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Note on the Ψ-asymptotic relationships between Ψ- bounded solutions of two Lyapunov matrix differential equations

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Abstract

There are proved existence results for Ψ-asymptotic relationships between Ψ-bounded solutions of two Lyapunov matrix differential equations.

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1 Introduction

In this presented paper, a several new sufficient conditions for Ψ-asymptotic relationships between Ψ-bounded solutions of two matrix differential equations

$$Z' = A(t)Z, \tag{1.1}$$

$$Z' = A(t)Z + F(t, Z) \tag{1.2}$$

and more, of two Lyapunov matrix differential equations

$$Z' = A(t)Z + ZB(t), \tag{1.3}$$

$$Z' = A(t)Z + ZB(t) + F(t, Z). \tag{1.4}$$

These conditions can be written in terms of fundamental matrices of the matrix differential equations (1.1), (1.3) and

$$Z' = ZB(t) \tag{1.5}$$

and of the function F. Here, Ψ is a matrix function who allows obtaining a mixed asymptotic behavior for the components of solutions of the differential equations.

A classical result in connection with boundedness of solutions of systems of ordinary differential equations

$$x' = A(t)x + f(t, x) \tag{1.6}$$

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was given by W. A. Coppel [5]. The problems of $\Psi-$bounded solutions for systems of ordinary differential equations or for Lyapunov matrix differential equations has been studied by many authors. See, for instance, [2], [3], [4], [6], [8], [9], [10], [11], [13] and the reference therein.

The results of this paper generalize and extend known results of W. A. Coppel [5], T. G. Hallam [11], F. Brauer and J. S. W. Wong [3], [4].

The main tools used in our paper are Schauder - Tychonoff fixed point theorem and the technique of variation of constants formula combined with Kronecker product of matrices, which has been successfully applied in various fields of matrix theory. See, for example, the cited papers and the references cited therein.

2 Preliminaries

In this section we present some basic notations, definitions, hypotheses and results which are useful later on.

Let R^d be the Euclidean $d-$ dimensional space. For $x = (x_1, x_2, ..., x_d)^T \in \mathrm{R}^d$, let $\| x \| = \max\{| x_1|, | x_2|, ..., | x_d|\}$ be the norm of x (here, ^{T} denotes transpose).

Let $\mathbb{M}_{d\times d}$ be the linear space of all real $d \times d$ matrices.

For $A = (a_{ij}) \in \mathbb{M}_{d\times d}$, we define the norm $| A |$ by formula $| A | = \sup\limits_{\|x\|\leq 1} \| Ax \|$. It is well-known that $| A | = \max\limits_{1\leq i\leq d}\{\sum\limits_{j=1}^{d} | a_{ij}|\}$.

In the previous equations, we assume that A and B are continuous $d \times d$ matrices on $\mathbb{R}_+$ and $F : R_+ \times \mathbb{M}_{d\times d} \longrightarrow \mathbb{M}_{d\times d}$ is continuous function.

By a solution of the equation (1.4) we mean a continuous differentiable $d \times d$ matrix function satisfying the equation (1.4) for all $t \in R_+ = [0, \infty)$. For the other equations is similar.

We consider the fundamental matrices $X(t)$ and $Y(t)$ for the equations (1.1) and (1.5) respectively.

Let $\Psi_i : R_+ \longrightarrow (0, \infty)$, $i = 1, 2, ..., d$, be continuous functions and

$$\Psi \ = \ \mathrm{diag}\ [\Psi_1, \Psi_2, \cdots \Psi_d].$$

Let $P_1,\ P_2 \in M_{d\times d}$ be supplementary projections and (for short) define

$$\Omega_i(t, s) = \Psi(t)X(t)P_iX^{-1}(s)\Psi^{-1}(s), \text{ for } i = 1, 2.$$

Definition 2.1. ([6, 8]). *A vector function $\varphi : R_+ \longrightarrow R^d$ is said to be $\Psi-$bounded on R_+ if $\Psi(t)\varphi(t)$ is bounded on R_+ (i.e. there exists $m > 0$ such that $\| \Psi(t)\varphi(t) \| \leq m$, for all $t \in R_+$). Otherwise, is said that the function φ is $\Psi-$unbounded on R_+.*

Definition 2.2. ([8]). *A matrix function $M : R_+ \longrightarrow M_{d\times d}$ is said to be $\Psi-$ bounded on R_+ if the matrix function $\Psi(t)M(t)$ is bounded on R_+ (i.e. there exists $m > 0$ such that $| \Psi(t)M(t) | \leq m$, for all $t \in R_+$). Otherwise, is said that the matrix function M is $\Psi-$ unbounded on R_+.*

Remark 2.3. 1. *These definitions extend the definition of boundedness of scalar functions.*
2. For $\Psi = I_d$, one obtain the notion of classical boundedness (see [5]).
3. It is easy to see that if Ψ and Ψ^{-1} are bounded on R_+, then the $\Psi-$boundedness is equivalent with the classical boundedness.

Definition 2.4. ([1]) *Let $A = (a_{ij}) \in M_{m\times n}$ and $B = (b_{ij}) \in M_{p\times q}$. The Kronecker product of A and B, written $A \otimes B$, is defined to be the partitioned matrix*

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

Obviously, $A \otimes B \in \mathbb{M}_{mp\times nq}$.

The important rules of calculation of the Kronecker product are given in [1], [12] and [9, Lemma 2.1]

Definition 2.5. ([12]) *The application* $\mathcal{V}ec : \mathbb{M}_{m\times n} \longrightarrow R^{mn}$, *defined by*

$$\mathcal{V}ec(A) = (a_{11}, a_{21}, \cdots, a_{m1}, a_{12}, a_{22}, \cdots, a_{m2}, \cdots, a_{1n}, a_{2n}, \cdots, a_{mn})^T,$$

where $A = (a_{ij}) \in \mathbb{M}_{m\times n}$, *is called the vectorization operator.*

For important properties and rules of calculation of the $\mathcal{V}ec$ operator, see [9, Lemmas 2.1, 2.2, 2.3, 2.5].

For "*corresponding Kronecker product system associated with (1.4)*", see [9, Lemma 2.4].

The [9, Lemmas 2.6 and 2.7], play an important role in the proofs of main results of present paper.

At the end of this section, we give a Lemma which is useful in the proof of our main results. This Lemma is a modification of [7, Lemma 1] and [5, Lemma 1, p. 68].

Lemma 2.1. *Let* $U(t)$ *be an invertible* $d\times d$ *matrix which is a continuous function of* t *on* R_+ *and let* P *a projection,* $P \in M_{d\times d}$.

Suppose that there exist a continuous function $\varphi : R_+ \to (0,\infty)$ *and the constants* $M > 0$ *and* $p > 1$ *such that*

$$\int_{t_0}^{t} \left(\varphi(s) \mid \Psi(t)U(t)PU^{-1}(s)\Psi^{-1}(s) \mid\right)^p ds \leq M, \text{ for } t \geq t_0 \geq 0$$

and $\int_{t_0}^{\infty} \varphi^p(s)ds = \infty$.

Then, there exists a constant $N > 0$ *such that*

$$\mid \Psi(t)U(t)P \mid \leq Ne^{-(pM)^{-1}\int_{t_0}^{t}\varphi^p(s)ds}, \text{ for } t \geq t_0.$$

Consequently, $\lim\limits_{t\to\infty} \mid \Psi(t)U(t)P \mid = 0$.

Proof. If P = 0, the conclusion is obvious. For P $\neq$ 0, let $h(t) = \varphi^p(t) \mid \Psi(t)U(t)P \mid^{-p}$, for $t \geq t_0$. From the identity (for $t \geq t_0 \geq 0$)

$$(\Psi(t)U(t)P)\int_{t_0}^{t} h(s)ds$$
$$= \int_{t_0}^{t} \left(\varphi^{-1}(s) \mid \Psi(s)U(s)P \mid\right)^{-p} \left(\varphi(s)\Psi(t)U(t)PU^{-1}(s)\Psi^{-1}(s)\right)\left(\varphi^{-1}(s)\Psi(s)U(s)P\right) ds$$

and Hölder inequality, we find the inequality (where $\frac{1}{p} + \frac{1}{q} = 1$)

$$\mid \Psi(t)U(t)P \mid \int_{t_0}^{t} h(s)ds \leq \left[\int_{t_0}^{t} \left(\varphi(s) \mid \Psi(t)U(t)PU^{-1}(s)\Psi^{-1}(s) \mid\right)^p ds\right]^{1/p} \cdot$$

$$\cdot \left[\int_{t_0}^{t} \left(\varphi^{-1}(s) \mid \Psi(s)U(s)P \mid\right)^{-pq} \cdot \left(\varphi^{-1}(s) \mid \Psi(s)U(s)P \mid\right)^q ds\right]^{1/q}.$$

It follows that

$$\mid \Psi(t)U(t)P \mid \int_{t_0}^{t} h(s)ds \leq M^{1/p}\left(\int_{t_0}^{t} h(s)ds\right)^{1/q}$$

and then

$$\mid \Psi(t)U(t)P \mid \leq M^{1/p}\left(\int_{t_0}^{t} h(s)ds\right)^{-1/p}, \text{ for } t \geq t_0.$$

Denoting $\int_{t_0}^{t} h(s)ds = u(t)$ for $t \geq t_0$, we have $\mid \Psi(t)U(t)P \mid (u(t))^{1/p} \leq M^{1/p}$ and then, $\mid \Psi(t)U(t)P \mid = (h(t))^{-1/p}\varphi(t) = (u'(t))^{-1/p}\varphi(t)$.

Thus, we have $(u'(t))^{-1/p}\varphi(t)(u(t))^{1/p} \leq M^{1/p}$ and then, $\frac{u'(t)}{u(t)} \geq M^{-1}\varphi^p(t)$, $t \ \geq t_0$.

Integrating from t_1 to t, $(t > t_1 > t_0)$, we obtain $\ln \frac{u(t)}{u(t_1)} \geq M^{-1}\int_{t_1}^{t} \varphi^p(s)ds$ and then $u(t) \geq u(t_1)e^{M^{-1}\int_{t_1}^{t} \varphi^p(s)ds}$. It follows that

$$\mid \Psi(t)U(t)P \mid \leq \left(\frac{M}{u(t_1)}\right)^{1/p} e^{-(pM)^{-1}\int_{t_1}^{t} \varphi^p(s)ds}.$$

Choosing $N \geq M^{1/p}\left(u(t_1)\right)^{-1/p} e^{(pM)^{-1}\int_{t_0}^{t_1} \varphi^p(s)ds}$ sufficiently large, we have the conclusion of the Lemma.

The proof of Lemma is complete.

3 Main results

The purpose of this section is to give new sufficient conditions for $\Psi-$asymptotic relationships between $\Psi-$bounded solutions of two pairs of Lyapunov matrix differential equations.

We begin with a result regarding $\Psi-$asymptotic relationships between $\Psi-$bounded solutions of equations (1.1) and (1.2). This result is motivated by a Theorems of Hallam [11] and of Brauer and Wong [3], [4].

Theorem 3.1. *Suppose that:*
1). There exist supplementary projections P_1, $P_2 \in M_{d\times d}$, a continuous function $\varphi : R_+ \to (0,\infty)$ and the constants $K \in (0,\infty)$, $p \in (1,\infty)$ such that the fundamental matrix X(t) for matrix differential equation (1.1) satisfies the inequality

$$\left[\int_{t_0}^{t} \left(\varphi(s) \mid \Omega_1(t,s) \mid\right)^p ds\right]^{1/p} + \left[\int_{t}^{\infty} \left(\varphi(s) \mid \Omega_2(t,s) \mid\right)^p ds\right]^{1/p} \leq K$$

for all $t \geq t_0 \geq 0$, where t_0 is sufficiently large;
Furthermore, the function φ satisfies the condition $\int_0^\infty \varphi^p(s)ds = +\infty$.
2). The continuous matrix function $F : R_+ \times \mathbb{M}_{d\times d} \longrightarrow \mathbb{M}_{d\times d}$ satisfies the inequality

$$\varphi^{-1}(t) \mid \Psi(t)F(t,Z) \mid \leq \omega(t, \mid \Psi(t)Z \mid)$$

for all $t \geq t_0$ and $Z \in \mathbb{M}_{d\times d}$, where $\omega(t,r) : R_+ \times R_+ \to R_+$ is a *continuous function and is nondecreasing in r, for each fixed $t \geq t_0$.*
Furthermore, the function ω satisfies the condition

$$\int_0^\infty \omega^q(t,\lambda)dt < +\infty,$$

for *a* $\lambda \in (0,\infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$.
Then, corresponding to each $\Psi-$bounded solution $Z_0(t)$ of (1.1), there exists a $\Psi-$*bounded solution Z(t) of (1.2) such that*

$$\lim_{t\to\infty} \mid \Psi(t)\left(Z(t) - Z_0(t)\right) \mid = 0. \tag{3.1}$$

Conversely, *to each $\Psi-$bounded solution Z(t) of (1.2),* there exists a $\Psi-$*bounded solution* Z_0(t) of (1.1) *such that (3.1) holds.*

Proof. We prove by means of the fixed point theorem of Schauder - Tychonoff (Coppel, [5], Chapter I, section 2).

Let C_Ψ denote the set of all contiuous matrix functions Z(t) defined on R_+ and $\Psi-$bounded on R_+. For $t_0 \geq 0$ sufficiently large so that $\left(\int_{t_0}^\infty \omega^q(s,\lambda)ds\right)^{1/q} < \lambda/2K$ and $\rho = \lambda/2$, let S_ρ be the subset formed by those functions Z(t) such that $\mid Z \mid_\Psi = \sup_{t\in[t_0,\infty)} \{\mid \Psi(t)Z(t) \mid\} \leq \rho$.

For Z $\in S_{2\rho}$, we define the operator T by

$$(TZ)(t) = Z_0(t) + \int_{t_0}^{t} X(t)P_1X^{-1}(s)F(s,Z(s))ds - \int_{t}^{\infty} X(t)P_2X^{-1}(s)F(s,Z(s))ds \tag{3.2}$$

for $t \geq t_0$, where $Z_0(t)$ is a $\Psi-$bounded solution of (1.1) such that $Z_0 \in S_\rho$.

From hypotheses, TZ exists and is continuous differentiable on R_+.

Indeed, for $v \geq t \geq t_0$,

$$| \int_t^v X(t)P_2X^{-1}(s)F(s,Z(s))ds |$$
$$=| \Psi^{-1}(t) \int_t^v \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)\Psi(s)F(s,Z(s))ds |$$
$$\leq| \Psi^{-1}(t) | \int_t^v \varphi(s) | \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) | \varphi^{-1}(s) | \Psi(s)F(s,Z(s)) | ds$$

$$\leq| \Psi^{-1}(t) | \int_t^v \varphi(s) | \Omega_2(t,s) | \omega(s, | \Psi(s)Z(s) |)ds$$
$$\leq| \Psi^{-1}(t) | \int_t^v \varphi(s) | \Omega_2(t,s) | \omega(s,2\rho)ds$$
$$\leq| \Psi^{-1}(t) | \left[\int_t^v (\varphi(s) | \Omega_2(t,s) |)^p \, ds\right]^{1/p} \cdot \left[\int_t^v \omega^q(s,2\rho)ds\right]^{1/q}$$
$$\leq \frac{\rho}{K} | \Psi^{-1}(t) | \left[\int_t^v (\varphi(s) | \Omega_2(t,s) |)^p \, ds\right]^{1/p}.$$

From the first assumption of Theorem, it follows that the integral

$$\int_t^\infty X(t)P_2X^{-1}(s)F(s,Z(s))ds$$

is convergent for all $Z \in S_{2\rho}$ and $t \geq t_0$.

From hypotheses, TZ exists and is continuous differentiable on $[t_0,\infty)$.

This operator T has the following properties:

a). T maps $S_{2\rho}$ into itself;

Indeed, for any $Z \in S_{2\rho}$, and for $t \geq t_0$, we have

$$| \Psi(t)(TZ)(t) |\leq| \Psi(t)Z_0(t) |$$
$$+ \int_{t_0}^t \varphi(s) | \Omega_1(t,s) | \varphi^{-1}(s) | \Psi(s)F(s,Z(s)) | ds$$
$$+ \int_t^\infty \varphi(s) | \Omega_2(t,s) | \varphi^{-1}(s) | \Psi(s)F(s,Z(s)) | ds$$
$$\leq| \Psi(t)Z_0(t) | +$$
$$+ \int_{t_0}^t \varphi(s) | \Omega_1(t,s) | \omega(s, | \Psi(s)Z(s) |)ds$$
$$+ \int_t^\infty \varphi(s) | \Omega_2(t,s) | \omega(s, | \Psi(s)Z(s) |)ds$$
$$\leq \rho + \int_{t_0}^t \varphi(s) | \Omega_1(t,s) | \omega(s,2\rho)ds$$
$$+ \int_t^\infty \varphi(s) | \Omega_2(t,s) | ds\omega(s,2\rho)ds$$
$$\leq \rho + \left[\int_{t_0}^t (\varphi(s) | \Omega_1(t,s) |)^p \, ds\right]^{1/p} \left[\int_{t_0}^t \omega^q(s,2\rho)ds\right]^{1/q}$$
$$+ \left[\int_t^\infty (\varphi(s) | \Omega_2(t,s) |)^p \, ds\right]^{1/p} \left[\int_t^\infty \omega^q(s,2\rho)ds\right]^{1/q}$$
$$\leq \rho + \rho/K \cdot K = 2\rho.$$

This proves the assertion.

b). T is continuous, in the sense that if $Z_n \in S_{2\rho}$, (n = 1, 2, ...) and $Z_n \to Z$ uniformly on every compact subinterval J of $[t_0,\infty)$, then $TZ_n \to TZ$ uniformly on every compact subinterval J of $[t_0,\infty)$;

Indeed, from (3.2), for any t $\in$ J $\subset [t_0, \infty)$, we have

$$\begin{aligned} &\mid (TZ_n)(t) - (TZ)(t) \mid \\ \leq &\mid \Psi^{-1}(t) \mid \int_{t_0}^{t} \varphi(s) \mid \Omega_1(t,s) \mid \varphi^{-1}(s) \mid \Psi(s) \left(F(s,Z_n(s)) - F(s,Z(s))\right) \mid ds \\ + &\mid \Psi^{-1}(t) \mid \int_{t}^{\infty} \varphi(s) \mid \Omega_2(t,s) \mid \varphi^{-1}(s) \mid \Psi(s) \left(F(s,Z_n(s)) - F(s,Z(s))\right) \mid ds \\ \leq &\mid \Psi^{-1}(t) \mid \left[\int_{t_0}^{t} (\varphi(s) \mid \Omega_1(t,s) \mid)^p \, ds\right]^{1/p} \cdot \\ &\cdot \left[\int_{t_0}^{t} \left(\mid \varphi^{-1}(s) \mid \Psi(s) \left(F(s,Z_n(s)) - F(s,Z(s))\right) \mid\right)^q ds\right]^{1/q} \\ + &\mid \Psi^{-1}(t) \mid \left[\int_{t}^{\infty} (\varphi(s) \mid \Omega_2(t,s) \mid)^p \, ds\right]^{1/p} \cdot \\ &\cdot \left[\int_{t}^{\infty} \left(\mid \varphi^{-1}(s) \mid \Psi(s) \left(F(s,Z_n(s)) - F(s,Z(s))\right) \mid\right)^q ds\right]^{1/q}. \end{aligned} \tag{3.3}$$

Let J = $[\alpha, \beta]$. For a fixed $\varepsilon > 0$, we choose $t_1 \geq t_0$ sufficientjy large $(t_1 \geq \beta)$ so that

$$\left[\int_{t_1}^{\infty} \omega^q(s, 2\rho) ds\right]^{1/q} < \frac{\varepsilon}{4K \sup\limits_{t \in J} \mid \Psi^{-1}(t) \mid}.$$

Since F, Ψ, φ are continuous and $Z_n \to$ Z uniformly on $[t_0, t_1]$, there exists an $n_0 \in \mathbb{N}$ such that, for $s \in [t_0, t_1]$ and n $\geq n_0$,

$$\varphi^{-1}(s) \mid \Psi(s) \left(F(s,Z_n(s)) - F(s,Z(s))\right) \mid < \frac{\varepsilon}{4K \sup\limits_{t \in J} \mid \Psi^{-1}(t) \mid} \cdot \frac{1}{(t_1 - t_0)^{1/q}}.$$

Now, for t $\in$ J and $n \geq n_0$, the first integral term of (3.3) becomes

$$\begin{aligned} &\mid \Psi^{-1}(t) \mid \int_{t_0}^{t} \varphi(s) \mid \Omega_1(t,s) \mid \varphi^{-1}(s) \mid \Psi(s) \left(F(s,Z_n(s)) - F(s,Z(s))\right) \mid ds \\ &\leq \mid \Psi^{-1}(t) \mid \left[\int_{t_0}^{t} (\varphi(s) \mid \Omega_1(t,s) \mid)^p \, ds\right]^{1/p} \cdot \\ &\quad \cdot \left[\int_{t_0}^{t} \left(\mid \varphi^{-1}(s) \mid \Psi(s) \left(F(s,Z_n(s)) - F(s,Z(s))\right) \mid\right)^q ds\right]^{1/q} \\ &\leq \mid \Psi^{-1}(t) \mid \left[\int_{t_0}^{t} (\varphi(s) \mid \Omega_1(t,s) \mid)^p \, ds\right]^{1/p} \cdot \left[\int_{t_0}^{t} \left(\frac{\varepsilon}{4K \sup\limits_{t \in J} |\Psi^{-1}(t)|} \cdot \frac{1}{(t_1 - t_0)^{1/q}}\right)^q ds\right]^{1/q} \\ &\leq \mid \Psi^{-1}(t) \mid \left[\int_{t_0}^{t} (\varphi(s) \mid \Omega_1(t,s) \mid)^p \, ds\right]^{1/p} \cdot \left[\int_{t_0}^{t} \frac{\varepsilon^q}{(4K)^q (\sup\limits_{t \in J} |\Psi^{-1}(t)|)^q} \cdot \frac{1}{t_1 - t_0} ds\right]^{1/q} < \frac{\varepsilon}{4}. \end{aligned}$$

For the second integral term of (3.3), for t $\in$ J and $n \geq n_0$, we have

$$\begin{aligned} &\mid \Psi^{-1}(t) \mid \int_{t}^{\infty} \varphi(s) \mid \Omega_2(t,s) \mid \varphi^{-1}(s) \mid \Psi(s) \left(F(s,Z_n(s)) - F(s,Z(s))\right) \mid ds \\ &= \mid \Psi^{-1}(t) \mid \Big\{\int_{t}^{t_1} \varphi(s) \mid \Omega_2(t,s) \mid \varphi^{-1}(s) \mid \Psi(s) \left(F(s,Z_n(s)) - F(s,Z(s))\right) \mid ds \\ &+ \int_{t_1}^{\infty} \varphi(s) \mid \Omega_2(t,s) \mid \varphi^{-1}(s) \mid \Psi(s) \left(F(s,Z_n(s)) - F(s,Z(s))\right) \mid ds\Big\} \\ &\leq \mid \Psi^{-1}(t) \mid \Big\{\left[\int_{t}^{t_1} (\varphi(s) \mid \Omega_2(t,s) \mid)^p \, ds\right]^{1/p} \cdot \\ &\cdot \left[\int_{t}^{t_1} \left(\mid \varphi^{-1}(s) \mid \Psi(s) \left(F(s,Z_n(s)) - F(s,Z(s))\right) \mid\right)^q ds\right]^{1/q} \end{aligned}$$

$+\left[\int_{t_1}^{\infty}\left(\varphi(s)\mid\Omega_2(t,s)\mid\right)^p ds\right]^{1/p}\cdot$

$\cdot\left[\int_{t_1}^{\infty}\left(\mid\varphi^{-1}(s)\mid\Psi(s)\left(F(s,Z_n(s))-F(s,Z(s))\right)\mid\right)^q ds\right]^{1/q}\}$

$<\mid\Psi^{-1}(t)\mid\cdot K\cdot\left[\int_t^{t_1}\left(\frac{\varepsilon}{4K\sup\limits_{t\in J}\mid\Psi^{-1}(t)\mid}\cdot\frac{1}{(t_1-t_0)^{1/q}}\right)^q ds\right]^{1/q}$

$+\mid\Psi^{-1}(t)\mid\cdot K\cdot\left[\int_{t_1}^{\infty}2^q\omega^q(s,2\rho)ds\right]^{1/q}$

$\leq\mid\Psi^{-1}(t)\mid\cdot K\cdot\{\left[\frac{\varepsilon^q}{(4K)^q(\sup\limits_{t\in J}\mid\Psi^{-1}(t)\mid)^q}\int_t^{t_1}\frac{1}{t_1-t_0}ds\right]^{1/q}+2\left[\int_{t_1}^{\infty}\omega^q(s,2\rho)ds\right]^{1/q}\}$

$\leq\mid\Psi^{-1}(t)\mid\cdot K\cdot\{\frac{\varepsilon}{4K\sup\limits_{t\in J}\mid\Psi^{-1}(t)\mid}+2\frac{\varepsilon}{4K\sup\limits_{t\in J}\mid\Psi^{-1}(t)\mid}\}\leq\frac{3\varepsilon}{4}.$

From the above results, we obtain that

$$\mid(TZ_n)(t)-(TZ)(t)\mid<\varepsilon,\text{ for any t}\in\text{J and n}\geq\text{n}_0.$$

Thus, the sequence (TZ$_n$) converges uniformly to TZ on compact subintervals of $[t_0,\infty)$.

We conclude that T is continuous.

c). the matrix functions in the image set TS$_{2\rho}$ are echicontinuous and uniformly bounded at every point of every compact subinterval J of $[t_0,\infty)$.

Indeed, from a), T maps S$_{2\rho}$ into itself. This shows that the matrix functions in the image set TS$_{2\rho}$ are uniformly bounded at every point of every compact subinterval J of $[t_0,\infty)$.

On the other hand, for the image V = TZ, we have

$V'(t)=Z_0'(t)+\int_{t_0}^t X'(t)P_1X^{-1}(s)F(s,Z(s))ds+X(t)P_1X^{-1}(t)F(t,Z(t))$

$-\int_t^{\infty}X'(t)P_2X^{-1}(s)F(s,Z(s))ds+X(t)P_2X^{-1}(t)F(t,Z(t))$

$=A(t)\left(Z_0(t)+\int_{t_0}^t X(t)P_1X^{-1}(s)F(s,Z(s))ds-\int_t^{\infty}X(t)P_2X^{-1}(s)F(s,Z(s))ds\right)$

$+X(t)\left(P_1+P_2\right)X^{-1}(t)F(t,Z(t))$

$=A(t)V(t)+F(t,Z(t))$, for $t\geq t_0$.

Since

$V'(t)=\left(A(t)\Psi^{-1}(t)\right)\left(\Psi(t)V(t)\right)+\left(\varphi(t)\Psi^{-1}(t)\right)\left(\varphi^{-1}(t)\Psi(t)F(t,Z(t))\right)$, for $t\geq t_0$,

and the matrices $A(t)\Psi^{-1}(t)$, $\Psi(t)V(t)$, $\varphi(t)\Psi^{-1}(t)$, $\varphi^{-1}(t)\Psi(t)F(t,Z(t))$ are uniformly bounded on every compact subinterval J of $[t_0,\infty)$, the derivatives of the functions in TS$_{2\rho}$ are uniformly bounded on every compact subinterval J of $[t_0,\infty)$. This shows that the functions in TS$_{2\rho}$ are echicontinuous on every compact subinterval J of $[t_0,\infty)$.

Thus, all the conditions of the fixed point theorem of Schauder - Tychonoff are satisfied. We conclude that the operator T has a fixed point Z in S$_{2\rho}$. This fixed point Z is evidently a $\Psi-$ bounded solution of *(1.2).*

To complete the proof, we must verify (3.1).

Acording to hypothesis 2) of Theorem, for each $\varepsilon>0$, we can choose $t_1>t_0$ such that

$$\left(\int_{t_1}^{\infty}\omega^q(s,2\rho)ds\right)^{1/q}<\frac{\varepsilon}{2K}.$$

By Lemma 2.1, there exists a $t_2>t_1$ so that

$$\mid\Psi(t)X(t)P_1\mid\int_{t_0}^{t_1}\mid X^{-1}(s)F(s,Z(s))\mid ds<\frac{\varepsilon}{2},\text{ for }t\geq t_2,$$

where Z(t) is the solution from above of TZ = Z.

Using definition of T, these inequalities and Hölder inequality, we obtain for $t \geq t_2$,

$$\begin{aligned}
&\mid \Psi(t)\left(Z(t) - Z_0(t)\right) \mid \\
&\leq \mid \textstyle\int_{t_0}^{t_1} \Psi(t)X(t)P_1X^{-1}(s)F(s,Z(s))ds + \int_{t_1}^{t} \Psi(t)X(t)P_1X^{-1}(s)F(s,Z(s))ds \\
&- \textstyle\int_t^{\infty} \Psi(t)X(t)P_2X^{-1}(s)F(s,Z(s))ds \mid \\
&\leq \mid \Psi(t)X(t)P_1 \mid \textstyle\int_{t_0}^{t_1} \mid X^{-1}(s)F(s,Z(s)) \mid ds \\
&+ \textstyle\int_{t_1}^{t} \varphi(s) \mid \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s) \mid \cdot\varphi^{-1}(s) \mid \Psi(s)F(s,Z(s)) \mid ds \\
&+ \textstyle\int_t^{\infty} \varphi(s) \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid \cdot\varphi^{-1}(s) \mid \Psi(s)F(s,Z(s)) \mid ds \\
&\leq \tfrac{\varepsilon}{2} + \textstyle\int_{t_1}^{t} \varphi(s) \mid \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s) \mid \omega(s,2\rho)ds \\
&+ \textstyle\int_t^{\infty} \varphi(s) \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid \omega(s,2\rho)ds \\
&\leq \tfrac{\varepsilon}{2} + \left[\textstyle\int_{t_1}^{t} \left(\varphi(s) \mid \Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s) \mid\right)^p ds\right]^{1/p} \cdot \left(\textstyle\int_{t_1}^{t} \omega^q(s,2\rho)ds\right)^{1/q} \\
&+ \left[\textstyle\int_t^{\infty} \left(\varphi(s) \mid \Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s) \mid\right)^p ds\right]^{1/p} \cdot \left(\textstyle\int_t^{\infty} \omega^q(s,2\rho)ds\right)^{1/q} \\
&\leq \tfrac{\varepsilon}{2} + K\left(\textstyle\int_{t_1}^{\infty} \omega^q(s,2\rho)ds\right)^{1/q} < \tfrac{\varepsilon}{2} + K \cdot \tfrac{\varepsilon}{2K} = \varepsilon,
\end{aligned}$$

which establishes (3.1).

To prove the last statement of the Theorem, consider a $\Psi-$ bounded solution Z(t) of equation (1.2). Define

$$Z_0(t) = Z(t) - \int_{t_0}^{t} X(t)P_1X^{-1}(s)F(s,Z(s))ds + \int_t^{\infty} X(t)P_2X^{-1}(s)F(s,Z(s))ds,\ t \geq t_0.$$

With the previous arguments, we can show that $Z_0(t)$ is a $\Psi-$ bounded solution of equation (1.1) that satisfies (3.1).

The proof of Theorem is complete.

Remark 3.2. *If we put*

$$Z = \begin{pmatrix} z_1 & z_1 & \cdots & z_1 \\ z_2 & z_2 & \cdots & z_2 \\ \cdots & \cdots & \cdots & \cdots \\ z_d & z_d & \cdots & z_d \end{pmatrix},\ F(t,Z) = \begin{pmatrix} f_1(t,z) & f_1(t,z) & \cdots & f_1(t,z) \\ f_2(t,z) & f_2(t,z) & \cdots & f_2(t,z) \\ \cdots & \cdots & \cdots & \cdots \\ f_d(t,z) & f_d(t,z) & \cdots & f_d(t,z) \end{pmatrix}$$

we get a version of Theorem 3.1 for systems of differential equations. In addition, putting $\Psi = diag\ [\psi, \psi, \cdots \psi]$, where $\psi : R_+ \to (0,\infty)$ is a continuous function, equation (1.2) becomes equation (2) from [11]. Thus, Theorem 3.1 generalizes Theorem 2, [11], in two directions: from systems of differential equations to matrix differential equations and the introduction of the matrix function Ψ which allows obtaining a mixed asymptotic behavior for the components of solutions of the above equations. In addition, the function φ satisfies the condition $\int_0^{\infty} \varphi^p(s)ds = +\infty$, better than the condition $\int_0^{\infty} \varphi^p(s)\psi^{-p}(s)ds = +\infty$.

The goal of next Theorem is to obtain a new result in connection with $\Psi-$asymptotic relationships between $\Psi-$bounded solutions of two Lyapunov matrix differential equations, namely (1.3) and (1.4).

Theorem 3.3. *Suppose that:*

1). There exist supplementary projections P_1, $P_2 \in M_{d\times d}$, a continuous function $\varphi : R_+ \to (0,\infty)$ and the constants

$K > 0$ and $p \in (1, \infty)$ such that the fundamental matrices X(t) and Y(t) for the linear matrix differential equations (1.1) and (1.5) respectively satisfy the inequality$_0$

$$\left[\int_{t_0}^{t} \left(\varphi(s) \mid \left(Y^T(t)(Y^T)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)P_1X^{-1}(s)\Psi^{-1}(s)\right) \mid\right)^p ds\right]^{1/p} + \tag{3.4}$$

$$+\left[\int_{t}^{\infty} \left(\varphi(s) \mid \left(Y^T(t)(Y^T)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)P_2X^{-1}(s)\Psi^{-1}(s)\right) \mid\right)^p ds\right]^{1/p} \leq K$$

for all $t \geq t_0 \geq 0$, where t_0 is sufficiently large;
Furthermore, the function φ satisfies the condition $\int_0^\infty \varphi^p(s)ds = +\infty$
2). The continuous matrix function $F : R_+ \times \mathbb{M}_{d\times d} \longrightarrow \mathbb{M}_{d\times d}$ satisfies the inequality

$$\varphi^{-1}(t) \mid \Psi(t)F(t, Z) \mid \leq \omega(t, \mid \Psi(t)Z \mid)$$

for all $t \geq t_0$ and $Z \in \mathbb{M}_{d\times d}$, where $\omega(t, r) : R_+ \times R_+ \to R_+$ is a *continuous function and is nondecreasing in r, for each fixed $t \geq t_0$.*
Furthermore, the function ω satisfies the condition

$$\int_0^\infty \omega^q(t, \lambda)dt < +\infty$$

for *a* $\lambda \in (0, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$.
Then, corresponding to each $\Psi-$bounded solution $Z_0(t)$ of (1.3), there exists a *$\Psi-$bounded solution Z(t) of Lyapunov matrix differential equation (1.4) such that*

$$\lim_{t\to\infty} \mid \Psi(t)\left(Z(t) - Z_0(t)\right) \mid = 0. \tag{3.5}$$

Conversely, *to each $\Psi-$bounded solution Z(t) of (1.4),* there exists a *$\Psi-$bounded solution* Z_0(t) *of (1.3) such that (3.5) holds.*
Proof. We will use the version of Theorem 3.1 for systems of differential equations and some results from [9].

From [9, Lemma 2.6], we know that Z(t) is a $\Psi-$bounded solution on R_+ of (1.4) iff $z(t) = \mathcal{V}ec(Z(t))$ is a $I \otimes \Psi(t)-$bounded solution of the corresponding Kronecker product system associated with (1.4), i.e. the system

$$z' = \left(I \otimes A(t) + B^T(t) \otimes I\right) z + f(t, z),\ t \geq t_0 \tag{3.6}$$

where $f(t, z) = \mathcal{V}ec(F(t, Z))$.

We verify the hypotheses of Theorem 3.1 (version for systems of differential equations).

a). From [9, Lemma 2.7], we know that $U(t) = Y^T(t) \otimes X(t)$ is a fundamental matrix for the homogeneous system associated to (3.6), i.e. the system

$$z' = \left(I \otimes A(t) + B^T(t) \otimes I\right) z. \tag{3.7}$$

With the help of [9, Lemmas 2.1, 2.3], we have

$$(\varphi(s) \mid \Omega_1(t, s) \mid)^p = \left(\varphi(s) \mid (I \otimes \Psi(t))\, U(t)(I \otimes P_i)U^{-1}(s)\, (I \otimes \Psi(s))^{-1} \mid\right)^p$$

$$= \left(\varphi(s) \mid (I \otimes \Psi(t)) \left(Y^T(t) \otimes X(t)\right)(I \otimes P_i)\left(Y^T(s) \otimes X(s)\right)^{-1}(I \otimes \Psi(s))^{-1} \mid\right)^p$$

$$= \left(\varphi(s) \mid \left(Y^T(t)(Y^T)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)P_iX^{-1}(s)\Psi^{-1}(s)\right) \mid\right)^p.$$

Thus, the hypothesis 1) ensures the hypothesis 1) of Theorem 3.1 (with $Y^T(t) \otimes X(t)$ in role of $X(t)$, $I \otimes \Psi(t)$ in the role of $\Psi(t)$ and $I \otimes P_i$ in role of P_i).

b). Similarly, by using [9, Lemma 2.5], we have, for $t \geq t_0$ *and* $Z \in \mathbb{M}_{d\times d}$,

$$\varphi^{-1}(t) \mid (I \otimes \Psi(t))\, f(t, z) \mid = \varphi^{-1}(t) \mid (I \otimes \Psi(t))\, \mathcal{V}ec\,(F(t, z)) \mid$$

$$\leq \varphi^{-1}(t) \mid \Psi(t)F(t, Z) \mid \leq \omega(t, \mid \Psi(t)Z \mid) \leq \omega(t, d \mid (I \otimes \Psi(t))\, \mathcal{V}ec(Z) \mid$$

$$= \omega(t, d \mid (I \otimes \Psi(t))\, z \mid.$$

Thus, the hypothesis 2) ensures the hypothesis 2) of Theorem 3.1.

Now, we finish the proof.

Let $Z_0(t)$ be a $\Psi-$bounded solution of (1.3). From [9, Lemma 2.6], the function $z_0(t) = \mathcal{V}ec(Z_0(t))$ is a $I \otimes \Psi(t)-$bounded solution of (3.7). From Theorem 3.1 (the version for systems), there exists a $I \otimes \Psi(t)-$bounded solution z(t) of (3.6) with the property that

$$\lim_{t\to\infty} \mid (I \otimes \Psi(t))\,(z(t) - z_0(t)) \mid = 0. \tag{3.8}$$

From [9, Lemmas 2.5 and 2.6], we obtain that (3.5) holds, where $Z(t) = \mathcal{V}ec^{-1}(z(t))$ is a $\Psi(t)-$bounded solution of (1.4).

For the last statement of Theorem, let Z(t) a $\Psi(t)-$bounded solution of (1.4). Then, $z(t) = \mathcal{V}ec(Z(t))$ is a $I \otimes \Psi(t)-$bounded solution of (3.6). From Theorem 3.1 (version for systems), there exists a $I \otimes \Psi(t)-$bounded solution $z_0(t)$ of (3.7) such that (3.8) holds. From [9, Lemmas 2.5 and 2.6], we obtain that (3.5) holds, where $Z_0(t) = \mathcal{V}ec^{-1}(z_0(t))$ is a $\Psi(t)-$bounded solution of (1.3).

The proof is now complete.

Remark 3.4. *If the hypothesis 1) of Theorem 3.3 is not satisfied, then the conclusion of Theorem 3.3 does not hold. This is shown by the next simple Example obtained after an example due to O. Perron, [14].*

Example 3.5. *In equations (1.3) and (1.4) consider*

$$A(t) = \begin{pmatrix} \sin\ln t + \cos\ln t & 0 \\ 0 & \frac{3}{2} \end{pmatrix},\ B(t) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

and

$$F(t, Z) = \begin{pmatrix} 0 & be^{-\frac{t}{2}} \\ 0 & 0 \end{pmatrix} Z,$$

where $t \geq 1$, $Z \in M_{2\times 2}$ and $b \in R$, $b \neq 0$.

In addition, consider

$$\Psi(t) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & e^{\frac{t}{2}} \end{pmatrix}.$$

The conditions 2) of Theorem 3.3 is satisfied. Indeed, for $t \geq 1$ and $Z \in M_{2\times 2}$, we have

$$|\Psi(t)F(t,Z)| = \left|\Psi(t)\begin{pmatrix} 0 & be^{-\frac{t}{2}} \\ 0 & 0 \end{pmatrix} Z\right| = \left|\Psi(t)\begin{pmatrix} 0 & be^{-\frac{t}{2}} \\ 0 & 0 \end{pmatrix}\Psi^{-1}(t)\Psi(t)Z\right| \leq$$

$$\leq \left|\Psi(t)\begin{pmatrix} 0 & be^{-\frac{t}{2}} \\ 0 & 0 \end{pmatrix}\Psi^{-1}(t)\right| |\Psi(t)Z| = \left|\begin{pmatrix} 0 & \frac{b}{2}e^{-t} \\ 0 & 0 \end{pmatrix}\right| |\Psi(t)Z| = \tfrac{|b|}{2}e^{-t}\,|\Psi(t)Z|$$

and then, $\omega(t,\lambda) = \frac{|b|}{2}e^{-t}\lambda$.

In addition, the condition $\int_0^\infty \omega^q(t,\lambda)dt < +\infty$ is satisfied for $q > 1$ and $\lambda \in (0,\infty)$.

Suppose that the condition 1) of Theorem is satisfied. Then the conclusion of Theorem holds. In particular, corresponding to each $\Psi-$bounded solution $Z_0(t)$ of (1.3), there exists a $\Psi-$bounded solution $Z(t)$ of Lyapunov matrix differential equation (1.4) such that (3.5) holds.

We find the general solutions of (1.3) and (1.4) in a particular case considered here.

Equation (1.3) becomes

$$Z' = \begin{pmatrix} \sin\ln t + \cos\ln t - 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} Z,\ t \geq 1.$$

A fundamental matrix for this equation is

$$X(t) = \begin{pmatrix} e^{t[\sin\ln t - 1]} & 0 \\ 0 & e^{-\frac{1}{2}t} \end{pmatrix},\ t \geq 1.$$

Then, the general solution of equation is $Z_{g_0} = X(t)C$*, where* C *is a real* 2×2 *constant matrix.*

Equation (1.4) becomes

$$Z' = \begin{pmatrix} \sin\ln t + \cos\ln t - 1 & be^{-\frac{1}{2}t} \\ 0 & -\frac{1}{2} \end{pmatrix} Z,\ t \geq 1.$$

A fundamental matrix for this equation is

$$Y(t) = \begin{pmatrix} v(t) & u(t) \\ e^{-\frac{t}{2}} & 0 \end{pmatrix},$$

where $u(t) = e^{t[\sin\ln t - 1]}$ and $v(t) = bu(t)\cdot\int_1^t e^{-s\sin\ln s}ds$, for $t \geq 1$.

The general solution of this equation is $Z_g = Y(t)C$*, where* C *is a real* 2×2 *constant matrix.*

Now, we consider the particular solution

$$Z_0(t) = X(t)\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ ce^{-\frac{t}{2}} & 0 \end{pmatrix}$$

of (1.3), where $c \neq 0$*. It is easy to see that this solution is* $\Psi-$ *bounded on* $[1,\infty)$.

From Theorem, there exists a $\Psi-$*bounded solution* $Z(t)$ *of Lyapunov matrix differential equation (1.4) such that (3.5) holds. We can take*

$$Z(t) = Y(t)\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \begin{pmatrix} c_1v(t) + c_3u(t) & c_2v(t) + c_4u(t) \\ c_1e^{-\frac{t}{2}} & c_2e^{-\frac{t}{2}} \end{pmatrix}$$

and then,

$$\Psi(t)Z(t) = \begin{pmatrix} \frac{1}{2}c_1v(t) + \frac{1}{2}c_3u(t) & \frac{1}{2}c_2v(t) + \frac{1}{2}c_4u(t) \\ c_1 & c_2 \end{pmatrix}.$$

Since v(t) is unbounded (see in [5], pp. 71) and u(t) is bounded on $[1,\infty)$*, the solution Z(t) is* $\Psi-$ *bounded on* $[1,\infty)$ *iff* $c_1 = c_2 = 0$*. In this case,*

$$\Psi(t)\left(Z(t) - Z_0(t)\right) = \begin{pmatrix} \frac{1}{2}c_3u(t) & \frac{1}{2}c_4u(t) \\ -c & 0 \end{pmatrix}.$$

But $c \neq 0$*, it being impossible to make* $\lim\limits_{t\to\infty} \mid \Psi(t)\left(Z(t) - Z_0(t)\right) \mid = 0$.

This proves the assertion.

Remark 3.6. *This Example shows that 1) is essential hypothesis in Theorem.*

Remark 3.7. *Theorem 3.3 generalizes Theorem 3.1, from matrix differential equations to Lyapunov matrix differential equations.*

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