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Some coefficient problems of a class of close-to-star functions of type α defined by means of a generalized differential operator

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Abstract

In this investigation, we studied a class of Bazilevič type close-to-star functions which is defined by a generalized differential operator. The new class generalizes many known and new subclasses of close-to-star functions. Some of the investigated properties are the coefficient bounds and the Fekete-Szegö functional. Our results extend some known and new ones.

Keywords: Analytic function, univalent function, starlike function, close-to-star function of type α, Carathéodory function, Opoola differential operator, coefficient bounds and Fekete-Szegö functional
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1 Introduction and Definitions

In this paper, we let $\mathcal{E} := \{z \in \mathbb{C} : |z| < 1\}$ represent the unit disk and we let $\mathcal{A}$ represent the class of complex-valued functions of the form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots + a_j z^j + \cdots \quad (z \in \mathcal{E}) \tag{1.1}$$

such that $f(0) = 0 = f'(0) - 1$. $\mathcal{A}$ is called the class of *normalized analytic* functions. Let $\mathcal{S}$, a subset of $\mathcal{A}$, represent the class of functions *analytic* and *univalent* in $\mathcal{E}$.

Also, let $\mathcal{S}^\star$, a subset of $\mathcal{S}$, represent the class of functions of the form:

$$s(z) = z + s_2 z^2 + s_3 z^3 + \cdots + s_j z^j + \cdots \quad (z \in \mathcal{E}) \tag{1.2}$$

such that $\mathcal{R}e(zs'(z)/s(z)) > 0$. The class $\mathcal{S}^\star$ is called the class of *starlike* functions.

Let $\mathcal{C}$ represent the class of analytic functions of the form:

$$c(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots + c_j z^j + \cdots \quad (z \in \mathcal{E}) \tag{1.3}$$

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such that $c(0) = 1$ and $\mathcal{R}e\, c(z) > 0$. The class $\mathcal{C}$ is called the class of *Carathéodory* functions. For additional details on classes $\mathcal{S}$, $\mathcal{S}^\star$ and $\mathcal{C}$, see [9, 21].

A function f in (1.1) is called *close-to-star* if there is a function $s(z) \in \mathcal{S}^\star$ such that

$$\mathcal{R}e\left(\frac{f(z)}{s(z)}\right) > 0 \quad (z \in \mathcal{E}). \tag{1.4}$$

The class of close-to-star was introduced by Reade [15] and it has been studied in various forms by some researchers in [3, 12, 18, 19, 22]. In particular, Babalola et al. [5] studied the class of close-to-star functions of type α defined by using the idea of Bazilevič functions of type α in [20].

Definition 1.1 ([5]). A function $f \in \mathcal{A}$ is called close-to-star of type α if

$$f(z) \in \mathcal{K}^\star(\alpha) := \left\{ f(z) \in \mathcal{A} : \mathcal{R}e\left(\frac{f(z)^\alpha}{s(z)^\alpha}\right) > 0,\ \alpha > 0,\ z \in \mathcal{E} \right\} \tag{1.5}$$

and all powers are regarded as principal determinations only.

Remark 1.2. The following are some remarks on $\mathcal{K}^\star(\alpha)$.

(i) $\mathcal{K}^\star(1)$ is the class introduced by Reade [15].
(ii) $f \in \mathcal{K}^\star(1)$ is not necessarily univalent in $\mathcal{E}$ (see [5, 15]), so also is $f \in \mathcal{K}^\star(\alpha)$.
(iii) The well-known Alexander duality theorem holds. If f is close-to-convex (see [21] for definition), then zf' is close-to-star (see [5, 15]).
(iv) Every starlike function is close-to-star (see [5, 15]).
(v) $\mathcal{R}e(f'(z)/s'(z) > 0 \implies \mathcal{R}e(f(z)/s(z)) > 0$ (see [5, 15, 16]).
(vi) If $f(z) \in \mathcal{K}^*(1)$, then $|a_j| \leqslant j^2$ ($j = \{2, 3, 4, \dots\}$) (see [15, 16]). Equality occurs for function $f_\delta(z) = (z + \delta z^2)/(1 - \delta z)^3$ ($\delta = \pm i$) (see [16]).

An interesting aspect of coefficient problems of functions f of form (1.1) is the study of the Fekete-Szegö functional defined by

$$\mathcal{F}(\gamma; f) = |a_3 - \gamma a_2^2|. \tag{1.6}$$

The functional was initiated by Fekete and Szegö [8] and it has received much attention for functions f in class $\mathcal{S}$ and its various subclasses. Some recent investigations in this direction can be found in [1, 4, 7, 11].

In 2017, Opoola [14] introduced a differential operator defined as follows.

Definition 1.3. Let $f \in \mathcal{A}$ be of the form (1.1), then the Opoola differential operator $\mathcal{D}^{n,\beta,\eta,\tau} : \mathcal{A} \longrightarrow \mathcal{A}$ is defined by

$$\begin{aligned}
\mathcal{D}^{0,\beta,\eta,\tau} f(z) &= f(z) \\
\mathcal{D}^{1,\beta,\eta,\tau} f(z) &= (1 + (\beta - \eta - 1)\tau) f(z) - z\tau(\beta - \eta) + z\tau f'(z) = \Delta_\tau f(z) \\
\mathcal{D}^{2,\beta,\eta,\tau} f(z) &= \Delta_\tau(\mathcal{D}^{1,\beta,\eta,\tau} f(z)) \\
\mathcal{D}^{3,\beta,\eta,\tau} f(z) &= \Delta_\tau(\mathcal{D}^{2,\beta,\eta,\tau} f(z)) \\
\vdots \qquad & \vdots \qquad \vdots \\
\mathcal{D}^{n,\beta,\eta,\tau} f(z) &= \Delta_\tau(\mathcal{D}^{n-1,\beta,\eta,\tau} f(z))
\end{aligned}$$

which implies that

$$\mathcal{D}^{n,\beta,\eta,\tau} f(z) = z + \sum_{j=2}^{\infty} (1 + (j + \beta - \eta - 1)\tau)^n a_j z^j \tag{1.7}$$

or for brevity,

$$\mathcal{D}^{n,\beta,\eta,\tau} f(z) = z + \sum_{j=2}^{\infty} \Lambda_j a_j z^j \quad (z \in \mathcal{E}) \tag{1.8}$$

where $\Lambda_j = (1 + (j + \beta - \eta - 1)\tau)^n$, $\tau \geqslant 0$, $0 \leqslant \eta \leqslant \beta$ and $n \in \mathbb{N} \cup \{0\}$.

Remark 1.4. The following properties hold in (1.7).

(i) $\mathcal{D}^{0,\beta,\eta,\tau}f(z) = \mathcal{D}^{n,\beta,\eta,0}f(z) = \mathcal{D}^{0,\beta,\eta,0}f(z) = f(z) \in \mathcal{A}$ in (1.1).

(ii) $\mathcal{D}^{n,\beta,\beta,1}f(z) = \mathcal{D}^{n,\eta,\eta,1}f(z) = \mathcal{D}^{n}f(z)$ is the Sălăgean differential operator introduced in [17].

(iii) $\mathcal{D}^{n,\beta,\beta,\tau}f(z) = \mathcal{D}^{n,\eta,\eta,\tau}f(z) = \mathcal{D}^{n,\tau}f(z)$ is the Al-Oboudi differential operator introduced in [2].

The new class investigated in this paper is therefore defined as follows.

Definition 1.5. If $n \in \mathbb{N} \cup \{0\}$, $0 \leqslant \eta \leqslant \beta$ and $\tau \geqslant 0$, then a function $f(z) \in \mathcal{A}$ is said to be a member of class $\mathcal{K}^\star(n, \beta, \eta, \tau; \alpha)$ if it satisfies the geometric condition

$$\mathcal{R}e\left(\frac{(\mathcal{D}^{n,\beta,\eta,\tau}f(z))^\alpha}{(\mathcal{D}^{n,\beta,\eta,\tau}s(z))^\alpha}\right) > 0 \quad (\alpha \geqslant 1,\ z \in \mathcal{E}) \tag{1.9}$$

where $\mathcal{D}^{n,\beta,\eta,\tau}$ is the Opoola differential operator in (1.7) and all powers are regarded as principal determinations only.

Remark 1.6. Note that $\mathcal{K}^\star(n, \beta, \eta, 0; \alpha) = \mathcal{K}^\star(0, \beta, \eta, \tau; \alpha) = \mathcal{K}^\star(0, \beta, \eta, 0; \alpha) = \mathcal{K}^\star(\alpha)$ is the class studied by Babalola et al. [5]. And that $\mathcal{K}^\star(n, \beta, \eta, 0; 1) = \mathcal{K}^\star(0, \beta, \eta, \tau; 1) = \mathcal{K}^\star(0, \beta, \eta, 0; 1) = \mathcal{K}^\star(1)$ is the class studied by Reade [15].

It is interesting to know that class $\mathcal{K}^\star(n, \beta, \eta, \tau; \alpha)$ is non-empty as shown in the following examples.

Example 1.7. *If $s(z) = z$, then for $f(z) \in \mathcal{A}$ of the form* (1.1),

$$\begin{aligned}
\mathcal{D}^{n,\beta,\eta,\tau}f_1(z) &= \{z^\alpha(1+z)\}^{\frac{1}{\alpha}} = z + \lambda(\alpha,1)z^2 + \lambda(\alpha,2)z^3 + \lambda(\alpha,3)z^4 + \lambda(\alpha,4)z^5 + \cdots,\\
\mathcal{D}^{n,\beta,\eta,\tau}f_2(z) &= \{z^\alpha(1-z)\}^{\frac{1}{\alpha}} = z - \lambda(\alpha,1)z^2 + \lambda(\alpha,2)z^3 - \lambda(\alpha,3)z^4 + \lambda(\alpha,4)z^5 - \cdots,\\
\mathcal{D}^{n,\beta,\eta,\tau}f_3(z) &= \{z^\alpha(1+z)^{-1}\}^{\frac{1}{\alpha}} = z + \lambda(\alpha,1)z^2 + \lambda(\alpha,2)z^3 + \lambda(\alpha,3)z^4 + \lambda(\alpha,4)z^5 + \cdots,\\
\mathcal{D}^{n,\beta,\eta,\tau}f_4(z) &= \{z^\alpha(1-z)^{-1}\}^{\frac{1}{\alpha}} = z - \lambda(\alpha,1)z^2 + \lambda(\alpha,2)z^3 - \lambda(\alpha,3)z^4 + \lambda(\alpha,4)z^5 - \cdots,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{D}^{n,\beta,\eta,\tau}f_5(z) = \left\{z^\alpha\frac{1+z}{1-z}\right\}^{\frac{1}{\alpha}} &= z + 2\lambda(\alpha,1)z^2 + [2\lambda(\alpha,1) + 4\lambda(\alpha,2)]z^3 + [2\lambda(\alpha,1) + 8\lambda(\alpha,2) + 8\lambda(\alpha,3)]z^4\\
&\quad + [2\lambda(\alpha,1) + 12\lambda(\alpha,2) + 24\lambda(\alpha,3) + 16\lambda(\alpha,4)]z^5 + \cdots
\end{aligned}$$

are in class $\mathcal{K}^\star(n, \beta, \eta, \tau; \alpha)$ such that for $m \in \mathbb{N}$,

$$\lambda(\alpha, m) = \begin{cases} \frac{1}{m!}\prod_{t=0}^{m-1}\left(\frac{1}{\alpha} - t\right) & for\ power\quad \frac{1}{\alpha} > 0\\[2ex] \frac{(-1)^m}{m!}\prod_{t=0}^{m-1}\left(\frac{1}{\alpha} + t\right) & for\ power\quad \frac{1}{\alpha} < 0. \end{cases}$$

Proof . From (1.9), we can say that

$$\frac{(\mathcal{D}^{n,\beta,\eta,\tau}f_k(z))^\alpha}{(\mathcal{D}^{n,\beta,\eta,\tau}s(z))^\alpha} = \begin{cases} 1+z & for\quad k=1\\ 1-z & for\quad k=2\\ \frac{1}{1+z} & for\quad k=3\\ \frac{1}{1-z} & for\quad k=4\\ \frac{1+z}{1-z} & for\quad k=5 \end{cases} \tag{1.10}$$

where all functions on the RHS of (1.10) are well-known functions in class $\mathcal{C}$ defined above. Now $s(z) = z$, implies that $(\mathcal{D}^{n,\beta,\eta,\tau}s(z))^\alpha = z^\alpha$ and $(\mathcal{D}^{n,\beta,\eta,\tau}f_k(z))^\alpha$ $(k = \{1, 2, 3, 4, 5\})$ in (1.10) give the functions in Examples 1.7 by simple calculation. This completes the proof. □

2 Relevant Lemmas

Lemma 2.1 ([21]). If $c(z) \in \mathcal{C}$, then $|c_j| \leqslant 2 \quad (j \in \mathbb{N})$.

Lemma 2.2 ([6]). If $c(z) \in \mathcal{C}$ and $u \in \mathbb{R}$, then

$$\left|c_2 - u\frac{c_1^2}{2}\right| \leqslant \begin{cases} 2(1-u) & for \quad u \leqslant 0, \\ 2 & for \quad 0 \leqslant u \leqslant 2, \\ 2(u-1) & for \quad u \geqslant 2. \end{cases}$$

Lemma 2.3 ([13]). If $c(z) \in \mathcal{C}$ and $i, j \in \mathbb{N}$, then $|c_{i+j} - uc_ic_j| \leqslant 2$ for $0 \leqslant u \leqslant 1$.

Lemma 2.4 ([21]). If $s(z) \in \mathcal{S}^\star$, then $|s_j| \leqslant j \quad (j \in \mathbb{N} \setminus \{1\})$.

Lemma 2.5 ([10]). If $s(z) \in \mathcal{S}^\star$ and $\rho \in \mathbb{R}$, then

$$|s_3 - \rho s_2^2| \leqslant \begin{cases} 3 - 4\rho & for \quad \rho \leqslant \frac{1}{2} \\ 1 & for \quad \frac{1}{2} \leqslant \rho \leqslant 1 \\ 4\rho - 3 & for \quad \rho \geqslant 1. \end{cases}$$

3 Main Results

In what follows, let $n \in \mathbb{N} \cup \{0\}$, $0 \leqslant \eta \leqslant \beta$, $\tau \geqslant 0$ and $\alpha \geqslant 1$ throughout this work unless otherwise mentioned. The following theorems are the results obtained.

Theorem 3.1. If $f(z) \in \mathcal{K}^\star(n, \beta, \eta, \tau; \alpha)$, then

$$|a_2| \leqslant \frac{2}{\alpha\Lambda_2}\left\{1 + \alpha\Lambda_2\right\}, \tag{3.1}$$

$$|a_3| \leqslant \frac{2}{\alpha\Lambda_3}\left\{1 + 2\Lambda_2 + \frac{3\alpha\Lambda_3}{2}\right\}, \tag{3.2}$$

$$|a_4| \leqslant \frac{2}{\alpha\Lambda_4}\left\{1 + 2\Lambda_2 + 3\Lambda_3 + 2\alpha\Lambda_4 + \frac{2(\alpha-1)(2\alpha-1)}{3\alpha^2}\right\}, \tag{3.3}$$

$$\begin{aligned}|a_5| \leqslant \frac{2}{\alpha\Lambda_5}\Bigg\{&1 + 2\Lambda_2 + 3\Lambda_3 + 2(\alpha-1)(2\alpha-1) + 2[2\Lambda_4 + 3(\alpha-1)|4-3\alpha|\Lambda_2\Lambda_3] \\ &+ \frac{4(\alpha-1)|\alpha-2||3\alpha-4|\Lambda_2}{3} + \frac{(\alpha-1)}{\alpha} + \frac{\alpha|\alpha-2|}{4}[16\Lambda_2\Lambda_4 + 9\Lambda_3^2] + \frac{5\alpha\Lambda_5}{2}\Bigg\}.\end{aligned} \tag{3.4}$$

Proof . The geometric expression in (1.9) can be expressed as

$$\frac{(\mathcal{D}^{n,\beta,\eta,\tau}f(z))^\alpha}{z^\alpha} = \frac{(\mathcal{D}^{n,\beta,\eta,\tau}s(z))^\alpha}{z^\alpha}c(z). \tag{3.5}$$

Using (1.1), (1.8) and binomially expanding LHS of (3.5) we obtain

$$\begin{aligned}\frac{(\mathcal{D}^{n,\beta,\eta,\tau}f(z))^\alpha}{z^\alpha} = 1 &+ \alpha\Lambda_2a_2z + \left\{\alpha\Lambda_3a_3 + \frac{\alpha(\alpha-1)}{2!}\Lambda_2^2a_2^2\right\}z^2 \\ &+ \left\{\alpha\Lambda_4a_4 + \frac{\alpha(\alpha-1)}{2!}2\Lambda_2\Lambda_3a_2a_3 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}\Lambda_2^3a_2^3\right\}z^3 \\ &+ \Bigg\{\alpha\Lambda_5a_5 + \frac{\alpha(\alpha-1)}{2!}(2\Lambda_2\Lambda_4a_2a_4 + \Lambda_3^2a_3^2) + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}3\Lambda_2^2\Lambda_3a_2^2a_3 \\ &+ \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{4!}\Lambda_2^4a_2^4\Bigg\}z^4 \\ &+ \cdots .\end{aligned} \tag{3.6}$$

And using (1.2), (1.3), (1.8) and binomially expanding RHS of (3.5) we obtain

$$\begin{aligned}\frac{(\mathcal{D}^{n,\beta,\eta,\tau}s(z))^{\alpha}}{z^{\alpha}}c(z) &= 1+\{c_1+\alpha\Lambda_2 s_2\}z+\left\{c_2+\alpha\Lambda_2 s_2 c_1+\alpha\Lambda_3 s_3+\frac{\alpha(\alpha-1)}{2!}\Lambda_2^2 s_2^2\right\}z^2\\
&+\left\{c_3+\alpha\Lambda_2 s_2 c_2+\left(\alpha\Lambda_3 s_3+\frac{\alpha(\alpha-1)}{2!}\Lambda_2^2 s_2^2\right)c_1\right.\\
&\quad\left.+\frac{\alpha(\alpha-1)}{2!}2\Lambda_2\Lambda_3 s_2 s_3+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}\Lambda_2^3 s_2^3+\alpha\Lambda_4 s_4\right\}z^3\\
&+\left\{c_4+\alpha\Lambda_2 s_2 c_3+\left(\alpha\Lambda_3 s_3+\frac{\alpha(\alpha-1)}{2!}\Lambda_2^2 s_2^2\right)c_2\right.\\
&\quad+\left(\alpha\Lambda_4 s_4+\frac{\alpha(\alpha-1)}{2!}2\Lambda_2\Lambda_3 s_2 s_3+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}\Lambda_2^3 s_2^3\right)c_1\\
&\quad+\frac{\alpha(\alpha-1)}{2!}(2\Lambda_2\Lambda_4 s_2 s_4+\Lambda_3^2 s_3^2)+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}(3\Lambda_2^2\Lambda_3 s_2^2 s_3)\\
&\quad\left.+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{4!}\Lambda_2^4 s_2^4+\alpha\Lambda_5 s_5\right\}z^4\\
&+\cdots. \end{aligned} \tag{3.7}$$

Now, if we equate the coefficients in (3.6) and (3.7), then we obtain

$$\alpha\Lambda_2 a_2 = c_1+\alpha\Lambda_2 s_2, \tag{3.8}$$

$$\alpha\Lambda_3 a_3+\frac{\alpha(\alpha-1)}{2!}\Lambda_2^2 a_2^2 = c_2+\alpha\Lambda_2 s_2 c_1+\alpha\Lambda_3 s_3+\frac{\alpha(\alpha-1)}{2!}\Lambda_2^2 s_2^2, \tag{3.9}$$

$$\begin{aligned}\alpha\Lambda_4 a_4+\frac{\alpha(\alpha-1)}{2!}2\Lambda_2\Lambda_3 a_2 a_3+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}\Lambda_2^3 a_2^3 &= c_3+\alpha\Lambda_2 s_2 c_2\\
+\left(\alpha\Lambda_3 s_3+\frac{\alpha(\alpha-1)}{2!}\Lambda_2^2 s_2^2\right)c_1&+\frac{\alpha(\alpha-1)}{2!}2\Lambda_2\Lambda_3 s_2 s_3+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}\Lambda_2^3 s_2^3+\alpha\Lambda_4 s_4\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\alpha\Lambda_5 a_5&+\frac{\alpha(\alpha-1)}{2!}(2\Lambda_2\Lambda_4 a_2 a_4+\Lambda_3^2 a_3^2)+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}3\Lambda_2^2\Lambda_3 a_2^2 a_3+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{4!}\Lambda_2^4 a_2^4\\
&= c_4+\alpha\Lambda_2 s_2 c_3+\left(\alpha\Lambda_3 s_3+\frac{\alpha(\alpha-1)}{2!}\Lambda_2^2 s_2^2\right)c_2\\
&+\left(\alpha\Lambda_4 s_4+\frac{\alpha(\alpha-1)}{2!}2\Lambda_2\Lambda_3 s_2 s_3+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}\Lambda_2^3 s_2^3\right)c_1\\
&+\frac{\alpha(\alpha-1)}{2!}(2\Lambda_2\Lambda_4 s_2 s_4+\Lambda_3^2 s_3^2)+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}3\Lambda_2^2\Lambda_3 s_2^2 s_3+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{4!}\Lambda_2^4 s_2^4+\alpha\Lambda_5 s_5.\end{aligned} \tag{3.11}$$

Simple calculation shows that (3.8) leads to

$$a_2=\frac{1}{\alpha\Lambda_2}c_1+s_2 \tag{3.12}$$

so that by applying triangle inequality and Lemmas 2.1 and 2.4 in (3.12) lead to inequality (3.1).

Using (3.12) in (3.9) leads to

$$a_3=\frac{1}{\alpha\Lambda_3}c_2+\frac{\Lambda_2 s_2}{\alpha\Lambda_3}c_1-\frac{\alpha-1}{2\alpha^2\Lambda_3}c_1^2+s_3 \tag{3.13}$$

so that by applying triangle inequality we obtain

$$|a_3|=\left|\frac{1}{\alpha\Lambda_3}c_2+\frac{\Lambda_2 s_2}{\alpha\Lambda_3}c_1-\frac{\alpha-1}{2\alpha^2\Lambda_3}c_1^2+s_3\right|\leqslant\frac{1}{\alpha\Lambda_3}\left\{\left|c_2-\frac{\alpha-1}{\alpha}\frac{c_1^2}{2}\right|+\Lambda_2|s_2||c_1|+\alpha\Lambda_3|s_3|\right\}$$

and using Lemmas 2.1, 2.2 and 2.4 we obtain (3.2).

Using (3.12) and (3.13) in (3.10) leads to

$$a_4 = \frac{1}{\alpha\varLambda_4}c_3 + \frac{\varLambda_2 s_2}{\alpha\varLambda_4}c_2 + \frac{\varLambda_3 s_3}{\alpha\varLambda_4}c_1 - \frac{\alpha-1}{\alpha^2\varLambda_4}c_1c_2 - \frac{(\alpha-1)\varLambda_2 s_2}{2\alpha^2\varLambda_4}c_1^2 + \frac{(\alpha-1)(2\alpha-1)}{6\alpha^3\varLambda_4}c_1^3 + s_4 \tag{3.14}$$

so that by applying triangle inequality,

$$\begin{aligned}|a_4| &= \left|\frac{1}{\alpha\varLambda_4}c_3 + \frac{\varLambda_2 s_2}{\alpha\varLambda_4}c_2 + \frac{\varLambda_3 s_3}{\alpha\varLambda_4}c_1 - \frac{\alpha-1}{\alpha^2\varLambda_4}c_1c_2 - \frac{(\alpha-1)\varLambda_2 s_2}{2\alpha^2\varLambda_4}c_1^2 + \frac{(\alpha-1)(2\alpha-1)}{6\alpha^3\varLambda_4}c_1^3 + s_4\right| \\ &\leqslant \frac{1}{\alpha\varLambda_4}\left\{\left|c_3 - \frac{\alpha-1}{\alpha}c_1c_2\right| + \varLambda_2|s_2|\left|c_2 - \frac{\alpha-1}{\alpha}\frac{c_1^2}{2}\right| + \varLambda_3|s_3||c_1| + \frac{(\alpha-1)(2\alpha-1)}{6\alpha^2}|c_1|^3 + \alpha\varLambda_4|s_4|\right\}\end{aligned}$$

and using Lemmas 2.1, 2.2, 2.3 and 2.4 we obtain (3.3).

Using (3.12), (3.13) and (3.14) in (3.11) leads to

$$\begin{aligned}a_5 = \frac{1}{\alpha\varLambda_5}\Big\{&c_4 + \varLambda_2 s_2 c_3 + \varLambda_3 s_3 c_2 + [\varLambda_4 s_4 + (\alpha-1)(4-3\alpha)\varLambda_2\varLambda_3 s_2 s_3]c_1 \\ &- \frac{(\alpha-1)\varLambda_3 s_3}{2\alpha}c_1^2 + \frac{(\alpha-1)(\alpha-2)(3\alpha-4)\varLambda_2 s_2}{6\alpha^2}c_1^3 - \frac{(\alpha-1)(2\alpha-1)}{24\alpha^3}c_1^4 \\ &- \frac{(\alpha-1)\varLambda_2 s_2}{\alpha}c_1c_2 - \frac{(\alpha-1)}{\alpha}c_1c_3 + \frac{(\alpha-1)(2\alpha-1)}{2\alpha^2}c_1^2c_2 - \frac{(\alpha-1)}{2\alpha}c_2^2 \\ &+ \frac{\alpha(\alpha-2)}{2}[2\varLambda_2\varLambda_4 s_2 s_4 + \varLambda_3^2 s_3^2] + \alpha\varLambda_5 s_5\Big\}\end{aligned}$$

and

$$\begin{aligned}|a_5| = \frac{1}{\alpha\varLambda_5}\Bigg|&\left(c_4 - \frac{(\alpha-1)}{\alpha}c_1c_3\right) + \left(\varLambda_2 s_2 c_3 - \frac{(\alpha-1)\varLambda_2 s_2}{\alpha}c_1c_2\right) + \left(\varLambda_3 s_3 c_2 - \frac{(\alpha-1)\varLambda_3 s_3}{2\alpha}c_1^2\right) \\ &+ \left(\frac{(\alpha-1)(2\alpha-1)}{2\alpha^2}c_1^2c_2 - \frac{(\alpha-1)(2\alpha-1)^2}{24\alpha^3}c_1^4\right) + \frac{(\alpha-1)(\alpha-2)(3\alpha-4)\varLambda_2 s_2}{6\alpha^2}c_1^3 \\ &+ [\varLambda_4 s_4 + (\alpha-1)(4-3\alpha)\varLambda_2\varLambda_3 s_2 s_3]c_1 - \frac{(\alpha-1)}{2\alpha}c_2^2 \\ &+ \frac{\alpha(\alpha-2)}{2}[2\varLambda_2\varLambda_4 s_2 s_4 + \varLambda_3^2 s_3^2] + \alpha\varLambda_5 s_5\Bigg|\end{aligned}$$

so that by applying triangle inequality we obtain

$$\begin{aligned}|a_5| \leqslant \frac{1}{\alpha\varLambda_5}\Bigg\{&\left|c_4 - \frac{(\alpha-1)}{\alpha}c_1c_3\right| + \varLambda_2|s_2|\left|c_3 - \frac{(\alpha-1)}{\alpha}c_1c_2\right| + \varLambda_3|s_3|\left|c_2 - \frac{(\alpha-1)}{2\alpha}c_1^2\right| \\ &+ \frac{(\alpha-1)(2\alpha-1)}{2\alpha^2}|c_1|^2\left|c_2 - \frac{(2\alpha-1)}{6\alpha}\frac{c_1^2}{2}\right| + \frac{(\alpha-1)|\alpha-2||3\alpha-4|\varLambda_2|s_2|}{6\alpha^2}|c_1|^3 \\ &+ [\varLambda_4|s_4| + (\alpha-1)|4-3\alpha|\varLambda_2\varLambda_3|s_2||s_3|]|c_1| + \frac{(\alpha-1)}{2\alpha}|c_2|^2 \\ &+ \frac{\alpha(\alpha-2)}{2}[2\varLambda_2\varLambda_4|s_2||s_4| + \varLambda_3^2|s_3|^2] + \alpha\varLambda_5|s_5|\Bigg\}\end{aligned}$$

and using Lemmas 2.1, 2.2, 2.3 and 2.4 we obtain (3.4). □

Remark 3.2. Setting $n = 0$ (or $\tau = 0$) makes inequalities (3.1), (3.2) and (3.3) to become the results of Babalola et al [5].

Theorem 3.3. If $f(z) \in \mathcal{K}^\star(n,\beta,\eta,\tau;\alpha)$, then for $x \in \mathbb{R}$,

$$|a_3 - xa_2^2| \leqslant \begin{cases} \frac{4}{\alpha\Lambda_3}\left(1-\frac{(\alpha-1)\Lambda_2^2+2\Lambda_3 x}{\alpha\Lambda_2^2}\right) & for \quad x \leqslant \frac{(1-\alpha)\Lambda_2^2}{2\Lambda_3}, \\ \frac{4}{\alpha\Lambda_3} & for \quad \frac{(1-\alpha)\Lambda_2^2}{2\Lambda_3} \leqslant x \leqslant \frac{2\alpha\Lambda_2^2+(1-\alpha)\Lambda_2^2}{2\Lambda_3}, \\ \frac{4}{\alpha\Lambda_3}\left(\frac{(\alpha-1)\Lambda_2^2+2\Lambda_3 x}{\alpha\Lambda_2^2}-1\right) & for \quad x \geqslant \frac{2\alpha\Lambda_2^2+(1-\alpha)\Lambda_2^2}{2\Lambda_3}, \\ \frac{2}{\alpha\Lambda_3}\left(\frac{4|\Lambda_2^2-2\Lambda_3 x|}{\Lambda_2}+\alpha\Lambda_3(3-4x)\right) & for \quad x \leqslant \frac{1}{2}, \\ \frac{2}{\alpha\Lambda_3}\left(\frac{4|\Lambda_2^2-2\Lambda_3 x|}{\Lambda_2}+\alpha\Lambda_3\right) & for \quad \frac{1}{2} \leqslant x \leqslant 1, \\ \frac{2}{\alpha\Lambda_3}\left(\frac{4|\Lambda_2^2-2\Lambda_3 x|}{\Lambda_2}+\alpha\Lambda_3(4x-3)\right) & for \quad x \geqslant 1. \end{cases} \tag{3.15}$$

Proof . Consider (3.12) and (3.13) in (1.6) and for $x \in \mathbb{R}$ implies that

$$\begin{aligned} |a_3 - xa_2^2| &= \left| \frac{1}{\alpha\Lambda_3}c_2 + \frac{\Lambda_2 s_2}{\alpha\Lambda_3}c_1 - \frac{\alpha-1}{2\alpha^2\Lambda_3}c_1^2 + s_3 - x\left(\frac{1}{\alpha\Lambda_2}c_1 + s_2\right)^2 \right| \\ &= \frac{1}{\alpha\Lambda_3}\left| c_2 - \left(\frac{(\alpha-1)\Lambda_2^2+2\Lambda_3 x}{\alpha\Lambda_2^2}\right)\frac{c_1^2}{2} + \frac{(\Lambda_2^2-2\Lambda_3 x)s_2}{\Lambda_2}c_1 + \alpha\Lambda_3(s_3 - xs_2^2)\right| \\ &\leqslant \frac{1}{\alpha\Lambda_3}\left| c_2 - \frac{(\alpha-1)\Lambda_2^2+2\Lambda_3 x}{\alpha\Lambda_2^2}\frac{c_1^2}{2}\right| + \frac{1}{\alpha\Lambda_3}\left|\frac{\Lambda_2^2-2\Lambda_3 x s_2}{\Lambda_2}c_1 + \alpha\Lambda_3(s_3 - xs_2^2)\right|. \end{aligned}$$

It is easy to see that

$$|a_3 - xa_2^2| \leqslant \frac{2}{\alpha\Lambda_3}\left| c_2 - \frac{(\alpha-1)\Lambda_2^2+2\Lambda_3 x}{\alpha\Lambda_2^2}\frac{c_1^2}{2}\right|$$
$$\text{if} \quad \left| c_2 - \frac{(\alpha-1)\Lambda_2^2+2\Lambda_3 x}{\alpha\Lambda_2^2}\frac{c_1^2}{2}\right| \geqslant \left|\frac{\Lambda_2^2-2\Lambda_3 x s_2}{\Lambda_2}c_1 + \alpha\Lambda_3(s_3 - xs_2^2)\right| \tag{3.16}$$

and

$$|a_3 - xa_2^2| \leqslant \frac{2}{\alpha\Lambda_3}\left|\frac{\Lambda_2^2-2\Lambda_3 x s_2}{\Lambda_2}c_1 + \alpha\Lambda_3(s_3 - xs_2^2)\right|$$
$$\text{if} \quad \left| c_2 - \frac{(\alpha-1)\Lambda_2^2+2\Lambda_3 x}{\alpha\Lambda_2^2}\frac{c_1^2}{2}\right| \leqslant \left|\frac{\Lambda_2^2-2\Lambda_3 x s_2}{\Lambda_2}c_1 + \alpha\Lambda_3(s_3 - xs_2^2)\right|. \tag{3.17}$$

Applying Lemma 2.2 in (3.16) shows that

$$|a_3 - xa_2^2| \leqslant \frac{2}{\alpha\Lambda_3}\left|c_2 - u\frac{c_1^2}{2}\right| \quad \text{where } u = \frac{(\alpha-1)\Lambda_2^2+2\Lambda_3 x}{\alpha\Lambda_2^2}$$

so that for u in the intervals $u \leqslant 0$, $0 \leqslant u \leqslant 2$ and $u \geqslant 2$ we obtain the first three results in (3.15).

On the other hand, (3.17) simplifies to

$$|a_3 - xa_2^2| \leqslant \frac{2}{\alpha\Lambda_3}\left\{\frac{|\Lambda_2^2-2\Lambda_3 x||s_2|}{\Lambda_2}|c_1| + \alpha\Lambda_3|s_3 - xs_2^2|\right\}$$

so that by applying Lemmas 2.1, 2.4 and 2.5 we obtain the last three results in (3.15) and the proof is complete. □

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