$

Since $\lim_{n\to\infty}V_n=0$ and the sequence $\{||x_n-p||\}$ converges, we get

$$\lim_{n\to\infty}\lambda_n\left(\frac{2k_1}{\alpha_1}-c\lambda_n\right)||w_n-Tw_n||^2=0.$$

Using (C2), we have

$$\lim_{n\to\infty}||w_n-Tw_n||^2=0.$$

Consequently,

$$\lim_{n\to\infty}||w_n-Tw_n||=0. \qquad (3.7)$$

From the definition of w_n and Remark 3.1(ii), we have

$$\begin{aligned}
||w_n-x_n|| &= ||x_n+\theta_n(x_n-x_{n-1})-x_n||\\
&= \theta_n||x_n-x_{n-1}||\to 0\ as\ n\to\infty. \qquad (3.8)
\end{aligned}$$

Using (3.7) and (3.8), we get

$$||x_n-Tw_n|| \le ||x_n-w_n||+||w_n-Tw_n||\to 0\ as\ n\to\infty. \qquad (3.9)$$

Since from Lemma 2.10(iii), T is nonexpansive, we get

$$\begin{aligned}
||x_n-Tx_n|| &= ||x_n-Tw_n+Tw_n-Tx_n||\\
&\le ||x_n-Tw_n||+||Tw_n-Tx_n||\\
&\le ||x_n-Tw_n||+||w_n-x_n||
\end{aligned}$$

Using (3.8) and (3.9), we obtain

$$\lim_{n\to\infty} ||x_n - Tx_n|| = 0. \tag{3.10}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to, say $x^* \in X$. Therefore from (3.10), it follows that $\lim_{i\to\infty} ||x_{n_i} - Tx_{n_i}|| = 0$ and consequently by Lemma 2.7 we have $Tx^* = x^*$. Therefore, we obtain that $\omega_w(x_n) \subset Fix(T) = \bigcap_{i=1}^{p} Fix(T_i)$.

Now, to prove that the sequence $\{x_n\}$ converges weakly to a point, say, $x^* \in Fix(T) = \bigcap_{i=1}^{p} Fix(T_i)$, it suffices to show that $\omega_w(x_n)$ is singleton. To do that, we proceed as follows;

By our assumption that X satisfies Opial's property, therefore using Lemma 2.8 and taking $p_1, p_2 \in \omega_w(x_n)$, let $\{x_{n_k}\}$ and $\{x_{n_j}\}$ be subsequences of $\{x_n\}$ such that $x_{n_k} \rightharpoonup p_1$ and $x_{n_j} \rightharpoonup p_2$. Then we have for $p_1 \neq p_2$ that

$$\begin{aligned}
\lim_{n\to\infty} ||x_n - p_1|| &= \lim_{k\to\infty} ||x_{n_k} - p_1|| \\
&< \liminf_{k\to\infty} ||x_{n_k} - p_2|| \\
&= \lim_{n\to\infty} ||x_n - p_2|| \\
&= \liminf_{j\to\infty} ||x_{n_j} - p_2|| \\
&< \liminf_{j\to\infty} ||x_{n_j} - p_1|| \\
&= \lim_{n\to\infty} ||x_n - p_1||,
\end{aligned}$$

which is a contradiction. This shows that $\omega_w(x_n)$ is a singleton. This completes the proof.
□

Theorem 3.3. If in addition to all the hypothesis of Theorem 3.2, the map T is semicompact, then the iterative sequence $\{x_n\}$ generated by (3.3) converges strongly to a fixed point of T.

Proof . Assume that T is semicompact. Since from step (ii) and step (iii) in the proof of Theorem 3.2, we know that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x^*$ as $k \to \infty$. Therefore $x_{n_k} \rightharpoonup x^*$ and so $x^* \in \omega_w(x_n) \subseteq Fix(T)$. Since from step (ii) in the proof of Theorem 3.2 $\lim_{n\to\infty} ||x_n - x^*||$ exists, then

$$\lim_{n\to\infty} ||x_n - x^*|| = \lim_{k\to\infty} ||x_{n_k} - x^*|| = 0,$$

which means that $x_n \to x^* \in Fix(T)$. This completes the proof.
□

If in Theorem 3.2, X is a real Hilbert space H, then we have the following corollary:

Corollary 3.4. Let H be a real Hilbert space and $T_i : H \to H$, $i = 1, 2, 3, \dots, p$ be a finite family of k_i - strictly pseudocontractive mappings with $\bigcap_{i=1}^{p} Fix(T_i) \neq \emptyset$. Define $T_{\alpha_i}x := (1 - \alpha_i)x + \alpha_i T_i x$. Then, the sequence $\{x_n\}$ generated by the algorithm (2) converges weakly to $x^* \in Fix(T)$, provided the following conditions hold;

(C1) $\alpha_i \in (0, \frac{\mu}{\sigma}]$, $\mu = min\big\{1, 2k\big\}$, where $k = max\{k_i : i = 1, 2, \dots, p\}$

(C2) $\liminf_{n\to\infty} \lambda_n \big(\frac{2k_1}{\alpha_1} - \lambda_n\big) > 0$.

Remark 3.5. Theorems 3.2 and 3.3 are improvements of the result of La Cruz [13] in the following sense:

- In La Cruz [13], the author proved a weak convergence theorem for a nonexpansive map in a real Hilbert space. In our theorems 3.2 and 3.3, weak and strong convergence theorems are respectively proved for a finite family of strictly pseudocontractive maps in a more general uniformly smooth real Banach space.

- Unlike the algorithm of La Cruz[13], our algorithm has an inertial term, which is well known to improve the speed of convergence of the algorithm.

Observe that if the inertial term is drop in algorithm 2, then the algorithm reduces to the following algorithm:
Algorithm 3 Step 0: Choose $x_0 \in X$, $\gamma, \sigma \in (0,1)$ and positive sequence $\{\eta_n\}$ such that;

$$\sum_{n=0}^{\infty} \eta_n < +\infty. \tag{3.11}$$

Let $n := 0$ and $y_0 = x_0 - Tx_0$.
Step 1: If $||x_n - Tx_n|| = 0$, define $x_{n+k} := x_n \; \forall k \geq 1$ and stop. Otherwise,
Step 2: Find the least positive integer m_n such that for all $m \geq m_n$, the following inequality holds.

$$||x_n - \sigma^m y_n - T(x_n - \sigma^m y_n)||^2 \leq (1 - \gamma\sigma^{2m})||y_n||^2 \; + \; \eta_n. \tag{3.12}$$

Step 3: Compute the step length λ_n using the formula

$$\lambda_n := \sigma^{max\{m_n,\, m_*\}}. \tag{3.13}$$

Step 4: Compute x_{n+1} and y_{n+1} using the following relations.

$$x_{n+1} = x_n - \lambda_n y_n, \tag{3.14}$$

$$y_{n+1} = x_{n+1} - Tx_{n+1}. \tag{3.15}$$

Step 5: Set $n := n + 1$ and go back to step 1.

4 Numerical Examples

In this section, we give a numerical example to show the computational performance of our proposed inertial algorithm and compare it with algorithm 3.
Example: Let $X = \mathbb{R}$ with its usual norm. For each $i = 1, 2, 3, \ldots, p$, let $T_i : X \to X$ be a map defined by for all $x \in X$,

$$T_i(x) = \frac{-(5+i)}{4+i} x,$$

then, it is easy to see that for each $i = 1, 2, 3, \ldots, p$, T_i is strictly pseudocontractive with $k_i = \frac{1}{9+2i}$ and $0 \in \bigcap_{i=1}^{p} F(T_i)$. We choose $c = \frac{3}{2}$, $\sigma = \frac{2}{9}$ and $\alpha_i = \frac{1}{2i}$, then $\mu = \frac{4}{33}$ and clearly $\alpha_i \in (0, \frac{\mu}{\sigma})$ and $T_{\alpha_i}(x) = \frac{(2i^2+6i-9)}{2i(4+i)} x$, for each $i = 1, 2, 3, \ldots, p$. So, for $p = 3$, we have $T(x) = \frac{-99}{3360} x$ and $m_* = 1$. Assume $x_0 = 2$, $x_1 = 5$, $\gamma = 10^{-5}$, $\sigma = 0.22$, $\eta_n = (0.888)^n(10^4 + \|w_1 - Tw_1\|^2)$ and $\delta_n = \frac{1}{n^2+1}$. If $\|x_n - x^*\| \leq 10^{-5}$ is chosen as stopping criteria, where $x^* = 0 \in \bigcap_{i=1}^{p} F(T_i)$, we obtain using MATLAB the numerical results for algorithm 2 and algorithm 3 in Figure 1, Figure 2 and Table 1

It is clear from Figure 1, Figure 2 and Table 1 that algorithm 2 with the inertial term converges faster than algorithm 3 with out the inertial term.

5 Conclusions

We studied an inertial redual algorithm in a real uniformly smooth Banach spaces. Weak and strong convergence Theorems were proved to approximate solutions of fixed points of strictly pseudocontractive mappings. Numerical example was presented to show the performance of our iterative scheme with a non-inertial algorithm.

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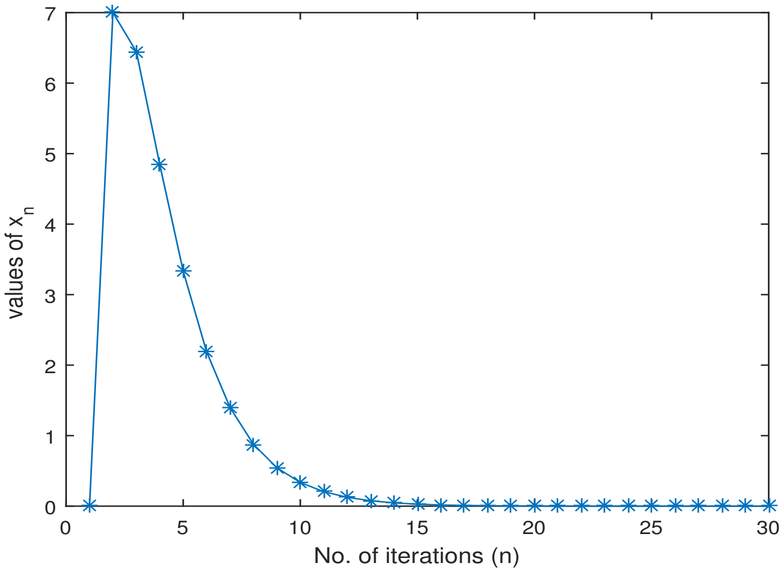

Figure 1: Convergence processes of algorithm 2

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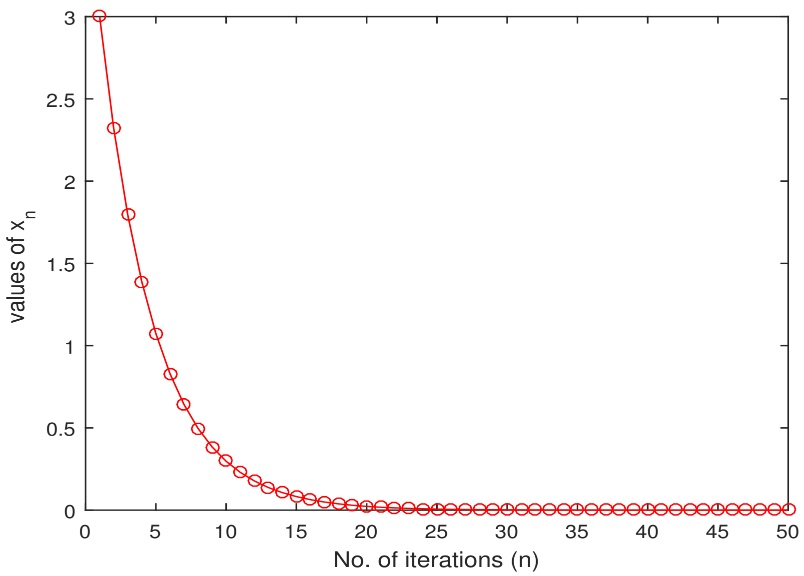

Figure 2: Convergence process of algorithm 3

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Table 1: Computational results for algorithm 2 and algorithm 3

Number of iteration	Agorithm 2	Algorithm 3
1	3	3
2	7	2.32055
3	6.44598	1.79498
4	4.84323	1.38845
5	3.33308	1.07399
6	2.18882	0.83075
7	1.39805	0.64260
8	0.87753	0.49706
9	0.54457	0.38448
10	0.33538	0.2974
11	0.20549	0.2300
12	0.12546	0.1779
13	0.07641	0.1376
14	0.04645	0.1064
15	0.02821	0.0823
16	0.01711	0.0637
17	0.01038	0.04927
18	0.00629	0.03811
19	0.00381	0.02948
20	0.00231	0.02280
21	0.00140	0.01764
22	0.00083	0.01364
23	0.00051	0.01055
24	0.00031	0.008164
25	0.00019	0.006315
26	0.00011	0.004885
27	6.909e-05	0.003778
28	4.184e-05	0.002922
29	2.534e-05	0.002260
30	1.534e-05	0.001748
31	0.000000	0.001352
32	0.000000	0.001046
33	0.000000	0.000809
34	0.000000	0.000626
⋮	⋮	⋮
47	0.000000	2.222e-05
48	0.000000	1.718e-05
49	0.000000	1.329e-05
50	0.000000	1.0286e-05
51	0.000000	0.000000