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Common fixed point theorems under implicit contractive condition using E. A. property on metric-like spaces employing an arbitrary binary relation with some application

Lucas Wangwe, Santosh Kumar*

Department of Mathematics, College of Natural and Applied Sciences, University of Dar es Salaam, Tanzania

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Abstract

In this paper, parallel to the ideas based on Ahmadullah *et al.* [4, 5, 6], and Eke *et al.* [18], we prove the existence and uniqueness of the common fixed point for a pair of self-mappings employing (E. A.)-property in metric-like spaces for implicit contractive mappings related to binary relation. Henceforth, results obtained will be verified with the help of illustrative examples. As an application of the results, we solve two boundary value problems of the second-order differential equation.

Keywords: Common fixed point, (E. A.)-property, Metric-like spaces, binary relation, implicit relation, integral equation
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1 Introduction

In 1906, Fréchet [19] introduced the study of metric fixed point theory in abstract spaces. In 1922, Banach [13] introduced a contraction principle in metric space that became a source for applying fixed point theory in pure and applied mathematics. Since then, several researchers are studying the existence and uniqueness of common fixed point for a pair of contractive mappings. We refer the reader to [24, 35, 37] and the references cited therein.

In 2002, Aamri and El-moutawakil [1] introduced the (E.A.)-property for a pair of self-mappings defined on metric spaces. It contains the class of compatible and non-compatible mappings in metric spaces and utilized the same to prove common fixed point theorems under strict contractive condition. For more information, we refer to readers [26, 27] and references there in.

In 2015, Alam and Imdad [8] gave a generalization of the Banach contraction principle in a complete metric space equipped with binary relation. Their results show that the contraction condition holds only for those elements linked with the binary relation, not for every pair of elements. For more results on binary relation, one can see [4, 5, 6, 18, 28, 34] and the references therein.

In 1999, Popa [33] gave the concept of implicit function in metric space which includes most of the well-known contractions of the existing literature besides several new ones. He proved some fixed point theorem for compatible

*Santosh Kumar
Email addresses: wangwelucas@gmail.com (Lucas Wangwe), drsengar2002@gmail.com (Santosh Kumar)

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mappings satisfying an implicit relation in metric spaces. These notion is still trending among the research community, and for more details, we refer to the readers [9, 15, 16, 22, 24, 23] and the references cited therein.

In 1994, Matthew [29] gave a generalization of Banach contraction principle to partial metric space using non-zero self distance notion (one can see in [31, 32, 36]). Further, Amini-Harandi [10] extended partial metric space notion to metric-like spaces by introducing some of its properties. One may consult in [12, 14, 17, 21], and the references therein.

In 2016, Ahmadullah *et al.* [4] proved a fixed point theorem for self mappings in metric-like spaces concerning binary relation. In 2019, Eke *et al.* [18] proved a common fixed point theorem for a pair of weakly compatible mappings under implicit contractive properties in metric spaces endowed with binary relation.

2 Preliminaries

We introduce some definitions, theorems and preliminary results, which will be helpful in developing the main result.

Definition 2.1. [29] A partial metric space is a pair (X, p) consisting of a non-empty set X together with a function $p : X \times X \to \mathbb{R}^+$, called the partial metric, such that for all $x, y, z \in X$ we have the following properties:

(P1) $p(x, y) = p(x, x) = p(y, y)$ if and only if $x = y$;

(P2) $p(x, x) \leq p(x, y)$;

(P3) $p(x, y) \leq p(y, x)$; and

(P4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

Then p is called a partial metric on X, and the pair (X, p) is called a partial metric space.

In partial metric space, it is not necessary that $p(x, x) = 0$, for every $x = y$, while in metric if $x = y$, then $p(x, x) = 0$. The following are fundamental properties of partial metric spaces.

Definition 2.2. [29] Let (X, p) be a partial metric space, then:

(i) a sequence $\{x_n\}$ in a partial metric space (X, p) converges to a point $x \in X$ if $\lim_{n\to\infty} p(x_n, x) = p(x, x)$.

(ii) a sequence $\{x_n\}$ of elements of X is called p-Cauchy if the $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.

(iii) a partial metric space (X, p) is said to be p-complete if every p- Cauchy sequence $\{x_n\}$ in X is p- convergent, with respect to τ_p, to a point $x \in X$ such that

$$\lim_{n,m\to\infty} p(x_n, x_m) = p(x, x).$$

An immediate example of partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$.

In 2012, Ami-Harandi [10] gave a generalization of partial metric spaces to metric-like spaces by introducing the following properties:

Definition 2.3. *[10]* A metric-like space is a pair (X, σ) consisting of a non-empty set X together with a function $\sigma : X \times X \to \mathbb{R}^+$, called the partial metric, such that for all $x, y, z \in X$, we have the following condition holds:

$(\sigma 1)$ $\sigma(x, y) = 0 \implies x = y$;

$(\sigma 2)$ $\sigma(x, y) = \sigma(y, x)$; and

$(\sigma 3)$ $\sigma(x, y) \leq \sigma(x, y) + \sigma(y, z)$.

Then σ is called a metric-like on X, so a pair (X, σ) is called a metric-like space.

The metric-like on X satisfies all of the conditions of metric except that $\sigma(x,x)$ may be positive for $x \in X$. Following are some characteristic of metric-like spaces:

Definition 2.4. [10] Let (X,σ) be a metric-like space.

(i) Each metric-like σ on X generates a topology τ_σ on X whose base is the family of open σ-balls

$$B_\sigma(x,\epsilon) = \{y \in X : |\sigma(x,y) - \sigma(x,x)| < \epsilon\}, \forall x \in X \ and \ \epsilon > 0.$$

(ii) A sequence $\{x_n\}$ in metric-like space X,σ converges to a point $x \in X$ if and only if

$$\lim_{n,m\to\infty} \sigma(x_n,x) = \sigma(x,x).$$

(iii) The sequence $\{x_n\}_{n=0}^{\infty}$ is said to be σ-Cauchy if the limit

$$\lim_{n,m\to\infty} \sigma(x_n,x_m)$$

exists and is finite.

(iv) The space (X,σ) is called complete if for every σ-Cauchy sequence in $\{x_n\}_{n=0}^{\infty}$, there exists some $x \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n,x) = \sigma(x,x) = \lim_{n\to\infty} \sigma(x_n,x_m).$$

(v) A sequence $\{x_n\}$ in (X,σ) is said to be $0-\sigma$-Cauchy sequence if $\lim_{n\to\infty} \sigma(x_n,x_m) = 0$. The space (X,σ) is said to be $0-\sigma$- complete if every $0-\sigma$- Cauchy sequence in X converges in (τ_σ) to a point $x \in X$ such that $\sigma(x,x) = 0$.

(vi) A mapping $T : X \longrightarrow X$ is continuous if the following limits exists (finite) and

$$\lim_{n\to\infty} \sigma(x_n,x) = \sigma(Tx_n,Tx).$$

Remark 2.5. [10] It is easy to see that a metric space is a partial metric space, and each partial metric space is a metric-like space, but the converse is not true.

Remark 2.6. [10] Every partial metric space is a metric-like space, and this can be illustrated by the use of the following example:

Example 2.7. [10] Let $X = \{0,1\}$ and $\sigma : X \times X \longrightarrow \mathbb{R}^+$ be defined by

$$\sigma(x,y) = \begin{cases} 2, & if \ x = y = 0, \\ 1, & otherwise. \end{cases}$$

Then (X,σ) is a metric-like space, but is not a partial metric space since $\sigma(0,0) \neq \sigma(0,1)$, then (X,σ).

Now, we introduce some definitions related to a common fixed point in metric-like space.

Definition 2.8. [24] Let S,T be self-mappings of a non empty set X. A point $x \in X$ is coincidence point of S and T if $x^* = Sx = Tx$. The set of coincidence point of S and T is denoted by $C(S,T)$.

Motivated from Jungck [25] and Sessa [35], we can have the following definitions:

Definition 2.9. Let (S,T) be a pair of self mappings on a metric-like space (X,σ). Then a point $x^* \in X$ is called coincidence point of the pair (S,T) if $Tx = Sx = x^*$. If $x^* = x$ then, x is said to be a common fixed point.

Definition 2.10. Let (S,T) be a pair of self mappings on a metric-like space (X,σ). Then the pair (S,T) is said to be:

(i) Commuting if, for all $x \in X$, $S(Tx) = T(Sx)$,

(ii) Weakly commuting if, for all $\sigma(S(Tx), T(Sx)) \leq \sigma(Sx, Tx)$,

(iii) Compatible if $\lim_{n\to\infty} \sigma(STx_{2n}, TSx_{2n}) = 0$, whenever x_{2n} is a sequence in X such that $\lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Sx_{2n} = x^*$,

(iv) Weakly compatible if, for all $S(Tx) = T(Sx)$, for every coincidence point $x \in X$.

Motivated from Aamri and Moutawakil [1], we can have the following definition:

Definition 2.11. A pair of self-mappings (T, S) of a metric-like space (X, σ) is said to satisfy the property (E.A) if there exist at least one sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Sx_{2n} = x^*,$$

for some $x^* \in X$.

Remark 2.12. It is known that two pairs of self-mappings S and T of metric-like spaces (X, σ), will be non compatible if there exist a sequence $\{x_{2n}\}$ in X such that

$$\lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Sx_{2n} = x^*,$$

but

$$\lim_{n\to\infty} \sigma(STx_{2n}, TSx_{2n})$$

is non zero or not exists.

2.1 Implicit relation and related concept

In 1999, Popa [33] initiated the idea of an implicit relation to cover several well-known contractions conditions of the existing literature in one go besides admitting several new ones. Imdad *et. al.* [23] modified results due to Popa [33] by removing the assumption of continuity, relaxing the 'compatibility' to 'coincidentally commuting property' and replacing the completeness of the space with a set of four alternative conditions.

Ahmadullah et al. [4], later proved the results for unified relation-theoretic metrical fixed point theorems of mappings satisfying implicit contractive conditions.

They considered the family $\mathcal{F}$ of all continuous real functions $F : \mathbb{R}_+^6 \longrightarrow \mathbb{R}+$ and the following conditions:

(F1) F is non-increasing in the fifth variable; and $F(u, v, v, u, u + v, 0) \leq 0$ for $u, v \geq 0$ implies that there exist $\lambda \in [0, 1)$ such that $u \leq \lambda v$;

(F2) $F(u, 0, u, 0, 0, u) > 0$, for all $u > 0$;

(F3) F is non-increasing in the sixth variable and $F(u, u, 0, 0, u, u) \leq 0$ for all $u > 0$.

In that way, Ahmadullah et al. used this to unify and extend various findings in the literature.

We give some examples of functions which satisfies the above implicit relation conditions.

Example 2.1 The function of $F \in \mathcal{F}$ satisfies the properties (F1) - (F3) (see, [4]).

(1) $F(u_1, u_2, u_3, u_4, u_5, u_6) = u_1 - ku_2$, where $k \in [0, 1)$;

(2) $F(u_1, u_2, u_3, u_4, u_5, u_6) = u_1 - k\{u_3 + u_4\}$, where $k \in [0, \frac{1}{2})$;

(3) $F(u_1, u_2, u_3, u_4, u_5, u_6) = u_1 - k\{u_5 + u_6\}$, where $k \in [0, \frac{1}{2})$;

(4) $F(u_1, u_2, u_3, u_4, u_5, u_6) = u_1 - a_1u_2 - a_2(u_3 + u_4) - a_3(u_5 + u_6)$, where $a_1, a_2, a_3 \in [0, 1)$ and $a_1 + 2a_2 + 2a_3 < 1$;

(5) $F(u_1, u_2, u_3, u_4, u_5, u_6) = u_1 - ku_2 - L\min\{u_3, u_4, u_5, u_6, u_6\}$, where $k \in [0, 1)$ and $L \geq 0$;

(6) $F(u_1, u_2, u_3, u_4, u_5, u_6) = u_1 - (a_1u_2 + a_2u_3 + a_3u_4 + a_4(u_5 + u_6))$, where $a_1, a_2, a_3, a_4 \geq 0$ and $a_1 + a_2 + a_3 + a_4 < 1$;

(7) $F(u_1, u_2, u_3, u_4, u_5, u_6) = u_1 - k\max\left\{u_2, u_3, u_4, \dfrac{u_5 + u_6}{2}\right\} - L\min\{u_3, u_4, u_5, u_6\}$, where $k \in [0, 1)$ and $L \geq 0$;

(8) $F(u_1, u_2, u_3, u_4, u_5, u_6) = u_1 - k\max\{u_2, u_3, u_4, u_5, u_6\}$ where $k \in [0, \frac{1}{2})$;

(9) $F(u_1, u_2, u_3, u_4, u_5, u_6) = u_1 - (a_1u_2 + a_2u_3 + a_3u_4 + a_4u_5 + a_5u_6))$, where $a_i'^s, \geq 0$ (for $i = 1, 2, 3, 4, 5$) and $\sum_{i=1}^{5} a_i < 1$;

(10) $F(u_1, u_2, u_3, u_4, u_5, u_6) = u_1 - k\max\left\{u_2, u_3, u_4, \frac{u_5}{2}, \frac{u_6}{2}\right\}$, where $k \in [0, 1)$;

(11) $F(u_1, u_2, u_3, u_4, u_5, u_6) = u_1 - k\max\{u_2, u_3, u_4\} - (1 - k)(au_5 + bu_6)$, where $k \in [0, 1)$ and $0 \leqq a, b < \frac{1}{2}$.

(12) $F(u_1, u_2, u_3, u_4, u_5, u_6) = u_1^2 - u_1(a_1u_2 + a_2u_3 + a_3u_4) - a_4u_5u_6)$, where $a_1 > 0$; $a_2, a_3, a_4 \geq 0$; $a_1 + a_2 + a_3 < 1$ and $a_1 + a_4 < 1$;

(13) $F(u_1, u_2, u_3, u_4, u_5, u_6) = u_1^2 - a_1\max\{u_2^2, u_3^2, u_4^2\} - a_2\max\{u_3u_5, u_4u_6\} - a_3u_5u_6$, where $a_i'^s, \geq 0$ (for $i = 1, 2, 3$); $a_1 + 2a_2 < 1$ and $a_1 + a_4 < 1$;

(14) $F(u_1, u_2, u_3, u_4, u_5, u_6) = u_1^3 - k\{u_2^3 + u_3^3 + u_4^3 + u_5^3 + u_6^3\}$, where $k \in [0, \frac{1}{11})$.

2.2 Relation-theoretic in metric-like spaces

In this part, we recall some definition in relation theoretic notion in metric-like space related to binary relation. We define $\mathbb{N}_0 = \{\{0\} \cup \mathbb{N}\}$, where $\mathbb{N}$ is a set of natural numbers in X.

Definition 2.13. *[8]* Let $\mathcal{R}$ be a binary relation defined on a non-empty set X. Then any pair of the point $x, y \in X$ is said to be $\mathcal{R}$-comparative if either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$, which is together written as $[x, y] \in \mathcal{R}$.

Definition 2.14. *[8]* Let $\mathcal{R}$ be a binary relation defined on a non-empty set X. Then a sequence $(x_n) \in X$ is called $\mathcal{R}$-preserving if $(x_n, x_{n+1}) \in \mathcal{R}$, $\forall$n$\in \mathbb{N}_0$.

Definition 2.15. *[7]* Let T and S be two self-mappings defined on a non-empty set X. Then a binary relation $\mathcal{R}$ defined on X is called (T, S)- closed if $(Sx, Sy) \in \mathcal{R} \Rightarrow (Tx, Ty) \in \mathcal{R}$, $\forall\ x, y \in X$.

Proposition 2.16. *[7]* Let X be non-empty set, $\mathcal{R}$ a binary relation on $X$and T, S two self mappings on X. If $\mathcal{R}$ is (T, S)-closed, then $\mathcal{R}^s$ is (T, S)-closed.

Next, we present some relevant relation-theoretic notions in metric-like spaces

Definition 2.17. *[4]* Let (X, σ) be a metric-like space and $\mathcal{R}$ a binary relation on X. We say that (X, σ)-is $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence $\{x_n\}$ in X, there is some $x \in X$ such that

$$\lim_{n,m\to\infty} \sigma(x_n, x_m) = \sigma(x, x) = \lim_{n\to\infty} \sigma(x_n, x).$$

Recall that the limit of convergent sequence in metric-like spaces is not necessary unique.

Definition 2.18. *[4]* Let (X, σ) be a metric-like space. The a mapping $f : X \to X$ is said to be continuous-like at x if $fx_n \xrightarrow{\tau_\sigma} fx$ for any sequence $\{x_n\}$ with $x_n \xrightarrow{\tau_\sigma} x$. As usual, f is said to be continuous-like if it is continuous-like in the whole space X.

Definition 2.19. *[4]* Let (X, σ) be a metric-like space and $\mathcal{R}$ a binary relation on X. Then a mapping $f : X \to X$ is said to be $\mathcal{R}$-continuous-like at x if $fx_n \xrightarrow{\tau_\sigma} fx$ for any $\mathcal{R}$-preserving sequence $\{x_n\}$ with $x_n \xrightarrow{\tau_\sigma} x$. As usual, f is said to be $\mathcal{R}$-continuous-like if it is $\mathcal{R}$- continuous-like in the whole space X.

Definition 2.20. *[8]* Let (X, σ) be a metric-like space and $\mathcal{R}$-binary relation on X. Then $\mathcal{R}$ is said to be σ-self closed if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ with $x_n \xrightarrow{\tau_\sigma} x$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $[x_{n_k}, x] \in \mathcal{R}$, for all $k \in \mathbb{N}$.

Definition 2.21. *[34]* Let (X, σ) be a metric-like space and $\mathcal{R}$-binary relation on X. Then a subset D of X is said to be $\mathcal{R}$-directed if for every pair of point $x, y \in D$, there is z in X such that $(x, z) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$.

Definition 2.22. *[28]* Let (X, d) be a metric-like space, $\mathcal{R}$ a binary relation defined on X and x, y a pair of points in X. Then a finite sequence $\{z_0, z_1, z_2, ..., z_l\} \in X$ is said to be a path of length $l(where\ l\ \in N)$ joining x to y in $\mathcal{R}$ if $z_0 = x, z_l = y$ and $[z_i, z_{i+1}] \in \mathcal{R}$ for each $i \in \{1, 2, 3, ...l - 1\}$.

Observe that, a path of length l involves $(l + 1)$ elements of X that need not be distinct in general. Given a metric-like space (X, σ), a self-mapping T on X and a binary relation $\mathcal{R}$ on X, we employ the following notations:

(i) $F(f)$: the set of all fixed points of f;

(ii) $X(f, \mathcal{R})$: the collection of all points $x \in X$ such that $(x, Tx) \in \mathcal{R}$;

(iii) $\gamma(x, y, R)$: the family of all paths joining x to y in $\mathcal{R}$.

Ahmadullah et al. [6] proved the results in metric-like space as well as partial metric spaces equipped with an arbitrary relation as follows:

Theorem 2.23. *[6]* Let $(X, \sigma,)$ be a metric-like spaces equipped with a binary relation $\mathcal{R}$ defined on X and f a self-mapping on X. Suppose that the following conditions are satisfied:

(a) there exists a subset $Y \subseteq X$ with $fX \subseteq Y$ such that (Y, σ) is $\mathcal{R}$-complete,

(b) there exists x_0 such that $(x_0, fx_0) \in \mathcal{R}$,

(c) $\mathcal{R}$ is f-closed,

(d) either f is $\mathcal{R}$-continuous-like or $\mathcal{R}|_Y$ is σ-self-closed,

(e) there exists a constant $k \in [0, 1)$ such that $(\forall x, y \in X$ with $x, y \in \mathcal{R})$

$$\sigma(fx, fy) \leq k\sigma(x, y). \tag{2.1}$$

Then f has a fixed point. Moreover, if

(f) $\Upsilon(fx, fy, \mathcal{R}^s)$ is non-empty, for each $x, y \in X$. Then f has a unique fixed point.

This paper aims to prove the results of common fixed point theorems for a pair of self-mappings satisfying (E. A.)-property under metric binary relation via implicit contractive condition in metric -like spaces by extending Theorem 2.23 proved in [6].

3 Main Results

In this section, we prove the following theorem which is a generalization and improvement of Theorem 2.23.

Theorem 3.1. Let (X, σ) be a metric-like spaces equipped with a binary relation $\mathcal{R}$ defined on X. Let T and S be a pair of self-mapping on X. Assume that the following conditions hold:

(a) there exists $TX \subseteq SX$ such that (X, σ) is $\mathcal{R}$- complete,

(b) there exists x_0 such that $(Sx_0, Tx_0) \in \mathcal{R}$,

(c) $X(T, S, \mathcal{R})$ is non-empty and satisfying (E. A.) property,

(d) either (T, S) is $\mathcal{R}$-continuous-like or $\mathcal{R}$ is σ-self-closed and weakly compatible,

(e) there exists an implicit function $F \in \mathcal{F}$ such that

$$F(\sigma(Tx,Ty),\sigma(Sx,Sy),\sigma(Sx,Tx),\sigma(Sy,Ty),\sigma(Sx,Ty),\sigma(Sy,Tx)) \leq 0. \tag{3.1}$$

$\forall x,y \in X$ such that $x,y \in \mathcal{R}$. Then T and S has a common fixed point. Moreover, if

(f) $\Upsilon_{T,S}\,(Tx,Sx,\mathcal{R}^s)$ is non-empty, for each $x,y \in X$, wherein F satisfies (F_3). Then T and S has a unique common fixed point.

Proof . Assume that $TX \subseteq SX$ and (X,σ) is $\mathcal{R}$-complete, for x_0 with $(Sx_0,Tx_0) \in \mathcal{R}$. We can construct a T-S-sequence $\{Tx_n\}$ with initial point x_0 satisfying

$$(Sx_0,Tx_0),(Sx_1,Tx_1),(Sx_2,Tx_2),(Sx_3,Tx_3)\ldots(Sx_{2n},Tx_n),(Sx_{2n+1},Tx_{n+1}),$$

$\forall n \in \mathbb{N}_0$, such that, $\{Tx_{2n}\}$, $\{Sx_{2n}\} \in T(X)$.

From assumption (c), let x_0 be an arbitrary element of $X(T,S,\mathcal{R})$, then $(Sx_0,Tx_0) \in \mathcal{R}$. If $Sx_0 = Tx_0$, then x_0 is a common fixed point of T and S and proof is completed. Otherwise, if $Tx_0 \neq Sx_0$, then $SX \subset TX$. Now, we choose $x_1 \in X$ such that $Sx_1 = Tx_0$. Again, we can choose $x_2 \in X$ such that $Sx_2 = Tx_1$. Proceeding the same way, we construct a sequence $\{x_n\} \subset X$, such that

$$Sx_{2n+1} = Tx_{2n}, \forall n \in \mathbb{N}_0, \tag{3.2}$$

Now, we claim that $\{Tx_{2n}\}$ is $\mathcal{R}$-preserving sequence, thus

$$(Tx_{2n},Tx_{2n+1}) \in \mathcal{R}, \forall n \in \mathbb{N}_0. \tag{3.3}$$

By mathematical induction, if $n = 0$ in (4.2) and using (c) such that $x_0 \in X(T,S,\mathcal{R})$, we have

$$(Sx_0,Sx_1) \in \mathcal{R}, \tag{3.4}$$

which proves that (4.2) is true for $n = 0$. Now, assume that (4.2) is true for $n = k > 0$ therefore

$$(Sx_{2k},Sx_{2k+1}) \in \mathcal{R}.$$

From condition (d), $\mathcal{R}$ is (T,S)-closed. Thus, we have

$$(Tx_{2k},Tx_{2k+1}) \in \mathcal{R},$$

which, on using (4.2) shows that

$$(Sx_{2k+1},Tx_{2k+2}) \in \mathcal{R},$$

therefore (4.3) holds for $n = 2k+1$. Hence, by induction, (4.3) is true for all $n \in \mathbb{N}$. In following (4.2) and (4.3), the sequence $\{Tx_n\}$ is also an $\mathcal{R}$-preserving, thus

$$(Tx_{2n},Tx_{2n+1}) \in \mathcal{R}, \forall n \in \mathbb{N}_0.$$

Also, assumption (c) claim that T and S satisfies $(E.A)$-property. Therefore, to prove the claim, let x_{2n} be a sequence in X, which is $\mathcal{R}$-preserving sequence. Using Definition (2.11), we have

$$\lim_{n\to\infty} Tx_{2n} \quad = \quad \lim_{n\to\infty} Sx_{2n} = x^*. \tag{3.5}$$

$$\lim_{n\to\infty} \sigma(Tx_{2n},x) \quad = \quad \lim_{n\to\infty} \sigma(Sx_{2n},x). \tag{3.6}$$

By (σ_3) we have

$$\sigma(Tx_{2n}, Sx_{2n}) \leqslant \sigma(Tx_{2n}, x) + \sigma(Sx_{2n}, x). \tag{3.7}$$

As $n \to \infty$, Equation (3.7) leads to

$$\sigma(Tx_{2n}, Sx_{2n}) \quad \leq \quad \sigma(Tx, x) + \sigma(Sx, x). \tag{3.8}$$

Which implies that

$$\lim_{n\to\infty} Tx_{2n} \quad = \quad \lim_{n\to\infty} Sx_{2n} = x^* = x. \tag{3.9}$$

For some $x \in X$, suppose that SX is complete, then there exists $a \in X$ such that $x = Sa$. Using Equation (3.5), we have the following :

$$\lim_{n\to\infty} \sigma(Tx_{2n}, Sa) \quad = \quad \lim_{n\to\infty} \sigma(Sx_{2n}, Sa) = x^*. \tag{3.10}$$

Let us show that $Ta = Sa$. Suppose that $Ta \neq Sa$, using (3.1), we get

$$\begin{aligned} F(\sigma(Tx_{2n}, Ta), \sigma(Sx_{2n}, Sa), \sigma(Tx_{2n}, Sx_{2n}), \sigma(Sa, Ta),& \\ \sigma(Sx_{2n}, Ta), \sigma(Sa, Tx_{2n})) \quad &\leq \quad 0. \end{aligned} \tag{3.11}$$

Letting $n \to \infty$ in (3.11), we get

$$\begin{aligned} F(\sigma(Sa, Ta), \sigma(Sa, Sa), \sigma(Ta, Sa), \sigma(Sa, Ta),& \\ \sigma(Sa, Ta), \sigma(Sa, Ta)) \quad &\leq \quad 0. \end{aligned} \tag{3.12}$$

From (3.12), we have

$$\begin{aligned} \sigma(Ta, Sa) \quad &\leq \quad 0, \\ \Rightarrow \sigma(Ta, Sa) \quad &= \quad 0. \end{aligned}$$

Hence Ta=Sa, which is a contradiction.

Suppose that $TX \subset SX$. For every $x_0 \in X$ we consider the sequence $\{x_{2n}\} \in X$ defined by

$$\begin{aligned} Sx_{2n} \quad &= \quad Tx_{2n-1}, \\ Sx_{2n+1} \quad &= \quad Tx_{2n}. \end{aligned}$$

Let $x_0 \in X$ be an arbitrary point. As $TX \subset SX$ one can choose T-S sequence $\{Tx_{2n}\}$ with initial point x_0.
$x = x_{2n}$ and $y = x_{2n+1}$ in Equation (3.1) and denote

$$u = \sigma(Tx_{2n}, Tx_{2n+1}),$$

$$v = \sigma(Tx_{2n-1}, Tx_{2n}),$$

$$\begin{aligned} F(\sigma(Tx_{2n}, Tx_{2n+1}), \sigma(Sx_{2n}, Sx_{2n+1}), \sigma(Sx_{2n}, Tx_{2n}), \sigma(Sx_{2n+1}, Tx_{2n+1}),& \\ \sigma(Sx_{2n}, Tx_{2n+1}), \sigma(Sx_{2n+1}, Tx_{2n})) \leq 0.& \end{aligned} \tag{3.13}$$

By substituting $Sx_n = Tx_{2n-1}$ and $Sx_{2n+1} = Tx_{2n}$ in Equation (3.13), we have

$$\begin{aligned} F(\sigma(Tx_{2n}, Tx_{2n+1}), \sigma(Tx_{2n-1}, Tx_{2n}), \sigma(Tx_{2n-1}, Tx_{2n}), \sigma(Tx_{2n}, Tx_{2n+1}),& \\ \sigma(Tx_{2n-1}, Tx_{2n+1}), \sigma(Tx_{2n}, Tx_{2n})) \leq 0.& \end{aligned} \tag{3.14}$$

By substituting u, v in Equation (3.14), we obtain

$$F(u, v, v, u, \sigma(Tx_{2n-1}, Tx_{2n+1}), 0) \leq 0. \tag{3.15}$$

Using (σ_3) and F_1, since is non-decreasing in the fifth variable, we get

$$\begin{aligned} \sigma(Tx_{2n-1}, Tx_{2n+1}) &\leq \sigma(Tx_{2n-1}, Tx_{2n}) + \sigma(Tx_{2n}, Tx_{2n+1}), \\ \sigma(Tx_{2n-1}, Tx_{2n+1}) &\leq v + u. \end{aligned} \tag{3.16}$$

Using Equation (3.16) in (3.15), we have

$$F(u, v, v, u, u + v, 0) \leq 0.$$

Which satisfies F_1, therefore

$$u \leq \lambda v.$$

Implies that

$$\begin{aligned} \sigma(Tx_{2n}, Tx_{2n+1}) &\leq \lambda\sigma(Tx_{2n-1}, Tx_{2n}), \\ &\leq \lambda^n \sigma(Tx_0, Tx_1). \\ &\leq \lambda^{n+1}\sigma(x_0, x_1) \end{aligned} \tag{3.17}$$

Using (3.17) and (σ_3), for all $n, m \in \mathbb{N}_0$ with $m > n$, we obtain

$$\begin{aligned} \sigma(Tx_{2n}, Tx_{2m}) &\leq \sigma(Tx_{2n}, Tx_{2n+1}) + \sigma(Tx_{2n+1}, Tx_{2n+2}) + ...\sigma(Tx_{2m-1}, Tx_{2m}), \\ &\leq (\lambda^n + \lambda^{n+1} + \lambda^{n+2} + ... + \lambda^{m-1})\sigma(Tx_0, Tx_1), \\ &\leq \lambda^n \sigma(Tx_0, Tx_1)(1 + \lambda^n + \lambda^{n+1} + \lambda^{n+2} + ... + \lambda^{m-1}), \\ &\leq \frac{\lambda^n}{1-\lambda}\sigma(Tx_0, Tx_1), \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

The results obey a Cauchy sequence properties of completeness. Hence $\{x_{2n}\}$ is $\mathcal{R}$-preserving Cauchy sequence. If the pair (T, S) is closed and weakly compatible. Using Definition 2.10, we have

$$\begin{aligned} TSz &= STz, \\ Tz &= Sz, \\ STz &= SSz, \\ TSz &= TTz. \end{aligned}$$

To show that Tz is a common fixed point of T and S, we use inequality (3.1) which gives

$$\begin{aligned} F(\sigma(Tz, TTz), \sigma(Sz, STz), \sigma(Sz, Tz), \sigma(TTz, STz), \\ \sigma(TTz, Sz), \sigma(Tz, STz)) &\leq 0. \end{aligned}$$

Which implies that

$$\begin{aligned} F(\sigma(Tz, TTz), \sigma(Tz, TTz), \sigma(Tz, Tz), \sigma(TTz, TTz), \\ \sigma(TTz, Tz), \sigma(Tz, TTz)) &\leq 0, \\ \sigma(Tz, TTz) &\leq 0. \end{aligned} \tag{3.18}$$

Thus $STz = TTz = Tz$. So Tz is a common fixed point of T and S.

For the uniqueness, take $z = Tz$ as a common fixed point of T and S. Assume that $w = Sw$ and $z \neq w$, using $x = z, y = w$ in Equation (3.1), we get

$$F(\sigma(Tz, Tw),\ \sigma(Sz, Sw),\ \sigma(Tz, Sz),\ \sigma(Tw, Sw),\ \sigma(Tw, Sz),\ \sigma(Tz, Sw)) \leq 0.$$

Hence, we get

$$\begin{aligned} \sigma(z, w) &\leq 0, \\ \Rightarrow \sigma(z, w) &= 0, \end{aligned}$$

which is a contradiction. Therefore, z is a unique common fixed point of T and S.

Using the assumption taken in Theorem 3.1, we prove assertion (f) as follows: we observe that $C(T, S)$ is non-empty, so let us take a pair of elements say (a, b) in $C(T, S)$ such that

$$\begin{aligned} &Ta = Sa = \bar{a}, \\ &Tb = Sb = \bar{b}. \end{aligned} \tag{3.19}$$

Next, we are required to show that $\bar{a} = \bar{b}$. By observing the above assertion, there exists a S-path (say,$z_0, z_1, z_2, ..., z_l$) of length l in $\mathcal{R}^s$ from Ta to Tb, with

$$\begin{aligned} &Sz_0 = Ta, \\ &Sz_l = Tb, \end{aligned} \tag{3.20}$$

such that

$$[Sz_{2i}, Sz_{2i+1}] \in \mathcal{R}^s \subseteq \mathcal{R}, \tag{3.21}$$

for all $i \in 0, 1, 2, 3, ...l - 1$.

And

$$[Sz_{2i}, Tz_{2i}] \in \mathcal{R}^s \subseteq \mathcal{R} \tag{3.22}$$

for every $i \in 0, 1, 2, 3, ...l - 1$.

Define two constant sequences such that

$$z_{2n}^0 = a \ and \ z_{2n}^l = b.$$

By using (3.20), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} Tz_{2n}^0 &= Ta = \bar{a}, \\ Tz_{2n}^l &= Tb = \bar{b}. \end{aligned}$$

By usual substitution for $z_0^i = z_i$ for each $i \in 0, 1, 2, ...l$, that is

$$\begin{aligned} z_0^1 &= z_1, \\ z_0^2 &= z_2, \\ z_0^3 &= z_3, \\ z_0^4 &= z_4, \\ z_0^{l-1} &= z_{l-1}. \end{aligned}$$

Recall that $TX \subseteq SX$. Thus we construct a sequence

$$\{z_{2n}^1\}, \{z_{2n}^2\}, \{z_{2n}^3\}, ...\{z_{2n}^i\} \in X.$$

In general, $\{z_n^1\} \in X$

$$\begin{aligned} Sz_{2n+1}^1 &= Tz_{2n}^1, \\ Sz_{2n+1}^2 &= Tz_{2n}^2, \\ Sz_{2n+1}^3 &= Tz_{2n}^3, \\ Sz_{2n+1}^4 &= Tz_{2n}^4, \\ Sz_{2n+1}^{l-1} &= Tz_{2n}^{l-1}, \forall n \in \mathbb{N}. \end{aligned}$$

We obtain

$$Sz_{2n+1}^i = Tz_{2n}^i,$$

for all $i \in [0,\ l-1]$. Corresponding to each z_i, we have $[Sz_0^i, Sz_1^i] \in \mathcal{R}$ from (3.20), (3.21) and (T,S)-compactness of $\mathcal{R}$, we get

$$\lim_{n\to\infty} \sigma(Sz_{2n}^i, Sz_{2n+1}^i) = 0,$$

for each $i \in 1, 2, 3, ...l-1$.

Thus, $\mathcal{R}$ is (T,S)-closed and we conclude that $[Tz_{2n}^i, Tz_{2n}^{i+1}] \in \mathcal{R}$, for each $i \in 0, 1, 2, 3, ...l-1$ and for all $n \in \mathbb{N}$.

Otherwise, $[Sz_{2n}^i, Sz_{2n}^{i+1}] \in \mathcal{R}$, for each $i \in 0, 1, 2, 3, ...l-1$ and for all $n \in \mathbb{N}$.

Define $\sigma_n^i = \sigma(Sz_{2n}^i, Sz_{2n}^{i+1})$, for each $i \in 0, 1, 2, 3, ...l-1$ and for all $n \in \mathbb{N}$. We assert that, $\lim_{n\to\infty} \sigma_{2n}^i > 0$. Assume that $\lim_{n\to\infty} \sigma_{2n}^i = \sigma > 0$.

Since $[Sz_{2n}^i, Sz_{2n}^{i+1}] \in \mathcal{R}$, either $[Sz_{2n}^i, Sz_n^{i+1}] \in \mathcal{R}$ or $[Sz_{2n}^{i+1}, Sz_{2n}^i] \in \mathcal{R}$.

If $[Sz_{2n}^i, Sz_{2n}^{i+1}] \in \mathcal{R}$, then applying the condition (e), we have

$$\begin{aligned} &F(\sigma(Tz_{2n}^i, Tz_{2n}^{i+1}), \sigma(Sz_{2n}^i, Sz_{2n}^{i+1}), \sigma(Sz_{2n}^i, Tz_{2n}^i), \sigma(Sz_{2n}^{i+1}, Tz_{2n}^{i+1}), \\ &\qquad \sigma(Sz_{2n}^i, Tz_{2n}^{i+1}), \sigma(Sz_{2n}^{i+1}, Sz_{2n}^i)) \leq 0. \end{aligned}$$

or

$$\begin{aligned} &F(\sigma(Sz_{2n+1}^i, Tz_{2n+1}^{i+1}), \sigma(Sz_{2n}^i, Sz_{2n}^{i+1}), \sigma(Sz_{2n}^i, Tz_{2n+1}^i), \sigma(Sz_{2n}^{i+1}, Sz_{2n+1}^{i+1}), \\ &\qquad \sigma(Sz_{2n}^i, Sz_{2n+1}^{i+1}), \sigma(Sz_{2n}^{i+1}, Sz_{2n}^i)) \leq 0. \end{aligned} \tag{3.23}$$

Taking lim as $n \to \infty$ and using $\lim_{n\to\infty} \sigma_{2n}^i = \sigma$, we get

$$F(\sigma, \sigma, 0, 0, \sigma, \sigma) \leq 0.$$

Which is contradiction and hence

$$\lim_{n\to\infty} \sigma_{2n}^i = \sigma = 0.$$

The same, if $(Sz_{2n}^{i+1}, Sz_{2n+1}^i) \in \mathcal{R}$, we have

$$\lim_{n\to\infty} \sigma_{2n}^i = \lim_{n\to\infty} \sigma(Sz_{2n}^{i+1}, Sz_{2n+1}^i) = 0,$$

for $i \in 0, 1, 2, ...l-1$.

Using (3.21), $\lim_{n\to\infty} \sigma_{2n}^i = 0$ and (σ_3), we have

$$\begin{aligned} \sigma(\bar{a}, \bar{b}) = \sigma(Sz_{2n}^0, Sz_{2n}^i) &\leq \sum_{i=0}^{l-1} \sigma(Sz_{2n}^i, Sz_{2n}^{i+1}) \\ &\leq \sum_{i=0}^{l-1} \sigma_{2n}^i, \\ &\to 0 \ \ as\ n \to \infty. \end{aligned}$$

So that

$$\begin{aligned}\sigma(\bar{a},\bar{b}) &= 0 \Longrightarrow \\ \bar{a} &= \bar{b}.\end{aligned}$$

Therefore

$$Sx = Sy.$$

Next we show the existence of common fixed point of T and S. Let $a \in C(T,S)$, i.e., by Definition 2.3, $Ta = Sa$. Proceeding using Definition 2.10, we have

(i) commuting if, for all $a \in X$,

$$\begin{aligned}S(Ta) &= T(Sa), \\ Sa &= Ta.\end{aligned}$$

(ii) Weakly commuting if, for all $d(S(Tx), T(Sx)) \leq d(Sx, Tx)$,

$$\sigma(S(Ta), T(Sa)) \leq \sigma(Sa, Ta).$$

(iii) Compatible if

$$\begin{aligned}\lim_{n\to\infty} d(STx_{2n}, TSx_{2n}) &= 0, \\ \sigma(S(Ta), T(Sa)) &= 0, \\ \sigma(Sa, Ta) &= 0, \\ Sa &= Ta.\end{aligned}$$

Also

$$\lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Sx_{2n} = x^*,$$
$$x_{2n} = a, Ta = Sa = x^* = a.$$

(iv) Weakly compatible if, for all $S(Tx) = T(Sx)$, for every coincidence point $x \in X$.

$$S(Ta) = T(Sa).$$

If we take another point say b, let $Ta = b = Sa$, we obtain

$$\begin{aligned}S(Ta) &= T(Sa). \\ Sb &= Tb.\end{aligned}$$

So that, b is a common fixed point of T and S.

For uniqueness of common fixed point of T and S,

$$b = Sb = Sz = z.$$

Thus z is a common fixed point of T and S, we have $z = b$. Thus the prove. $\square$

From Theorem 3.1, we can deduce several corollary which appeared in the following:

Corollary 3.2. The results of Theorem 3.1 remain true for all $x, y \in X$ with $(Tx, Sy) \in \mathsf{R}$), the implicit relation (d) is replaced by one of the following:

(i)

$$\sigma(Tx, Ty) \leq k\sigma(Sx, Sy), \tag{3.24}$$

where $k \in [0, 1)$.

(ii)

$$\sigma(Tx,Ty) \leq k[\sigma(Sx,Tx)+\sigma(Sy,Ty)], \tag{3.25}$$

where $k\in[0,\frac{1}{2})$.

(iii)

$$\sigma(Tx,Ty) \leq k[\sigma(Sx,Ty)+\sigma(Sy,Tx)], \tag{3.26}$$

where $k\in[0,\frac{1}{2})$.

(iv)

$$\begin{aligned}\sigma(Tx,Ty) \leq\ & a_1\sigma(Sx,Sy)+a_2[\sigma(Sx,Tx)+\sigma(Sy,Ty)]+\\ & a_3[\sigma(Sx,Ty)+\sigma(Sy,Tx)],\end{aligned} \tag{3.27}$$

where $a_1,a_2,a_3\in[0,1)$ and $a_1+2a_2+2a_3<1$.

(v)

$$\begin{aligned}\sigma(Tx,Ty) \leq\ & kd(Sx,Sy)+L\min\{\sigma(Sx,Tx),\sigma(Sy,Ty),\\ & \sigma(Sx,Ty),\sigma(Sy,Tx)\},\end{aligned} \tag{3.28}$$

where $k\in[0,1)$ and $L\geq 0$.

(vi)

$$\begin{aligned}\sigma(Tx,Ty) \leq\ & (a_1\sigma(Sx,Sy)+a_2\sigma(Sx,Tx)+a_3\sigma(Sy,Ty)+\\ & a_4(\sigma(Sx,Ty)+\sigma(Sy,Tx)))\end{aligned} \tag{3.29}$$

where $a_1,a_2,a_3,a_4\geq 0$ and $a_1+a_2+a_3+2a_4<1$.

(vii)

$$\begin{aligned}\sigma(Tx,Ty) \leq\ & k\max\Big\{\sigma(Sx,Sy),\sigma(Sx,Tx),\sigma(Sy,Ty),\\ & \frac{\sigma(Sx,Tx)+\sigma(Sy,Ty)}{2}\Big\}+L\min\Big\{\sigma(Sx,Tx),\sigma(Sy,Ty),\\ & \sigma(Sx,Ty),\sigma(Sy,Tx)\Big\},\end{aligned} \tag{3.30}$$

where $k\in[0,1)$ and $L\geq 0$.

(viii)

$$\begin{aligned}\sigma(Tx,Ty) \leq\ & k\max\{\sigma(Sx,Sy),\sigma(Sx,Tx),\sigma(Sy,Ty),\\ & \sigma(Sx,Ty),\sigma(Sy,Tx)\},\end{aligned} \tag{3.31}$$

where $k\in[0,\frac{1}{2})$.

(ix)

$$\begin{aligned}\sigma(Tx,Ty) \leq\ & k\max\{a_1\sigma(Sx,Sy)+a_2\sigma(Sx,Tx)+a_3\sigma(Sy,Ty)+\\ & a_4\sigma(Sx,Ty)+a_5\sigma(Sy,Tx)\},\end{aligned} \tag{3.32}$$

where $a_i'^s,\geq 0$ (for $i=1,2,3,4,5$) and $\sum_{i=1}^{5}a_i<1$.

(x)

$$\sigma(Tx,Ty) \leq k\max\Big\{\sigma(Sx,Sy)+\sigma(Sx,Tx)+\sigma(Sy,Ty)+ \frac{\sigma(Sx,Ty)}{2}+\frac{\sigma(Sy,Tx)}{2}\Big\}, \tag{3.33}$$

where $k \in [0,1)$.

(xi)

$$\sigma(Tx,Ty) \leq k\max\big\{\sigma(Sx,Sy),\sigma(Sx,Tx),\sigma(Sy,Ty)\big\}+ (1-k)\big\{a\sigma(Sx,Ty)+b\sigma(Sy,Tx)\big\}, \tag{3.34}$$

where $k \in [0,1)$ and $a,b < \frac{1}{2}$.

(xii)

$$\sigma(Tx,Ty)^2 \leq \sigma(Tx,Ty)\big\{a_1\sigma(Sx,Sy),a_2\sigma(Sx,Tx),a_3\sigma(Sy,Ty)\big\}+ a_4\sigma(Sx,Ty)\sigma(Sy,Tx), \tag{3.35}$$

where $a_1 > 0$; $a_2,a_3,a_4 \geq 0$; $a_1+a_2+a_3 < 1$ and $a_1+a_4 < 1$.

(xiii)

$$\sigma(Tx,Ty)^2 \leq a_1\max\big\{\sigma(Sx,Sy)^2,\sigma(Sx,Tx)^2,\sigma(Sy,Ty)^2\big\}+ a_2\max\Big\{\sigma(Sx,Tx)\sigma(Sx,Ty),\sigma(Sy,Ty)\sigma(Sy,Tx)\Big\} -a_3\sigma(Sx,Ty)\sigma(Sy,Tx), \tag{3.36}$$

where $a_i'^s, \geq 0$ (for $i=1,2,3$); $a_1+2a_2 < 1$ and $a_1+a_4 < 1$.

(xiv)

$$\sigma(Tx,Ty)^3 \leq k\big\{\sigma(Sx,Sy)^3+\sigma(Sx,Tx)^3+\sigma(Sy,Ty)^3+ \sigma(Sx,Ty)^3+\sigma(Sy,Tx)^3\big\}, \tag{3.37}$$

where $k \in [0,\frac{1}{11})$.

Example 3.3. Consider $X=[0,2]$ endowed with complete metric-like, defined by metric $\sigma(x,y)=(x-y)^2$ in $\mathbb{R}^2$ with binary relation

$$\mathcal{R}=\{(0,0),(0,1),(0,2),(1,1),(1,2),(2,2)\} \textit{ on } X.$$

Then X is either complete or $\mathcal{R}$-complete.

Define a pair of mappings $T,S : X \longrightarrow X$ by

$$Tx = \frac{x}{2}, \forall x \in X,$$

and

$$Sx = x^2, \forall x \in X.$$

Then $TX=\{0\} \subset [0,\dfrac{1}{2}] \subseteq [0,2) = SX$.

Clearly, $\mathcal{R}$ is (T, S)-closed, and $x_0 = 0, (S0, T0) \in \mathcal{R}$.

Define continuous function $F : \mathbb{R}_+^6 \to \mathbb{R}$ by

$$F(u_1, u_2, u_3, u_4, u_5, u_6) = u_1 - \frac{1}{2}u_5 - \frac{1}{2}u_6.$$

i.e.,

$$\sigma(Tx, Ty) \leq \frac{1}{2}\sigma(Sx, Ty) + \frac{1}{2}\sigma(Sy, Tx).$$

For

$$(x, y) \in \{(0,0)(0,1), (0,2), (1,1), (1,2), (2,2)\}, \forall x, y \in \mathcal{R}.$$

$$\sigma(Tx, Ty) = 0,$$

hence obvious.

For $(x, y) \in (0, 1)$

$$\begin{aligned}
\sigma(Tx, Ty) &= \sigma(T0, T1) = \frac{1}{4}. \\
\sigma(Sx, Ty) &= \sigma(S0, T1) = \frac{1}{4}. \\
\sigma(Sy, Tx) &= \sigma(S1, T0) = 1.
\end{aligned}$$

$$\begin{aligned}
\sigma(T0, T1) &\leq \frac{1}{2}\sigma(S0, T1) + \frac{1}{2}\sigma(S1, T0). \\
\frac{1}{4} &\leq \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times 1. \\
\frac{1}{4} &\leq \frac{1}{8} + \frac{1}{2}. \\
\frac{1}{4} &\leq \frac{5}{8}.
\end{aligned}$$

For $(x, y) \in (0, 2)$

$$\begin{aligned}
\sigma(Tx, Ty) &= \sigma(T0, T2) = 1. \\
\sigma(Sx, Ty) &= \sigma(S0, T2) = 1. \\
\sigma(Sy, Tx) &= \sigma(S2, T0) = 16.
\end{aligned}$$

$$\begin{aligned}
\sigma(T0, T2) &\leq \frac{1}{2}\sigma(S0, T2) + \frac{1}{2}\sigma(S2, T0) \\
1 &\leq \frac{1}{2} \times 1 + \frac{1}{2} \times 16 \\
1 &\leq \frac{1}{2} + 8 \\
1 &\leq \frac{17}{2}.
\end{aligned}$$

For $(x, y) \in (1, 1)$

$$\begin{aligned}
\sigma(Tx, Ty) &= \sigma(T1, T1) = 0. \\
\sigma(Sx, Ty) &= \sigma(S1, T1) = \frac{1}{4}. \\
\sigma(Sy, Tx) &= \sigma(S1, T1) = \frac{1}{4}.
\end{aligned}$$

$$\begin{aligned}
\sigma(T1, T1) &\leq \frac{1}{2}\sigma(S1, T1) + \frac{1}{2}\sigma(S1, T1) \\
0 &\leq \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{4} \\
1 &\leq \frac{1}{8} + \frac{1}{8} \\
0 &\leq \frac{1}{4}.
\end{aligned}$$

For $(x, y) \in (1, 2)$

$$\begin{aligned}
\sigma(Tx, Ty) &= \sigma(T1, T2) = \frac{1}{4}. \\
\sigma(Sx, Ty) &= \sigma(S1, T2) = 0. \\
\sigma(Sy, Tx) &= \sigma(S2, T1) = \frac{49}{4}. \\
\sigma(T1, T2) &\leq \frac{1}{2}\sigma(S1, T2) + \frac{1}{2}\sigma(S2, T1) \\
\frac{1}{4} &\leq \frac{1}{2} \times 0 + \frac{1}{2} \times \frac{49}{4} \\
\frac{1}{4} &\leq 0 + \frac{49}{8} \\
\frac{1}{4} &\leq \frac{49}{8}.
\end{aligned}$$

For $(x, y) \in (2, 2)$

$$\begin{aligned}
\sigma(Tx, Ty) &= \sigma(T2, T2) = 0. \\
\sigma(Sx, Ty) &= \sigma(S2, T2) = 9. \\
\sigma(Sy, Tx) &= \sigma(S2, T2) = 9. \\
\sigma(T2, T2) &\leq \frac{1}{2}\sigma(S2, T2) + \frac{1}{2}\sigma(S2, T2) \\
0 &\leq \frac{1}{2} \times 9 + \frac{1}{2} \times 9 \\
0 &\leq \frac{9}{2} + \frac{9}{2} \\
0 &\leq 9.
\end{aligned}$$

Which shows that all assertion of Theorem 3.1 is satisfied. Hence $x = 0$ is a fixed point of T. Furthermore, using Equation 3.1, we deduce an implicit function as shown below

$$F(\sigma(Tx, Ty), \sigma(Sx, Sy), \sigma(Sx, Tx),$$
$$\sigma(Sy, Ty), \sigma(Sx, Ty), \sigma(Sy, Tx)) = \sigma(Tx, Ty) - \frac{1}{2}[\sigma(Sx, Ty) + \sigma(Sy, Tx)]$$

$$\begin{aligned}
&= \sigma(Tx, Ty) - \frac{1}{2}[\sigma(Sx, Ty) + \sigma(Sy, Tx)] \\
&= (\frac{x}{2}, \frac{y}{2}) - \frac{1}{2}[\sigma(x^2, \frac{y}{2}) + \sigma(y^2, \frac{x}{2})], \\
&= (\frac{x}{2} - \frac{y}{2})^2 - \frac{1}{2}[(x^2 - \frac{y}{2})^2 + (y^2 - \frac{x}{2})^2], \\
&= \frac{1}{4}x^2 - \frac{xy}{2} + \frac{y^2}{4} - \frac{1}{2}x^4 + \frac{x^2y}{2} - \frac{y^2}{8} - \frac{y^4}{2} + \frac{xy^2}{2} - \frac{x^2}{8}, \\
&= \frac{1}{8}x^2 + \frac{1}{8}y^2 + \frac{x^2y}{2} + \frac{xy^2}{2} - \frac{xy}{2} - \frac{x^4}{2} - \frac{y^4}{2}, \\
&= \frac{1}{8}[x^2 + y^2 + 4x^2y + 4xy^2 - 4xy - 4x^4 - 4y^4].
\end{aligned}$$

which is the implicit function satisfies Theorem 3.1.

4 Some applications

We will use our results proved in previous section to solve the second order differential nonlinear two boundary value problem and the existence of solution for Volterra-Fredholm type integral equation.

4.1 The existence of solution for the second-order differential non-linear two boundary value problem.

In this subsection, we consider a second-order differential non-linear two boundary value problem. The following problem motivated by [2, 20, 30].

$$\begin{cases} u^{''}(t) = f(t, u(t), u'(t)),\ t \in (0,1), \\ u(a) = u_1, \\ u(b) = u_2,\ a, b \in [0,1], \end{cases} \tag{4.1}$$

where $f : [0,1] \times X \times X \longrightarrow X$ is a continuous function.

This problem is equivalent to the integral equation

$$u(t) \quad = \quad h(t) + \int_a^b G(t,s) f(s, u(s), u'(s)) ds, \forall\ t, s \in [a,b], \tag{4.2}$$

where the Green's function associated with the above integral equation is given by

$$G(t,s) = \begin{cases} \dfrac{(b-t)(s-a)}{b-a}, & a \le s \le t \le b, \\ \dfrac{(b-s)(t-a)}{b-a}, & a \le t \le s \le b, \end{cases}$$

and $h(t)$ satisfies $h'' = 0$, $h(a) = u_1$, $h(b) = u_2$.

By Theorem 3.1 (a), $TX \subseteq SX$. The fixed point of S is also a solution of (4.1).

Now we prove our results by establishing the existence of a common fixed point for a pair of self mappings:

Theorem 4.1. Let $T, S : C([a,b]) \longrightarrow C([a,b])$ be self maps of a metric-like space (X, σ) such that the following condition holds:

(1) f: $[0,1] \times X \times X \longrightarrow X$ is a nonincreasing function in the fifth and sixth variable,

(2) There exist a functions $f : [0,1] \times X \times X \longrightarrow X$ with constants α and β such that

$$|f(t, u(t), v'(t))| - |f(t, u(t), v'(t))| \quad \le \quad L|\alpha|u - v| + \beta|u' - v'||,$$

for all $t \in [0,1]$ and $u, v \in C^1([a,b], X)$,

(3) there exists a path $a, b \in [0,1]$ and $\alpha, \beta > 0$ such that

$$k = \frac{\alpha + 4\beta}{8}, k \in L \textit{ and } L \le 1.$$

Then, the non linear integral equation has a common solution in $C^1([a,b], X)$ and (4.2) has a solution. Also it is the solution of differential equation (4.1).

Proof: Consider $C^1([a,b], X)$ with the metric

$$\sigma(x,y) = \max_{a \le t \le b} \left\{ \alpha|u - v| + \beta|u' - v'| \right\}.$$

The (X, σ) is a complete metric-like space.

Let $T, S : X \longrightarrow X$ be two operator defined as

$$Tu(t) \quad = \quad h(t) + \int_a^b G(t,s) f(s, u(s), u'(s)) ds, \forall\ t, s \in [a,b],$$

and

$$Su(t) \quad = \quad h(t) + \int_a^b G(t,s)f(s,u(s),v'(s))ds, \forall\ t,s \in [a,b],$$

where f and h are continuous functions. Now, u is a solution of (4.2) if and only if u is a common fixed point of T and S. Since T and S are increasing in the fifth and sixth variables and other assertion of Theorem 3.1 are satisfied.

By using condition (3) of Theorem 4.1 we obtain

$$\begin{aligned} |Tu(t) - Sv(t)| &= \int_a^b |G(t,s)||f(s,u(s),u'(s)) - f(s,v(s),v'(s))|ds, \\ &= \int_a^b |G(t,s)|ds(\alpha|u-v| + \beta|u'-v'|), \\ &= L\sigma(u,v)\int_a^b |G(t,s)|ds. \end{aligned}$$

For each $a,b \in [0,1]$, we have

$$\int_a^b G(t,s)ds \quad = \quad \max_{a\le t\le b} \frac{(b-a)(b-a)}{8} = \frac{(b-a)^2}{8},$$

$$\alpha|u(t)-v(t)| \quad = \quad \alpha\frac{(b-a)^2}{8}\sigma(u,v) = \frac{\alpha}{8}\sigma(u,v), \tag{4.3}$$

Similarly,

$$\begin{aligned} |Tu'(t) - Sv'(t)| &= \int_a^b |G(t,s)||f(s,u(s),u'(s)) - f(s,v(s),v'(s))|ds, \\ &= L\int_a^b |G(t,s)|ds(\alpha|u(t)-v(t)| + \beta|u'(t)-v'(t)|), \\ &= L\sigma(u,v)\int_a^b |G(t,s)|ds, \end{aligned}$$

$$\int_a^b G(t,s)ds \quad = \quad \max_{a\le t\le b} \frac{(b-a)^2(b-a)^2}{2(b-a)} = \frac{b-a}{2},$$

$$\beta|u'-v'| \quad = \quad \beta\frac{(b-a)}{2}\sigma(u',v') = \frac{\beta}{2}\sigma(u',v'), \tag{4.4}$$

From adding (4.3) and (4.4) we obtain

$$\begin{aligned} \sigma(Tu,Sv) &= \frac{(b-a)^2}{8}\sigma(u,v) + \frac{(b-a)}{2}\sigma(u,v), \\ &= \left[\frac{\alpha}{8} + \frac{\beta}{2}\right]\sigma(u,v). \end{aligned}$$

Since

$$k \quad = \quad \frac{\alpha+4\beta}{8} and\ k \in L < 1,$$

we have

$$\sigma(Tu,Sv) \quad = \quad L\sigma(u,v).$$

Therefore $u \in X$, hence u is a common fixed of T and S, also a solution to integral equation (4.2). Thus, a differential equation (4.1) has a solution.

4.2 Existence of the solution of Volterra-Fredholm type integral equation

Now, in this subsection, we investigate the existence of the solution to the Volterra-Fredholm type integral equation, which is used to illustrate the use of Theorem 3.1 for the existence of a common fixed point of a pair of maps in metric space. The following integral equation inspired by [30, 3]. The equation arise from the theory of parabolic boundary valued problems, which is the mathematical modelings of the spatiotemporal development of epidemic and various physical and biological models.

$$u(t,x) = h(t,x) + \int_0^t \int_{\mathbb{R}^2} K(t,x,s,y,u(s,y))dyds, \forall\ t,x \in D, \tag{4.5}$$

where $h : D \to \mathbb{R}^{\mathbb{N}}$, K : D $\times D \to \mathbb{R}^{\mathbb{N}}$, $D = [0,T] \times \Omega, T > 0$ and $\Omega = \mathbb{R}^{\mathbb{N}}$ is the non empty and closed set of Euclidean space $\mathbb{R}^{\mathbb{N}}$ eqquiped with norm $\|.\|$, $\forall$ $\mathbb{N} \geq 1$.

Let $(X, \|.\|)$ be a Banach space. Define the mapping $d : X \times X \to [0,\infty)$ by

$$d(x,y) = \|x - y\|.$$

Then (X, d) is a complete metric space.

By Theorem 3.1 (a), $TX \subseteq SX$. The fixed point of S is also a solution of (4.5).

Now we prove our results by establishing the existence of a common fixed point for a pair of self mappings:

Theorem 4.2. Let $T, S : C^{\mathbb{N}}([a,b]) \longrightarrow C^{\mathbb{N}}([a,b])$ be self maps of a metric-like space (X,σ). Suppose the following assumptions hold:

(i) the function $h :\to \mathbb{R}_+$ and $K : D \times D \times \mathbb{R}_+ \longrightarrow X$ are continuous,

(ii) there exist a continuous function $L : D \times D \to [0,\infty)$ such that

$$\|K(t,x,s,y,u(s,y)) - K(t,x,s,y,v(s,y))\| \leq L(t,x,s,y)\|u - v\|,$$

for all $t,x,s,y,u(s,y) \in D \times D \times \mathbb{R}^{\mathbb{N}}$,

(iii) there exists a path $a, b \in [0,1]$ with a constant $\gamma \in [0,1)$ such that

$$\int_0^t \int_{\mathbb{R}^2} L(t,x,s,y)\|u - v\|dyds \leq \frac{1+t}{6+7t^2}[\ln(1 + \frac{1}{3}|x|) - \ln(1 + \frac{1}{3}|x|)],$$

where

$$L(t,x,s,y) = \gamma = \frac{1+t}{6+7t^2} < 1.$$

Then, the Volterra-Fredholm integral equation (4.5) has a unique common solution in $C^{\mathbb{N}}([a,b],X)$.

Proof . Consider $C^{\mathbb{N}}([a,b],X)$ with the metric

$$\sigma(x,y) = \|x - y\|.$$

The (X,σ) is a complete metric-like space.

Let $T, S : X \to S$ be two operators such that S$\in X$ and $u \in S$. Defined as

$$Tu(t,x) = h(t,x) + \int_0^t \int_{\mathbb{R}^2} K(t,x,s,y,u(s,y))dyds, \forall\ t,x \in D. \tag{4.6}$$

Since $TX \subset SX$,we prove that T maps S into itself. So, suppose that $Tu : [0,T] \times \mathbb{R} \to \mathbb{R}^2$ is continuous mapping. Now, on contrary to that, we claim that $T : S \to S$ is not a contraction. So, let (u,v) be a pair of elements in S. For all $(t,x) \in D$ and using condition (ii) of Theorem 4.1 we get

$$\begin{aligned}
\|Tu - Sv\| &= \int_0^t \int_{\mathbb{R}^2} \|K(t,x,s,y,u(s,y)) - K(t,x,s,y,v(s,y))\| dyds, \\
&\leq \int_0^t \int_{\mathbb{R}^2} L(t,x,s,y)\|u - v\| dyds, \\
&\leq \frac{1+t}{6+7t^2}\left[\ln(1+\frac{1}{3}\|u\|) - \ln(1+\frac{1}{3}\|v\|)\right], \\
&\leq \frac{1+t}{6+7t^2}\ln\left[\frac{1+\frac{1}{3}\|u\|}{1+\frac{1}{3}\|u\|}\right], \\
&\leq \frac{1+t}{6+7t^2}\ln\left[1+\frac{\frac{1}{3}\|u\| - \frac{1}{3}\|v\|}{1+\frac{1}{3}\|v\|}\right], \\
&\leq \gamma\|u - v\|, \\
\|Tu - Sv\| &\leq \gamma\|u - v\|, \\
\sigma(Tu, Sv) &\leq \gamma\sigma(u,v),
\end{aligned}$$

which is a contradiction. Hence u is a common fixed of T and S, also a solution to integral equation (4.5). □

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