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Global existence and decay estimates for the semilinear heat equation with memory in $\mathbb{R}^n$

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Abstract

In this paper, we study the initial value problem for a semi-linear heat equation with memory in n-dimensional space $\mathbb{R}^n$. Under a smallness conditions on the initial data, the global existence and decay estimates of the solutions are established. Furthermore, time decay estimates in higher Sobolev space of the solution are provided. The proof is carried out by means of the point-wise decay estimates of the solution in the Fourier space and a fixed point-contraction mapping argument.

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1 Introduction

In this paper we consider the initial value problem of the following semi-linear Volterra integro-differential equations of the first order posed in the whole space $\mathbb{R}^n$

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) + g * (-\Delta)^\theta u = f(u), \; t>0, \; x \in \mathbb{R}^n, \\ u(0,x) = u_0\left(x\right), \; x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

here $u = u(t,x)$ is an unknown real valued function of $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $t > 0$, $u_0\left(x\right)$ is a given initial data and the function f is an external nonlinear force. The fractional Laplace operator $(-\Delta)^\theta$ may be defined through its Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ by

$$(-\Delta)^\theta h(x) = \mathcal{F}^{-1}\left(|\xi|^{2\theta}\mathcal{F}\left(h\right)\left(\xi\right)\right)\left(x\right), \, x \in \mathbb{R}^n,$$

or by its representation $(-\Delta)^\theta h(x) = C(n,\theta)\int_{\mathbb{R}^n}\frac{h(x)-h(y)}{|x-y|^{n+2\theta}}dy$, with $0 < \theta < 1$. In the limit $\theta \to 1$ the standard Laplace operator, $-\Delta$, is recovered (see Section 4 of [22]).

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The convolution $g * (-\Delta)^{\theta} u := \int_0^t g(t-s)(-\Delta)^{\theta} u(s) ds$ corresponds to the memory term g, that satisfies the following assumptions :

a) The kernel g is a nonnegative summable function having the explicit form

$$g(t) = \int_t^{+\infty} \mu(s)\, ds$$

for some (nonnegative) non-increasing piecewise absolutely continuous function $\mu \in L^1(\mathbb{R}^+)$ of total mass

$$g = \int_0^{\infty} \mu(s) ds < \infty.$$

b) Moreover, we require that

$$g(s) \leq K\mu(s)$$

for some positive constant K and every $s > 0$. As shown in [4], this is equivalent to the requirement that there exist $C \geq 1$ and $\delta > 0$ such that for any $t \geq 0$ and almost every $s > 0$

$$\mu(t+s) \leq Ce^{-\delta t}\mu(s).$$

In particular, the kernel μ is allowed to exhibit (infinitely many) jumps. For example, a typical kernels μ considered in the papers [4, 6, 12, 21] where the authors assumed that the set of jump points of μ is a strictly increasing sequence $\{s_i\}$, with $s_0 = 0$, either finite (possibly reduced to s_0 only) or converging to $s_\infty \in (0, \infty]$ such that, for all $i \geq 1$, μ has jumps at $s = s_i$, and it is absolutely continuous on each interval $I_i = (s_{i-1}, s_i)$ and on the interval $I_\infty = (s_\infty, \infty)$, unless I_∞ is not defined. If $s_\infty < \infty$, then μ may or may not have a jump at $s = s_\infty$. Thus, μ may be singular at $s = 0$, and μ' exists almost everywhere .

Assumption [B] on f Assume that $f \in C^\infty(\mathbb{R})$, and $f(u) = O(|u|^\alpha)$ as $|u| \to 0$, here $\alpha > \alpha_n$ and $\alpha_n := 1 + \frac{2}{n}$, $n = 1, 2$, and α is assumed to be an integer for $n \geq 3$.

Equation (1.1) can be viewed as an abstract version of an evolution model with fading memory describing the dynamic behavior of different phenomena like, e.g., population dynamics, heat conduction in materials with memory or diffusion in fractured media in materials (see [7, 17, 23] and the references therein). This is an important variant of the classical diffusion case because there are many situations in which the evolution of the model is not only affected by the present state of the system but for its past history.

From the mathematical point of view, the study of the qualitative properties of evolution equations involving a finite or infinite memory is also important as such systems occur in various problems of applied science, and it has attracted some of attention of many mathematicians for instance, see ([6, 7, 8, 12, 14, 15, 17, 21, 23, 24, 26]) and references therein. These works deal with the questions of existence and uniqueness, asymptotic behavior, global attractor and so forth as well as a variety of methods used to study these questions. In particular, stability and boundedness results of the solutions of the homogeneous part of abstract forms of (1.1) have been investigated widely; see, for instance ([1, 2, 3, 10, 11, 26, 28]) by means of representing the solution in terms of the resolvent operator.

Recent contributions on the existence of global attractor and exponential attractor with some of the previously enumerated properties or another type of Volterra integro-differential equations in a bounded domain have been made (see [4, 5, 6, 9, 13, 14, 15, 16, 21]) and the reference therein). Decay properties of the semigroup generated by a linear parabolic integro-differential equation with memory functions in a Hilbert space arising from heat conduction with memory has been studied by [2, 3, 4, 6, 21].

It is worth noticing that, the most previous works dealing about global existence and uniform decay rates of solution for parabolic equation with memory in a bounded domains Ω in $\mathbb{R}^n$. In this paper, we are interested in the case when Ω is the whole space $\mathbb{R}^n$. More precisely, we are interested to the parabolic Volterra integro-differential equation in which some kind of finite memory is taken into account and fractional Laplacian operator is included in the memory term.

However, to the best of the authors' knowledge, the decay results for semilinear heat equations with memory in whole space $\mathbb{R}^n$, becomes much more complicated. One major difficulty is the loss of Poincaré's inequality, which is indispensable for obtaining the existence of global solution. On the other hand, the condition on the kernel k is more general than that existing in the literature where a differential inequality is necessary required for μ. In particular the relaxation function μ may be unbounded in a neighborhood of the origin, and may exhibit infinitely many jumps which constitute a rather demanding limitation on the choice of the possible kernels.

Motivated by this observation, we intend to study the decay property of solution for problem (1.1) in the energy space in the presence of fractional Laplacian operator. To do this, we shall extend the solution u to negative times by zero, and then we introduce a new auxiliary *past history variable* η in which dissipative semigroups theory is invoked. In order to derive the decay estimates property of such semigroup in a suitable Sobolev space, first we make use the energy multiplier techniques in the Fourier space see, eg. [12, 18, 20, 27]. Appealing to this pointwise estimates, the corresponding uniform decay estimates of solution and their properties are obtained.

Consequently, the global existence and optimal decay estimates of solution to (1.1) are achieved by means of contraction mapping theorem. As for the semi-linear problem one point worthy to be mentioned is that we obtain the results for $\alpha > \alpha_n$ in the case $n = 1, 2$, while $\alpha_n = 1 + \frac{2}{n}$ is the well-known critical Fujita exponent in dealing with the global existence of solutions to some semi-linear parabolic differential equations.

The rest of the paper is organized as follows. In Section 2, we obtain the fundamental solution formula of linear Cauchy problem corresponding to the nonlinear problem (1.1). In Section 3, we obtain the point-wise estimates and decay properties of the fundamental solution operator. In Section 4, we prepare such lemmas leads to the decay estimates of solutions to the problem (2.1). In Section 5, we prove the global existence and the optimal decay estimates of solution to the problem (1.1).

Notations. We give some notations which are used in this paper. Let $\mathcal{F}[f]$ denote the Fourier transform of f defined by

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix.\xi} f(x)dx,$$

and we denote by F^{-1} for its Fourier inverse transform.

Let $\mathcal{L}[f]$ the laplace transform of f defined by

$$\mathcal{L}[f](\lambda) := \int_0^{+\infty} e^{-\lambda t} f(t)dt,$$

and its inverse transform denoted by $\mathcal{L}^{-1}$.

Throughout the paper $L^p = L^p(\mathbb{R}^n)$ $(1 \le p < \infty)$ denotes the usual Lebesgue space with the norm $\|f\|_{L^p}^p = \int_{\mathbb{R}^n} |f(x)|^p \, dx$ and $W^{s,p}(\mathbb{R}^n)$, (with $s \ge 1$ is an integer and $p \in [1, \infty)$ denotes the usual Sobolev space with its norm

$$\|f\|_{W^{s,p}} := (\sum_{k=0}^{s} \|\partial_x^k f\|_{L^p}^p)^{\frac{1}{p}}.$$

In particular, we use $W^{s,2} = H^s$. For a nonnegative integer k, ∂_x^k denotes the totality or each of all the k-th order derivatives with respect to $x \in \mathbb{R}^n$. Also, $C^k(I, H^s(\mathbb{R}^n))$ denotes the space of k-times continuously differentiable functions on the interval I with values in the Sobolev space $H^s = H^s(\mathbb{R}^n)$.

Finally, throughout this paper, C or c denote positive generic constants, not necessarily the same at different places.

2 Solution formula

In this section we try to obtain the solution formula for the following linear problem corresponding (1.1)

$$\begin{cases} \partial_t u - \Delta u + g * (-\Delta)^\theta u = 0, & t > 0, \ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{2.1}$$

By applying Fourier transform and Laplace transform to Eq. (2.1), we obtain the solution $\bar{u}$ which expressed in terms of H, by $\bar{u} = H(t) * u_0$ where H is a fundamental solution to the following problem,

$$\begin{cases} \partial_t H - \Delta H + g * (-\Delta)^\theta H = 0, \ x \in \mathbb{R}^n, \ t > 0, \\ H(0,x) = \delta, \ x \in \mathbb{R}^n. \end{cases} \tag{2.2}$$

Here δ is the Dirac distribution in $x = 0$ with respective to the spatial variables.

The Fourier transform of the fundamental solution of (2.2) is given formally through Laplace inverse transform by

$$\hat{H}(t,\xi) = \hat{H}_0(\xi)\,\mathcal{L}^{-1}[\frac{1}{\lambda+\beta|\xi|^2+|\xi|^{2\theta}\mathcal{L}[g]}](t,\xi). \tag{2.3}$$

Lemma 2.1. The fundamental solution $\hat{H}(t,\xi)$ given by (2.3) exists.

Proof . Denote $F(\lambda) := \lambda+|\xi|^2+|\xi|^{2\theta}\mathcal{L}[g](\lambda)$. To prove $\mathcal{L}^{-1}[\frac{1}{F(\lambda)}]$ exists, we need to consider the zero points of $F(\lambda)$. Denote $\lambda = \sigma + i\nu,\ \ \sigma > -\frac{\delta}{C}$, then $\mathcal{L}[g](\lambda)$ exists. Assume that $\lambda_1 = \sigma_1 + i\nu_1$ is a zero point of $F(\lambda)$ and $\sigma_1 > -\frac{\delta}{C}$, then σ_1 and ν_1 satisfy

$$\begin{cases} \Re F(\lambda_1) = \sigma_1 + |\xi|^2 + |\xi|^{2\theta}\displaystyle\int_0^\infty \cos(\nu_1 t)e^{-\sigma_1 t}g(t)dt = 0, \\ \Im F(\lambda_1) = \nu_1 - |\xi|^{2\theta}\displaystyle\int_0^\infty \sin(\nu_1 t)e^{-\sigma_1 t}g(t)dt = 0. \end{cases} \tag{2.4}$$

In order to show that $F(\lambda)$ does not vanish in the region $\{\lambda\in\mathbb{C}; \Re(\lambda)\geq \max\{g,\sqrt{g}\}\}$, we distinguish to cases

case 1 : If $|\xi|<1$, we assume that $\sigma_1\geq\sqrt{g}$, then

$$\begin{aligned} \Re F(\lambda_1) &= \sigma_1 + |\xi|^2 + |\xi|^{2\theta}\int_0^\infty e^{-\sigma_1 t}g(t)dt \\ &\geq \sigma_1 + |\xi|^2 - |\xi|^{2\theta}g\int_0^\infty e^{-\sigma_1 t}dt, \end{aligned}$$

$$\Re F(\lambda_1) \geq \sigma_1 + |\xi|^2 - |\xi|^{2\theta}\frac{g}{\sigma_1},$$

which implies

$$\Re F(\lambda_1) \geq \frac{g}{\sigma_1} + |\xi|^2 - |\xi|^{2\theta}\frac{g}{\sigma_1}.$$

Consequently

$$\Re F(\lambda_1) \geq |\xi|^2 + \frac{g}{\sigma_1}(1-|\xi|^{2\theta}) > 0.$$

It yields contradiction with $(2.4)_1$. Then $\sigma_1<\sqrt{g}$.

case 2 : In $|\xi|\geq 1$ we assume that $\sigma_1\geq g$, then we have

$$\begin{aligned} \Re F(\lambda_1) &= \sigma_1 + |\xi|^2 - |\xi|^{2\theta}\int_0^\infty e^{-\sigma_1 t}g(t)dt \\ &\geq \sigma_1 + |\xi|^2 - |\xi|^{2\theta}g\int_0^\infty e^{-\sigma_1 t}dt \\ &\geq \sigma_1 + |\xi|^2 - |\xi|^{2\theta}\frac{g}{\sigma_1} \\ &\geq \sigma_1 + \frac{g}{\sigma_1}|\xi|^2 - |\xi|^{2\theta}\frac{g}{\sigma_1}. \end{aligned}$$

Which gives

$$\Re F(\lambda_1) \geq \sigma_1 + \frac{g}{\sigma_1}(|\xi|^2 - |\xi|^{2\theta}) \geq \sigma_1 > 0,$$

which it yields a contradiction with $(2.4)_1$. Therefore $\sigma_1<g$.

Combining the two cases, we know that $\frac{1}{F(\lambda)}$ is analytic in $\{\lambda\in\mathbb{C}; \Re(\lambda)\geq\sqrt{g}\}$ if $|\xi|<1$ and in $\{\lambda\in\mathbb{C}; \Re(\lambda)\geq g\}$ if $|\xi|\geq 1$.

Take $\lambda=\sigma+i\nu,\ \ \sigma>\max\{\Re\lambda_s\}$, here $\{\lambda_s\}$ is the set of all the singular points of $F(\lambda)$, then we have that

$$\begin{aligned} \mathcal{L}^{-1}[\frac{1}{F(\lambda)}](t) &= \int_{\sigma-i\infty}^{\sigma+i\infty}\frac{e^{\lambda t}}{F(\lambda)}d\lambda = \int_{-\infty}^{+\infty} i\frac{e^{(\sigma+i\nu)t}}{F(\sigma+i\nu)}d\nu \\ &= \left(\int_{\{\nu;|\nu|\leq R\}} + \int_{\{\nu;|\nu|>R\}}\right)\left(i\frac{e^{(\sigma+i\nu)t}}{F(\sigma+i\nu)}d\nu\right) \\ &=: J_1 + J_2. \end{aligned}$$

The integral J_1 converges, so we only need to consider J_2. Notice that $\frac{1}{F(\lambda)} = \frac{1}{\lambda} - \frac{|\xi|^2+|\xi|^{2\theta}\mathcal{L}[g](\lambda)}{\lambda F(\lambda)}$ and $|\mathcal{L}[g](\lambda)| \leq C$, then it is not difficult to prove that J_2 converges, then we proved that J_2 converges, so far we complete the proof. □

By Duhamel principle, the solution to the problem (1.1) could be expressed as following :

$$u(t) = H(t) * u_0 + \int_0^t H(t-\tau) * f(u)(\tau) d\tau. \tag{2.5}$$

We denote

$$\bar{u}(t) := H(t) * u_0, \tag{2.6}$$

then $\bar{u}(t)$ is the solution to the linear problem (2.2).

3 Decay properties of solution operators

We look at (1.1) as an ordinary differential equation in a proper Hilbert space accounting for the past history of the variable u. Extending the solution to (1.1) for all times, by setting $u(t) = 0$ when $t < 0$, and considering for $t \geq 0$ the auxiliary variable

$$\eta^t(s,x) = \int_{t-s}^t u(r,x)dr, \qquad t \geq 0, \quad s > 0.$$

Note immediately that $\eta^0(s) = 0$ for all $s > 0$, the integro-differential equation of problem (2.1) reads

$$\partial_t u(t) - \Delta u(t) + \int_0^\infty \mu(s) \left(-\Delta\right)^\theta \eta^t(s) ds = 0, \;\; t > 0. \tag{3.1}$$

The past history variable η is the unique mild solution (in the sense of [25]) of an abstract Cauchy problem in the μ-weighted space $\mathcal{M} = L^2_\mu(\mathbb{R}^+, H^1(\mathbb{R}^n))$, that is,

$$\begin{cases} \partial_t \eta^t = T\eta^t + u(t), & t > 0, \\ \eta^0 = 0, \end{cases} \tag{3.2}$$

where, as a consequence of the basic assumption (see [16]), the linear operator T is the infinitesimal generator of the right-translation C_0-semigroup on $\mathcal{M}$,defined as

$$T\eta = -\eta', \;\; \text{with domain } \mathcal{D}(T) = \{\eta \in \mathcal{M} : \eta' \in \mathcal{M}, \;\; \eta(0) = 0\}.$$

Here the prime "′"symbol denotes the distributional derivative with respect to the internal variable s.

Applying the Fourier transform to (3.1) and (3.2), we obtain, for every $\xi \in \mathbb{R}^n$, the following system

$$\begin{cases} \partial_t \hat{u} + |\xi|^2 \hat{u} + |\xi|^{2\theta} \displaystyle\int_0^\infty \mu(s)\hat{\eta}^t(s)ds = 0, & t > 0, \\ \partial_t \hat{\eta}^t = T\hat{\eta}^t + \hat{u}(t), \quad t > 0, \\ \hat{u}(0) = \hat{u}_0, \;\; \hat{\eta}^0 = 0, \end{cases} \tag{3.3}$$

in the transformed variables $\hat{u}(t,\xi)$ and $\hat{\eta}^t(t,\xi)$, where now T is the infinitesimal generator of the right-translation semigroup on $L^2_\mu(\mathbb{R}^+;\mathbb{R}^n)$, and $|.|$ stands for the standard euclidian norm in $\mathbb{R}^n$.

The energy density function is given by

$$\mathcal{E}(t,\xi) = |\hat{u}(t,\xi)|^2 + |\xi|^{2\theta} \int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2 ds. \tag{3.4}$$

In particular,

$$\mathcal{E}(0,\xi) = |\hat{u}_0(\xi)|^2.$$

Moreover, by the Plancherel theorem, we have a relation between the energy and the density

$$E(t) = \int_{\mathbb{R}^n} \mathcal{E}(t,\xi) d\xi.$$

Performing standard multiplication the first equation in (3.3) by $\overline{\hat{u}}$, using the second equation and then taking real parts, it can be seen as in [6] that the functional density satisfies for every fixed $\xi \in \mathbb{R}^n$ the following differential equality

$$\frac{d}{dt}\mathcal{E}(t,\xi) + 2|\xi|^2|\hat{u}(t,\xi)|^2 - |\xi|^{2\theta}\int_0^\infty \mu'(s)|\hat{\eta}^t(s,\xi)|^2 ds + |\xi|^{2\theta}\sum_{i\geq 1}(\mu(s_i^-) - \mu(s_i^+))|\hat{\eta}^t(s_i,\xi)|^2 = 0. \tag{3.5}$$

where the sum includes the value $i = \infty$ if $s_\infty < \infty$. We notice that

$$-|\xi|^{2\theta}\int_0^\infty \mu'(s)|\hat{\eta}^t(s,\xi)|^2 ds + |\xi|^{2\theta}\sum_{i\geq 1}(\mu(s_i^-) - \mu(s_i^+))|\hat{\eta}^t(s_i,\xi)|^2 \geq 0.$$

This means that the functional density $\mathcal{E}(t,\xi)$ is a non-increasing function of t.

Theorem 3.1. Let μ satisfy the assumptions a), b), and let $\theta \in [0,1]$. There exists a positive constant $C > 0$, such that the solution u of (2.2) satisfies the following pointwise estimate in the Fourier space:

$$\mathcal{E}(t,\xi) \leq C\mathcal{E}(0,\xi)e^{-c\rho(\xi)t}, \text{ for any } t > 0.$$

We begin by introducing the following functionals :

$$\Upsilon(t,\xi) = |\xi|^{2\theta}\int_0^\infty g(s)|\hat{\eta}^t(s,\xi)|^2 ds,$$

$$\Theta(t,\xi) = |\xi|^{2\theta}\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2 ds.$$

Lemma 3.2. The functionals Υ and Θ are well defined and fulfill the following inequality

$$\Upsilon(t,\xi) \leq K\Theta(t,\xi). \tag{3.6}$$

Proof . Inequality (3.6) follows directly by assumption b). Moreover, by using (3.4) we have

$$\Theta(t,\xi) \leq \mathcal{E}(t,\xi) \leq \mathcal{E}(0,\xi) < \infty,$$

the well-definedness of Θ is achieved whereas the one for Υ is a consequence of (3.6). □

Lemma 3.3. The following differential inequality holds:

$$\frac{d}{dt}\Upsilon(t,\xi) + \frac{1}{2}|\xi|^{2\theta}\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2 ds \leq 2K^2 g|\xi|^{2\theta}|\hat{u}(t,\xi)|^2, \tag{3.7}$$

Proof . By means of the equation for the past history variable, a direct calculation leads to the following differential equality for $\Upsilon(t,\xi)$:

$$\frac{d}{dt}\Upsilon(t,\xi) + |\xi|^{2\theta}\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2 ds = 2\Re|\xi|^{2\theta}\int_0^\infty g(s)\hat{u}(t,\xi)\overline{\hat{\eta}^t(s,\xi)}ds. \tag{3.8}$$

Concerning the right-hand side of (3.8), we have

$$\Re\int_0^\infty g(s)\hat{u}(t,\xi)\overline{\hat{\eta}^t(s,\xi)}ds \leq \nu(\int_0^\infty g(s)|\hat{\eta}^t(s,\xi)|ds)^2 + \frac{1}{4\nu}|\hat{u}(t,\xi)|^2,$$

for some $\nu > 0$. Using assumption b) and Cauchy-Shwarz's inequality, we get

$$\begin{aligned}
\int_0^\infty g(s)|\hat{\eta}^t(s,\xi)|ds &\leq K\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|ds \\
&\leq K(\int_0^\infty \mu(s)ds)^{\frac{1}{2}}(\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2 ds)^{\frac{1}{2}} \\
&\leq K\sqrt{g}(\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2 ds)^{\frac{1}{2}}.
\end{aligned}$$

Therefore

$$2\Re|\xi|^{2\theta}\int_0^\infty k(s)\hat{u}(t,\xi)\overline{\hat{\eta}^t(s,\xi)}ds \le 2\nu K^2 g|\xi|^{2\theta}\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2ds + \frac{1}{2\nu}|\xi|^{2\theta}|\hat{u}(t,\xi)|^2. \tag{3.9}$$

Now, we go back to equality (3.8), we infer from (3.9)

$$\frac{d}{dt}\Upsilon(t,\xi) + |\xi|^{2\theta}\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2ds \le 2\nu K^2 g|\xi|^{2\theta}\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2ds + \frac{1}{2\nu}|\xi|^{2\theta}|\hat{u}(t,\xi)|^2,$$

taking $\nu = \frac{1}{4K^2 g}$, we obtain the desired inequality

$$\frac{d}{dt}\Upsilon(t,\xi) + \frac{1}{2}|\xi|^{2\theta}\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2ds \le 2K^2 g|\xi|^{2\theta}|\hat{u}(t,\xi)|^2.$$

□

Proof .[Proof of Theorem3.1] We define the additional functional

$$\mathcal{L}(t,\xi) = \mathcal{E}(t,\xi) + \beta\rho(\xi)\,\Upsilon(t,\xi),$$

for some $\beta > 0$ and $0 \le \rho(\xi) \le 1$ to be determined later.

Clearly we have $\mathcal{L}(t,\xi) \ge \mathcal{E}(t,\xi)$. On the other hand, we have

$$\begin{aligned}
\mathcal{L}(t,\xi) &= \mathcal{E}(t,\xi) + \beta\rho(\xi)\,\Upsilon(t,\xi)\\
&\le \mathcal{E}(t,\xi) + \beta\Upsilon(t,\xi)\\
&\le \mathcal{E}(t,\xi) + \beta|\xi|^{2\theta}\int_0^\infty g(s)|\hat{\eta}^t(s,\xi)|^2ds\\
&\le \mathcal{E}(t,\xi) + \beta K|\xi|^{2\theta}\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2ds\\
&\le (1+\beta K)\,\mathcal{E}(t,\xi).
\end{aligned}$$

From (3.5) and (3.7), we get

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t,\xi) &= \frac{d}{dt}\mathcal{E}(t,\xi) + \beta\rho(\xi)\,\frac{d}{dt}\Upsilon(t,\xi)\\
&\le -2|\xi|^2|\hat{u}(t,\xi)|^2 + |\xi|^{2\theta}\int_0^\infty \mu'(s)|\hat{\eta}^t(s,\xi)|^2ds\\
&\quad +\beta\rho(\xi)\left(2K^2\kappa|\xi|^{2\theta}|\hat{u}(t,\xi)|^2 - \frac{1}{2}|\xi|^{2\theta}\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2ds\right).
\end{aligned}$$

From which, it follows

$$\begin{aligned}
&\frac{d}{dt}\mathcal{L}(t,\xi) + 2|\xi|^2|\hat{u}(t,\xi)|^2 - |\xi|^{2\theta}\int_0^\infty \mu'(s)|\hat{\eta}^t(s,\xi)|^2ds + \frac{\beta}{2}\rho(\xi)\,|\xi|^{2\theta}\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2ds\\
\le\;& 2K^2 g\beta\rho(\xi)\,|\xi|^{2\theta}|\hat{u}(t,\xi)|^2.
\end{aligned}$$

Therefore

$$\frac{d}{dt}\mathcal{L}(t,\xi) + 2\left(|\xi|^2 - K^2 g\beta\rho(\xi)\,|\xi|^{2\theta}\right)|\hat{u}(t,\xi)|^2 + \frac{\beta}{2}\rho(\xi)\,|\xi|^{2\theta}\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2ds \le 0.$$

We choose $\beta \le 1/\left(\frac{1}{4} + K^2 g\right)$ so that $2\left(|\xi|^2 - K^2 g\beta\rho(\xi)\,|\xi|^{2\theta}\right) \ge \rho(\xi)\,\frac{\beta}{2}$. In fact by substituting $\rho(\xi) = \frac{|\xi|^2}{1+|\xi|^2}$ and using that $\frac{|\xi|^{2\theta}}{1+|\xi|^2} \le 1$, we obtain that

$$2\left(|\xi|^2 - K^2 g\beta\rho(\xi)\,|\xi|^{2\theta}\right) \ge 2\left(1 - K^2 g\beta\right)|\xi|^2 \ge \frac{\beta}{2}|\xi|^2 \ge \rho(\xi)\,\frac{\beta}{2}, \text{ for any } \xi \in \mathbb{R}^n.$$

Hence, we arrive at

$$\frac{d}{dt}\mathcal{L}(t,\xi) + \frac{\beta}{2}\rho(\xi)\,|\hat{u}(t,\xi)|^2 + \frac{\beta}{2}\rho(\xi)\,|\xi|^{2\theta}\int_0^\infty \mu(s)|\hat{\eta}^t(s,\xi)|^2ds \le 0.$$

From the definition of $\mathcal{E}$, we obtain

$$\frac{d}{dt}\mathcal{L}(t,\xi)+\frac{\beta}{2}\rho\left(\xi\right)\mathcal{E}(t,\xi)\le 0. \tag{3.10}$$

Making use the equivalence between functionals $\mathcal{E}$ and $\mathcal{L}$ in (3.10), we infer

$$\frac{d}{dt}\mathcal{L}(t,\xi)+\frac{\beta}{2\left(1+\beta K\right)}\rho\left(\xi\right)\mathcal{L}(t,\xi)\le 0,$$

with $\rho(\xi)=\frac{|\xi|^2}{|\xi|^2+1}$. By invoking the Gronwall Lemma, we get

$$\mathcal{E}(t,\xi)\le\mathcal{L}(t,\xi)\le C\mathcal{E}(0,\xi)e^{-c\rho(\xi)t}, \tag{3.11}$$

which concludes the proof. □

Now we study the decay estimates of solutions to the linear problem (2.1).

Theorem 3.4 (Energy estimate for linear problem). Let $s\ge 1$ be an integer, and $\theta\in[0,1]$. Assume that $u_0\in H^s(\mathbb{R}^n)$, and put

$$I_0:=||u_0||_{H^s(\mathbb{R}^n)}.$$

Then the solution $\bar{u}$ to the problem (2.1) given by (2.6) satisfies

$$\bar{u}\in C^0([0,+\infty);H^s(\mathbb{R}^n)),$$

and the following energy estimate :

$$||\bar{u}(t)||^2_{H^s}+\int_0^t||\partial_x\bar{u}(t)||^2_{H^{s-1}}d\tau\le cI_0^2$$

Proof . From (3.10) we have that

$$\frac{d}{dt}\mathcal{L}(t,\xi)+c\rho(\xi)\mathcal{E}(t,\xi)\le 0.$$

Integrate the previous inequality with respect to t and appeal to (3.11), then we obtain

$$\mathcal{E}(t,\xi)+c\int_0^t\rho(\xi)\mathcal{E}(\tau,\xi)d\tau\le c\mathcal{E}(0,\xi). \tag{3.12}$$

Multiply (3.12) by $(1+|\xi|^2)^s$ and integrate the resulting inequality with respect to $\xi\in\mathbb{R}^n$, then we have that

$$||\bar{u}(t)||^2_{H^s}+\int_0^t||\partial_x\bar{u}(t)||^2_{H^{s-1}}d\tau\le cI_0^2 \tag{3.13}$$

(3.13) guarantees the regularity of the solution (2.6). So far we complete the proof of Theorem 3.4. □

Lemma 3.5. The fundamental solution $H(t,x)$ satisfies :

$$|\hat{H}(t,\xi)|\le Ce^{-c\rho(\xi)t}$$

where $\rho(\xi)=\frac{|\xi|^2}{1+|\xi|^2}$.

Proof . From the representation formula of solution to the linear problem, we have

$$\bar{u}(t,x):=H(t,.)*u_0\left(x\right).$$

Using the expression of $\bar{u}$ in Theorem 3.1, we find

$$|\widehat{\bar{u}}(t,\xi)|^2=\left|\hat{H}(t,\xi)\right|^2|\hat{u}_0(\xi)|^2\le Ce^{-c\rho(\xi)t}|\hat{u}_0(\xi)|^2,$$

which gives the desired estimate. □

Lemma 3.6 (Pointwise estimate). Assume $\bar{u}$ is the solution of (2.1) and if $\theta \in [0,1]$, then it satisfies the following point-wise estimate in the Fourier space:

$$|\widehat{\bar{u}}(t,\xi)|^2 \leq Ce^{-c\rho(\xi)t}|\hat{u}_0(\xi)|^2, \tag{3.14}$$

where $\rho(\xi) = \frac{|\xi|^2}{1+|\xi|^2}$.

Proof . we have

$$\bar{u}(t,x) := H(t,.) * u_0\,(x)\,.$$

Then by Fourier transform property and from Lemma 3.5, we get

$$|\widehat{\bar{u}}(t,\xi)|^2 = |\hat{H}(t,\xi)|^2|\hat{u}_0(\xi)|^2 \leq Ce^{-c\rho(\xi)t}|\hat{u}_0(\xi)|^2.$$

□

Proposition 3.7. Let $s \geq 1$ be an integer and $1 \leq p \leq 2$. Let $\varphi \in H^s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. If $\theta \in [0,1]$ then the following estimates hold :

$$||\partial_x^k H(t) * \varphi||_{L^2} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}||\varphi||_{L^p} + Ce^{-ct}||\partial_x^k\varphi||_{L^2}, \tag{3.15}$$

for $0 \leq k \leq s$.

Proof . In view of Lemma 3.6 we have that

$$\begin{aligned} ||\partial_x^k H(t) * \varphi||_{L^2}^2 &\leq C\int_{\mathbb{R}^n} |\xi|^{2k} e^{-2c\rho(\xi)t}|\hat{\varphi}(\xi)|^2 d\xi \\ &\leq C\int_{\{\xi,|\xi|\leq 1\}} |\xi|^{2k}e^{-c|\xi|^2 t}|\hat{\varphi}|^2 d\xi + C\int_{\{\xi,|\xi|\geq 1\}} |\xi|^{2k}e^{-2ct}|\hat{\varphi}|^2 d\xi \leq k_1 + k_2. \end{aligned}$$

Assume that p' satisfies $\frac{1}{p} + \frac{1}{p'} = 1$, then by Hausdorf–Young's inequality, we obtain

$$k_1 \leq C(1+t)^{-n(\frac{1}{p}-\frac{1}{2})-k}||\hat{\varphi}||_{p'}^2 \leq C(1+t)^{-n(\frac{1}{p}-\frac{1}{2})-k}||\varphi||_p^2.$$

On the other hand

$$k_2 \leq Ce^{-2ct}||\partial_x^k\varphi||_{L^2}^2, \text{ for } 0 \leq k \leq s,$$

then

$$||\partial_x^k H(t) * \varphi||_{L^2} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}||\varphi||_p + Ce^{-ct}||\partial_x^k\varphi||_{L^2}, \text{ for } 0 \leq k \leq s.$$

□

By using Proposition 3.7 with $p = 2$, we obtain the following decay estimates of $\bar{u}$ given by (2.6), if initial data $u_0 \in H^s(\mathbb{R}^n)$.

4 Decay estimates for linear problem

Theorem 4.1 (Decay estimates for linear problem). Under the same assumptions as in Theorem 3.4, the solution $\bar{u}$ given by (2.6) satisfies the decay estimates:

$$||\partial_x^k \bar{u}(t)||_{H^{s-k}} \leq cI_0(1+t)^{-\frac{k}{2}}$$

for $0 \leq k \leq s$.

Also, if initial data $u \in H^s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ then by using (3.15) we have that sharp decay estimates of the solution $\bar{u}$ to (2.1) . Therefore the theorem 4.1 can be stated as follows.

Theorem 4.2 (Sharp decay estimates for linear problem). Let $s \geq 1$ be an integer and $\theta \in [0,1]$. Assume that $u_0 \in H^s(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ and put $I_p := ||u_0||_{H^s} + ||u_0||_{L^p}$ with $1 \leq p < 2$. Then the solution $\bar{u}$ to (2.1) given by (2.6) satisfies the following decay estimates :

$$||\partial_x^k \bar{u}(t)||_{H^{s-k}} \leq cI_p(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}, \tag{4.1}$$

for $0 \leq k \leq s$.

Since proof of Theorem 4.1 and Theorem 4.2 are similar, here we only prove Theorem 4.2.

Proof . Let $k \geq 0$, $m \geq 0$ are an integers. In view of (2.6), by using (3.15), we have that

$$\begin{aligned} ||\partial_x^{k+m}\bar{u}(t)||_{L^2} &\leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}||u_0||_{L^p} + Ce^{-ct}||\partial_x^{k+m}u_0||_{L^2} \\ &\leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}(||u_0||_{L^p} + ||\partial_x^{k+m}u_0||_{L^2}) \end{aligned} \tag{4.2}$$

for $k+m \leq s$ then $m \leq s-k$ we have that

$$||\partial_x^k\bar{u}(t)||_{H^{s-k}} \leq CJ_p(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}, \text{ for } 0 \leq k \leq s.$$

Thus (4.1) is proved. □

Remark 4.3. The estimates in Theorem 4.1 and Theorem 4.2 indicate that the decay structure of solutions to the linear problem (2.1) is not of regularity-loss type.

5 Global existence and decay for semi-linear problem

In this section we will first introduce a set of time-weighted Sobolev spaces and employ the contraction mapping theorem to prove the global existence and optimal decay of solution to the semi-linear problem.

First we give some useful lemmas.

Lemma 5.1. Assume that p, q, r and k are integers, $1 \leq p, q, r \leq \infty$, $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ and $k \geq 0$, then

$$||\partial_x^k(uv)||_{L^p} \leq C(||u||_{L^q}||\partial_x^k v||_{L^r} + ||v||_{L^q}||\partial_x^k u||_{L^r}).$$

Proof of Lemma 5.1 can be found in [19].

By using Lemma 5.1, we have

Lemma 5.2. Assume that p, q, r, k, α and β are integers, $1 \leq p, q, r \leq \infty$, $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ and $k \geq 0, \alpha \geq 1$ and $\beta \geq 1$, then

$$||\partial_x^k(u^\alpha v^\beta)||_{L^p} \leq C||u||_{L^\infty}^{\alpha-1}||v||_{L^\infty}^{\beta-1}(||u||_{L^q}||\partial_x^k v||_{L^r} + ||v||_{L^q}||\partial_x^k u||_{L^r}).$$

Recall Assumption [B], we know that $f \in C^\infty(\mathbb{R}\backslash\{0\})$, and $f(u) = O(|u|^\alpha)$ as $|u| \to 0$, here $\alpha > \alpha_n$ and $\alpha_n := 1+\frac{2}{n}$, $n \geq 1$, and α is assumed to be an integer for obtain the following result about the global existence and optimal decay estimates of solution to the semi-linear problems (1.1).

Theorem 5.3 (existence and decay estimates for semi-linear problem). Let s be an integer such that $s \geq [n/2]+1$ for $n \geq 1$ and $\theta \in [0, 1]$. Let $u_0 \in H^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, and put

$$I_0 := ||u_0||_{H^s} + ||u_0||_{L^1}.$$

If I_0 is suitably small, then there exists a unique solution $u(t, x) \in C^0([0, \infty); H^s(\mathbb{R}^n))$ of (1.1) satisfying the following decay estimates:

$$||\partial_x^k u(t)||_{H^{s-k}} \leq cI_0(1+t)^{-\frac{n}{4}-\frac{k}{2}}, \text{ for } 0 \leq k \leq s. \tag{5.1}$$

Proof . Let us define the following space

$$X := \{u \in C([0, \infty), H^s(\mathbb{R}^n));\ ||u||_X < \infty\},$$

where

$$||u||_X := \sum_{k \leq s} \sup_{t \geq 0}(1+t)^{\frac{n}{4}+\frac{k}{2}}||\partial_x^k u(t)||_{H^{s-k}}.$$

We introduce the closed ball

$$B_R := \{u \in X; ||u||_X \leq R\},\ R > 0;$$

Denote

$$\Phi[u](t) := \Phi_{\text{lin}}(t) + \int_0^t H(t-\tau) * f(u)(\tau)d\tau,$$

where

$$\Phi_{\text{lin}}(t) := H(t) * u_0.$$

We have

$$\forall v, w \in X,\ \Phi[v](t) - \Phi[w](t) = \int_0^t H(t-\tau) * (f(v) - f(w))(\tau)d\tau.$$

Noticing that $f(v) = O(|v|^\alpha)$ and using Lemma 5.2. We have the following inequalities for $k \geq 0$

$$\begin{aligned} & ||\partial_x^k (f(v) - f(w))(\tau)||_{L^1} \\ \leq\ & C||(v,w)(\tau)||_{L^\infty}^{\alpha-2} \\ & \left(||(v,w)(\tau)||_{L^2}||\partial_x^k (v-w)(\tau)||_{L^2} + ||\partial_x^k (v,w)(\tau)||_{L^2}||(v-w)(\tau)||_{L^2}\right), \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} & ||\partial_x^k (f(v) - f(w))(\tau)||_{L^2} \\ \leq\ & C||(v,w)(\tau)||_{L^\infty}^{\alpha-2} \\ & \left(||(v,w)(\tau)||_{L^\infty}||\partial_x^k (v-w)(\tau)||_{L^2} + ||\partial_x^k (v,w)(\tau)||_{L^2}||(v-w)(\tau)||_{L^\infty}\right). \end{aligned} \tag{5.3}$$

Also, if $v \in X$, then the following estimate holds

$$||v(\tau)||_{L^\infty} \leq C||v||_X (1+\tau)^{-\frac{n}{4}}. \tag{5.4}$$

In fact, by using the Gagliardo-Nirenberg inequality, we get

$$||v(\tau)||_{L^\infty} \leq ||v(\tau)||_{L^2}^{1-\lambda} \left\|\partial_x^{s_0} v(\tau)\right\|_{L^2}^{\lambda},$$

where $s_0 = \left[\frac{n}{2}\right] + 1$, $\lambda = \frac{n}{2s_0}$. We have for any $n \geq 1$,

$$||v(\tau)||_{L^2} \leq C(1+\tau)^{-\frac{n}{4}}||v(\tau)||_X$$

and since $s \geq s_0$, we obtain

$$\|\partial_x^{s_0} v(\tau)\|_{L^2} \leq \|v(\tau)\|_{H^{s_0}} \leq \|v(\tau)\|_{H^s} \leq C(1+\tau)^{-\frac{n}{4}}||v(\tau)||_X.$$

Hence, we deduce

$$\|v(\tau)\|_{L^\infty} \leq C(1+\tau)^{-\frac{n}{4}(1-\lambda)-\frac{n}{4}\lambda}||v(\tau)||_X = C(1+\tau)^{-\frac{n}{4}}||v(\tau)||_X.$$

Now, we prove the estimate

$$||\partial_x^k (\Phi[v] - \Phi[w])(t)||_{H^{s-k}} \leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}}||(v,w)||_X^{\alpha-1}||v-w||_X, \tag{5.5}$$

for $0 \leq k \leq s$.

Assume that k, m are non-negative integers, such that $k \leq s$.

By applying ∂_x^{k+m} to $\Phi[v] - \Phi[w]$, we have that

$$\begin{aligned} ||\partial_x^{k+m}(\Phi[v] - \Phi[w])(t)||_{L^2} \ &\leq\ \left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t\right)||\partial_x^{k+m} H(t-\tau) * (f(v) - f(w))(\tau)||_{L^2} d\tau \\ &=\ : I_1 + I_2. \end{aligned} \tag{5.6}$$

By virtue of (3.15) with $p=1$, we have

$$\begin{aligned} I_1 &\leq C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{4}-\frac{k+m}{2}}||(f(v)-f(w))(\tau)||_{L^1}d\tau \\ &+C\int_0^{\frac{t}{2}}e^{-c(t-\tau)}||\partial_x^{k+m}(f(v)-f(w))(\tau)||_{L^2}d\tau \\ &=: I_{11}+I_{12}. \end{aligned} \tag{5.7}$$

By using (5.2) with $k=0$ and (5.4), we have that

$$||(f(v)-f(w))(\tau)||_{L^1} \leq C||(v,w)||_X^{\alpha-1}||(v-w)||_X(1+\tau)^{-\frac{n}{4}(\alpha-1)-\frac{n}{4}}. \tag{5.8}$$

Since $\alpha > \alpha_n = 1+\frac{2}{n}$ for $n=1,2$ and $\alpha \geq 2$ for $n \geq 3$, appealing to (5.8), we have

$$I_{11} \leq C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{4}-\frac{k+m}{2}}(1+\tau)^{-\frac{n}{4}(\alpha-1)-\frac{n}{4}}d\tau||(v,w)||_X^{\alpha-1}||(v-w)||_X$$

then

$$I_{11} \leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}}||(v,w)||_X^{\alpha-1}||v-w||_X. \tag{5.9}$$

If $k+m \leq s$, by virtue of (5.3) with k replaced by $k+m$ and (5.4), it yields that

$$||\partial_x^{k+m}(f(v)-f(w))(\tau)||_{L^2} \leq C||(v,w)||_X^{\alpha-1}||(v-w)||_X(1+\tau)^{-\frac{n}{2}(\alpha-1)-\frac{n}{4}-\frac{k}{2}}. \tag{5.10}$$

By appealing to (5.10) and notice that $\alpha \geq 2$, we obtain

$$I_{12} \leq C\int_0^{\frac{t}{2}}e^{-c(t-\tau)}(1+\tau)^{-\frac{n}{2}(\alpha-1)-\frac{n}{4}-\frac{k}{2}}d\tau||(v,w)||_X^{\alpha-1}||(v-w)||_X.$$

Hence

$$I_{12} \leq Ce^{-ct}||(v,w)||_X^{\alpha-1}||v-w||_X,$$

Therefore by putting estimates I_{11} and I_{12} into (5.7), we get

$$I_1 \leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}}||(v,w)||_X^{\alpha-1}||v-w||_X + Ce^{-ct}||(v,w)||_X^{\alpha-1}||v-w||_X. \tag{5.11}$$

Thus

$$I_1 \leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}}||(v,w)||_X^{\alpha-1}||v-w||_X \tag{5.12}$$

with $0 \leq m \leq s-k$. Also by employing (3.15) with $p=1$ to the term I_2, we obtain

$$\begin{aligned} I_2 &\leq C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{n}{4}-\frac{m}{2}}||\partial_x^k(f(v)-f(w))(\tau)||_{L^1}d\tau \\ &+C\int_{\frac{t}{2}}^{t}e^{-c(t-\tau)}||\partial_x^{k+m}(f(v)-f(w))(\tau)||_{L^2}d\tau \\ &:= I_{21}+I_{22}. \end{aligned} \tag{5.13}$$

In view of (5.2) and (5.4), we have that

$$||\partial_x^k(f(v)-f(w))(\tau)||_{L^1} \leq C||(v,w)||_X^{\alpha-1}||(v-w)||_X(1+\tau)^{-\frac{n}{2}(\alpha-2)-\frac{n}{2}-\frac{k}{2}}, \tag{5.14}$$

for $0 \leq k \leq s$.

Since $\alpha > \alpha_n = 1+\frac{2}{n}$ for $n=1,2$ and $\alpha \geq 2$ for $n \geq 3$, appealing to (5.14), we have

$$I_{21} \leq C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{n}{4}-\frac{m}{2}}(1+\tau)^{-\frac{n}{2}(\alpha-2)-\frac{n}{2}-\frac{k}{2}}d\tau||(v,w)||_X^{\alpha-1}||(v-w)||_X$$

then

$$I_{21} \leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}}||(v,w)||_X^{\alpha-1}||v-w||_X,$$

with $0 \leq k \leq s$. We use (5.10) to have that

$$I_{22} \leq C \int_{\frac{t}{2}}^{t} e^{-c(t-\tau)}(1+\tau)^{-\frac{n}{2}(\alpha-1)-\frac{n}{4}-\frac{k}{2}} d\tau ||(v,w)||_X^{\alpha-1} ||(v-w)||_X.$$

Therefore

$$I_{22} \leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}} ||(v,w)||_X^{\alpha-1} ||v-w||_X,$$

with $0 \leq m \leq s-k$. We substitute the estimates for I_{21} and I_{22} into (5.13), we infer

$$I_2 \leq C(1+t)^{-\frac{n}{4}-\frac{k}{2}} ||(v,w)||_X^{\alpha-1} ||v-w||_X, \tag{5.15}$$

with $0 \leq m \leq s-k$.

Using estimates (5.12) and (5.15) into (5.6) and take the sum with respect to m, $0 \leq m \leq s-k$, we get the estimate (5.5). Hence, we obtain that

$$||\Phi[v] - \Phi[w]||_X \leq C||(v,w)||_X^{\alpha-1} ||v-w||_X.$$

So far we proved that $||\Phi[v] - \Phi[w]||_X \leq C_1 R^{\alpha-1} ||v-w||_X$ for $v, w \in B_R$.

On the other hand $\Phi[0](t) = \Phi_{\text{lin}}(t) = \bar{u}(t)$ and from Theorem 4.2 we know that $||\Phi_{\text{lin}}||_X \leq C_2 I_0$ if I_0 suitably small. We put $R = 2C_2 I_0$. Now, if I_0 suitably small such that $R < 1$ and $C_1 R \leq \frac{1}{2}$, then we infer that

$$||\Phi[v] - \Phi[w]||_X \leq \frac{1}{2}||v-w||_X.$$

So, it yields for $v \in B_R$ that

$$||\Phi[v]||_X \leq ||\Phi_0||_X + \frac{1}{2}||v||_X \leq C_2 I_0 + \frac{1}{2}R = R,$$

i.e. $\Phi[v] \in B_R$. Thus $v \to \Phi[v]$ is a contraction mapping on B_R, which implies that there exists a unique $u \in B_R$ satisfying $\Phi[u] = u$, and it is a solution to the semi-linear problem (1.1) satisfying the decay estimate (5.1). So we complete the proof of Theorem 5.3.□

Example 5.4. Consider the following initial value problem

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) + g * (-\Delta)^{1/2} u = (\sin t)\, e^{-u^2} u^3, \ t > 0, \ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \ x \in \mathbb{R}^n, \end{cases} \tag{5.16}$$

here, $\theta = 1/2$, $g(t) = \int_t^{+\infty} \mu(s)\, ds$ where $\mu(t) = t^{-\gamma} e^{-\beta t} \chi_{(0,s_0]} + \sum_{i=1}^{+\infty} a_i \chi_{[s_{i-1},s_i]}(t)$, with $s_0 = 1$, $0 \leq \gamma < 1$, $\beta \geq 0$ and where $\{a_i\}$ is a strictly decreasing positive sequence such that $a_1 \leq e^{-\beta}$. Note that $\mu'(t) \leq 0$ for almost everywhere $t > 0$, and $g = g(0) = \int_0^1 s^{-\gamma} e^{-\beta s} ds + \sum_{i=1}^{+\infty} a_i (s_i - s_{i-1})$. Now, we check that $g(t) \leq K\mu(t)$ holds for any $t > 0$ and for some $K > 0$. In fact, for $t < 1$, we have

$$\begin{aligned} g(t) &= \int_t^{+\infty} \mu(s)\, ds = \int_t^1 s^{-\gamma} e^{-\beta s} ds + \sum_{i=1}^{+\infty} a_i (s_i - s_{i-1}) \\ &= \int_t^1 s^{-\gamma} e^{-\beta s} ds + \left[\sum_{i=1}^{+\infty} a_i (s_i - s_{i-1})\right] t^{\gamma} e^{\beta t} t^{-\gamma} e^{-\beta t} \\ &\leq (1-t)\, t^{-\gamma} e^{-\beta t} + g e^{\beta} t^{-\gamma} e^{-\beta t} \\ &\leq \left(1 + g e^{\beta}\right) t^{-\gamma} e^{-\beta t} = K\mu(t). \end{aligned}$$

When $t \geq 1$, there exists $i_0 \geq 1$ such that $t \in [s_{i_0-1}, s_{i_0}]$, so we have

$$\begin{aligned} g(t) &= \int_t^{+\infty} \mu(s)\, ds = a_{i_0}(s_{i_0} - t) + \sum_{i=i_0+1}^{+\infty} a_i (s_i - s_{i-1}) \\ &\leq a_{i_0}\left(s_{i_0} - 1 + \frac{g}{a_{i_0}}\right) = K\mu(t). \end{aligned}$$

On the other hand the function $f(u) = (\sin t)\, e^{-u^2} u^3 \in C^\infty(\mathbb{R})$ satisfies $f(u) = O\left(|u|^3\right)$ as $u \to 0$. Thus, all the assumptions in Theorem 5.3 satisfied, and, hence, the initial value problem (5.16) has a unique global solution $u(t,x) \in C([0,\infty); H^s(\mathbb{R}^n))$, $s > n/2$.

Conclusion

In this paper, we studied the initial value problem for certain semilinear parabolic equation with effect of memory involving fractional Laplacian term in the whole space $\mathbb{R}^n$. It is well-known that this type of problems can be considered as nonclassical diffusion in which the Fourier law and the standard heat conduction equations are replaced by more general equations. As first aim of this study, we extended the energy multiplier techniques in the Fourier space to prove optimal decay estimates of the fundamental solution of the linear homogeneous equation under some conditions on the memory term. Further by using Fourier–Laplace transforms the solution formula of the problem is obtained. Then, by employing the same time-weighted energy method together with the semigroup argument, we show the optimal decay estimate of solutions for small initial data. Our results improve and generalize many earlier related works where the system is not necessarily dissipative and the kernel μ may be unbounded at the origin and contain jumping discontinuities. Finally, an example is provided to illustrate the main results.

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References

[1] Y.K. Chang and R. Ponce. *Uniform exponential stability and its applications to bounded solutions of integro-differential equations in Banach spaces*, J. Integral Equations Appl. **30** (2018), 347–369.

[2] J. Chen, T. Xiao and J. Liang. *Uniform exponential stability of solutions to abstract Volterra equations*, J. Evol. Equ. 4 (2009), no. 9, 661–674.

[3] J. Chen, J. Liang, T. Xiao. *Stability of solutions to integro-differential equations in Hilbert spaces*, Bull. Belg. Math. Soc. **18** (2011), 781–792.

[4] V.V. Chepyzhov, E. Mainini and V. Pata, *Stability of abstract linear semigroups arising from heat conduction with memory*, Asymptot. Anal. **46** (2006), 251–273.

[5] V.V. Chepyzhov and A. Miranville, *On trajectory and global attactors for semilinear heat equations with fadding memory*, Indiana Univ. Math. J. **55** (2006), 119–168.

[6] V.V. Chepyzhov and V. Pata, *Some remarks on stability of semigroups arising from linear viscoelasticity*, Asymptotic Anal. **46** (2006), 251–273.

[7] B.D. Coleman and M.E. Gurtin, *Equipresence and constitutive equations for rigid heat conductors*, Z. Angew. Math. Phys. **18** (1967), 199–208.

[8] C.M. Dafermos, *Asymptotic stability in viscoelasticity*, Arch. Ration. Mech. Anal. **37** (1970), 297–308

[9] V. Danese, P.G. Geredeli and V. Pata, *Exponential attractors for abstract equations with memory and applications to viscoelasticity*, J. Discrete Contin. Dyn. Syst. **35** (2015), 2881–2904.

[10] G. Da Prato and M. Iannelli, *Existence and regularity for a class of integro-differential equations of parabolic type*, J. Math. Anal. Appl. **112** (1985), no. 1, 36–55.

[11] G. Da Prato and A. Lunardi, *Solvability on the real line of a class of linear Volterra integro-differential equations of parabolic type*, Ann. Mat. Pura Appl. **150** (1988), no. 4, 67–117.

[12] M. Conti, S. Gatti and V. Pata, *Decay rates of volterra equations on $\mathbb{R}^n$*, Cent. Eur. J. Math. **5** (2007), 720–732.

[13] S. Gatti, A. Miranville, V. Pata and S. Zelik, *Attractors for semilinear equations of viscoelasticity with very low dissipation*, Rocky Mountain J. Math. **38** (2008), 1117–1138.

[14] M. Fabrizio and A. Morro, *Mathematical problems in linear viscoelasticity*, Society for Industrial and Applied Mathematics, Philadelphia, 1992.

[15] C. Giorgi, M.G. Naso and V. Pata, *Exponential stability in linear heat conduction with memory: A semigroup approach*, Comm. Appl. Anal. **5** (2001), 121–134.

[16] M. Grasselli and V. Pata, *Uniform attractors of nonautonomous dynamical systems with memory*, Evolution equations, semigroups and functional analysis. Birkhäuser, Basel, 2002, pp. 155–178.

[17] M.E. Gurtin and A.C. Pipkin, *A general theory of heat conduction with finite wave speeds*, Arch. Rational Mech. Anal. **31** (1968), 113–126.

[18] Y. Liu and S. Kawashima, *Decay property for a plate equation with memory-type dissipation*, Kinet. Relat. Models **4** (2011), no. 2, 531–547.

[19] Y. Liu and S. Kawashima, *Global Existence and Asymptotic Behavior of Solution for Quasi-linear Dissipation Plate Equation*, Discrete Contin. Dyn. Syst. **29** (2011), 1113–1139.

[20] R. Ikehata and M. Natsume, *Energy decay estimates for wave equations with a fractional damping*, Differ. Integral Equ. **25** (2012), no. 9-10, 939–956.

[21] E. Mainini and G. Mola, *Exponential and polynomial decay for first order linear Volterra evolution equations*, Quart. Appl. Math. **67** (2009), 93–111.

[22] E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), no. 5, 521–573.

[23] J.W. Nunziato, *On heat conduction in materials with memory*, Quart. Appl. Math. **29** (1971), 187–204.

[24] R.K. Miller, *An integrodifferential equation for rigid heat conductors with memory*, J. Math. Anal. Appl. **66** (1978), 331–332.

[25] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, 1983.

[26] J. Prüss, *Evolutionary integral equations and applications*, Monographs in Mathematics, vol. 87, Birkhäuser Verlag, Basel, 1993.

[27] B. Said-Houari and S.A. Messaoudi, *General decay estimates for a Cauchy viscoelastic wave problem*, Comm. Pure Appl. Anal. **13** (2014), no. 4, 1541–1551.

[28] G.F. Webb, *An abstract semilinear Volterra integrodifferential equation*, Proc. Amer. Math. Soc. **69** (1978), no. 2, 255–260