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Introducing n-sequences and study of their topological properties

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Abstract

In this article, the concept of n-sequences in topological spaces has been introduced which is new to the existing literature. We study the sequential limit aspect of statistical convergence for such sequences. Besides, the notions of subsequences, limit points, statistical limits points, and statistical cluster points in topological spaces have been given for n-sequences. This material contains a detailed explanation of inclusion relations between these points spaces and their basic properties. We also introduce s_n and s_n^*-convergent spaces and discuss some of their properties.

Keywords: n-sequence, statistical convergence, statistical limit point, statistical cluster point, topological space
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1 Introduction and Preliminaries

Fast [1] and Steinhaus [3] introduced the idea of statistical convergence and several other visualized different approaches of this concept, e.g., sequential limit, summability method etc., (for reference see [1], [2], [3]). But previously most of the work in this domain was done on real or complex number system. Kolk [4] was the first to study it on Banach spaces which was further developed in topological spaces and established the topological notion of this convergence [5]. In other direction, some authors also explored this idea in double and triple sequences and some other spaces (see [6], [7], [8], [9], [14] , [15], [16], [18] and references therein).

However, the literature still lacks the concept in the setting of n-sequences. Before proceeding further we give the definition of n-sequence.

Definition 1.1. An n-sequence is a function whose domain is either $\mathbb{N}^n$ or subset of $\mathbb{N}^n$.

In the sequel, we take $\mathbb{N}^n$ as an ordered set, it can easily be proven by using the lexicographical order on $\mathbb{N}^n$, i.e., we compare it component-wise.

For the sake of completeness, we recall the definition of statistical convergence in a topological space.

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Definition 1.2. [10] A sequence $(x_n), n \in \mathbb{N}$ in a topological space X is said to be converge statistically (s-convergent) to $x \in X$, if for every neighbourhood U of $x, \delta(\{n \in \mathbb{N} : x_n \notin U\}) = 0$. We write $x = s\text{-}\lim x_n$.

If $X = \mathbb{R}$, then one can say that above definition is equivalent to the following statement:

A topological space X is s-convergent to $x \in X$, if there exists a subset A of $\mathbb{N}$ with $\delta(A) = 1$ such that the sequence (x_n) in A converges to x, i.e., for every neighbourhood V of x, there exists $n_0 \in \mathbb{N}$ such that $x_n \in V$ for every $n \geq n_0$.

This fact leads to a different type of convergence in topological space.

Definition 1.3. A sequence $(x_n), n \in \mathbb{N}$ in a topological space X is said to be s_*- convergence to $x \in X$, if there is $A \subset \mathbb{N}$ with $\delta(A) = 1$ such that $\lim\limits_{m\to\infty, m\in A} x_m = x$. We write $x = s_*\text{-}\lim x_n$.

The incentive of this paper is the realization of remarkable application of n-sequence in different sectors such as physical, computer and biological sciences. It could help in the study of multiple sequence alignment and protein topology. Multiple sequence alignments are important in many application including tree elements, secondary structure prediction and critical residue identification.

In this paper, we redefine the concept of s and s_*-convergence for n-sequences in a topological space. We give some inclusion relation and necessary and sufficient condition for its convergence. Also, we introduce n-statistical limit and cluster points and discuss their inclusion relations and some of their sequential properties. We exemplify our study to explain the theory better. The topological terminologies used in this paper have been taken from [11].

2 Statistical n-limit and n-cluster Point spaces

Fridy [12] introduced statistical convergence of real numbers. Some authors also defined it in topological spaces. Here we introduce limit points, statistical limit points and cluster points of n-sequence in a topological space and establish some inclusion relation among them. We also discuss some important topological properties of n-sequence space. But first we give the notations which we will be using throughout the paper. The asymptotic (or natural) density $d(A)$ for sets A of natural numbers is a central tool in number theory:

$$d(A) = \lim_{n\to\infty} \frac{|A \cap [1,n]|}{n}, (provided\ the\ limit\ exists).$$

Notations: The topological terminologies used in this paper have been taken from [11]. Throughout the paper, our topological space is a Hausdorff space. It is to be noted that the work is in n-sequences. So from the definition of n-sequence, in the sequence $(x_{i_1,i_2,...,i_n})$, $i_1, i_2, ..., i_n \in \mathbb{N}$, unless otherwise stated.

Definition 2.1. Let X be a topological space. A sequence $x = (x_{i_1,i_2,...,i_n})$, $i_1, i_2, ..., i_n \in \mathbb{N}$, in X is said to have a limit point L if each neighbourhood of L contains infinite number of member of x. The set of all limit point of x is denoted by $L_n(x)$.

On merging statistical concept and above definition, we intend to define statistical limit point. But for that, we need to define n-dimensional subsequence and upper density.

Definition 2.2. Let $A \subseteq \mathbb{N}^n$, for $i_1, i_2, ..., i_n \in \mathbb{N}$ set

$$A(i_1, i_2, ..., i_n) = \{(k_1, k_2, ..., k_n) \in \mathbb{N}^n : k_1 \leq i_1, k_2 \leq i_2, ..., k_n \leq i_n\}.$$

We define upper n-density of A as follows:

$$\bar{\delta_n}(A) = \limsup_{i_1,i_2,...,i_n} \frac{|A(i_1, i_2, ..., i_n)|}{\prod\limits_{j=1}^{n} i_j} = \limsup_{i_1,i_2,...,i_n} \frac{|A \cap \{[1, i_1] \times [1, i_2] \times ... \times [1, i_n]\}|}{\prod\limits_{j=1}^{n} i_j}.$$

The lower n-density of A can be defined in a similar manner as of upper density by taking lim inf instead of lim sup. We denote it as $\underline{\delta}_n(A)$. If limits exist and $\bar{\delta}_n(A) = \underline{\delta}_n(A)$ then n-density of A is defined as

$$\delta_n(A) = \lim_{i_1,i_2,...,i_n} \frac{|A(i_1, i_2, ..., i_n)|}{\prod\limits_{j=1}^{n} i_j}.$$

Example 2.3. Consider the set $A = \{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : i_j \text{ is even}, j = 1, 2, ..., n\}$. Then upper n-density of A will be

$$\bar{\delta_n}(A) = \frac{1}{2^n}.$$

Definition 2.4. Let $x = (x_{i_1,i_2,...,i_n})$, $i_1, i_2, ..., i_n \in \mathbb{N}$ be an n-sequence. Choose

$$K = \{((m_1)_{i_1}, (m_2)_{i_2}, ..., (m_n)_{i_n}) \in \mathbb{N}^n : (i_1, i_2, ..., i_n) \in \mathbb{N}^n\},$$

such that K is an strictly increasing subset of $\mathbb{N}^n$ and $\bar{\delta}_n(K) > 0$. Then the sequence $(x_{(m_1)_{i_1},(m_2)_{i_2},...,(m_n)_{i_n}})$ or $(x_{i_1,i_2,...,i_n})_{(i_1,i_2,...,i_n)\in K}$ is called a subsequence of x.

Example 2.5. Let $x = (x_{i_1,i_2,...,i_n})$, $i_1, i_2, ..., i_n \in \mathbb{N}$, be an n-sequence. Take

$$K = \{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : i_1 \text{ is even}\},$$

so that order of elements of K is same as of $\mathbb{N}^n$. Then $(x_{i_1,i_2,...,i_n})$, $i_1, i_2, ..., i_n \in K$ is a subsequence of x.

Definition 2.6. A sequence $x = (x_{i_1,i_2,...,i_n})$, $i_1, i_2, ..., i_n \in \mathbb{N}$, is said to have a statistical limit point L, if $\exists$ a subsequence K of $\mathbb{N}^n$, $\delta_n(K) \neq 0$ (i.e. its upper density is positive or density does not exist) such that

$$\lim_{(i_1,i_2,...,i_n)\in K} x_{i_1,i_2,...,i_n} = L.$$

The set of statistical limit points of x is denoted by $\wedge_n(x)$.

Example 2.7. Let $X = \mathbb{R}$ with the usual topology. Define

$$x_{i_1,i_2,...,i_n} = \begin{cases} 1, \text{if } i_j \text{ is prime}, j = 1, 2, ..., n; \\ 2, \ otherwise. \end{cases}$$

Then, $L_n(x) = \{1, 2\}$ and $\wedge_n(x) = \{2\}$. From Definition 2.1 & 2.6 it is clear that $\wedge_n(x) \subseteq L_n(x)$. Now, we give examples which shows that these two could be same or very different.

Example 2.8. Consider $X = \mathbb{R}^n$ with Euclidean topology. Define

$$x_{i_1,i_2,...,i_n} = \begin{cases} (i_1, 0, ..., 0), \text{if } i_j \text{ is prime}, j = 1, 2, ..., n; \\ (i_1, i_2, ..., i_n), \ otherwise. \end{cases}$$

Then, $\wedge_n(x) = \phi$ and $L_n(x) = \phi$.

Now take $\mathbb{Q}_n = \{((r_1)_{i_1}, (r_2)_{i_2}, ..., (r_n)_{i_n}) : (i_1, i_2, ..., i_n) \in \mathbb{N}^n\}$ as an increasing subset of $\mathbb{R}^n$ and define

$$x_{i_1,i_2,...,i_n} = \begin{cases} ((r_1)_{i_1}, (r_2)_{i_2}, ..., (r_n)_{i_n}), \text{if } i_j \text{ is prime}, j = 1, 2, ..., n; \\ (i_1, i_2, ..., i_n), \ otherwise. \end{cases}$$

Then, $\wedge_n(x) = \varnothing$, but $L_n(x) = \mathbb{R}^n$, because $\mathbb{Q}^n$ is a dense subset of $\mathbb{R}^n$.

Definition 2.9. Let $x = (x_{i_1,i_2,...,i_n}) \in X$, $i_1, i_2, ..., i_n \in \mathbb{N}$. Then L is said to be a statistical cluster point of x if for each neighbourhood U of L,

$$\delta_n(\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : x_{i_1,i_2,...,i_n} \in U\}) \neq 0.$$

The set of all statistical cluster point of x is denoted by $\Gamma_n(x)$.

Consider Example 2.7, $\Gamma_n(x) = \{2\}$. It is clear that $\Gamma_n(x) \subseteq L_n(x)$. It may seem that statistical limit points and statistical cluster points are equivalent, but its not the case.

Example 2.10. Consider the sets $A_j = \{(2^{j-1}i_1, i_2, ..., i_n) \in \mathbb{N}^n : i_1 \text{ is odd}\},\ j \in \mathbb{N}$. Then $\delta_n(A_j) = \dfrac{1}{2^j}$. Let

$$x_{i_1,i_2,...,i_n} = \begin{cases} \frac{1}{j},\ (i_1, i_2, ..., i_n) \in A_j; \\ 1,\ otherwise. \end{cases}$$

Then,

$$L_n(x) = \{0\} \cup \left\{\frac{1}{j}\right\}, \wedge_n(x) = \left\{\frac{1}{j}\right\} \text{ and } \Gamma_n(x) = \{0\} \cup \left\{\frac{1}{j}\right\}; j \in \mathbb{N}.$$

Let X be a topological space. Then,

1. an F_σ set in X, is a set which can be written as countable union of closed subsets of X.

2. X is a Lindelöf space if every open cover in X has a countable subcover.

We now establish a theorem to support the inclusion relation between $\wedge_n(x)$ and $L_n(x)$.

Theorem 2.11. Let X be a topological space and $x = (x_{i_1,i_2,...,i_n}),\ i_1, i_2, ..., i_n \in \mathbb{N}$, be a sequence in X. Then, $\wedge_n(x) \subseteq \Gamma_n(x)$.

Proof . Suppose $L \in \wedge_n(x)$. Then there exists an increasing subset K of $\mathbb{N}^n$ such that

$$K = \{((m_1)_{i_1}, (m_2)_{i_2}, ..., (m_n)_{i_n}) : (i_1, i_2, ..., i_n) \in \mathbb{N}^n\}$$

with $\delta_n(K) \neq 0$ such that

$$\lim_{(i_1,i_2,...,i_n)\in K} x_{i_1,i_2,...,i_n} = L.$$

Then for each neighbourhood U of L,

$$\begin{aligned} &\delta_n(\{(i_1, i_2, ..., i_n) \in K : x_{i_1,i_2,...,i_n} \notin U\}) = 0 \\ &\qquad \Rightarrow \delta_n(\{(i_1, i_2, ..., i_n) \in K : x_{i_1,i_2,...,i_n} \in U\}) \neq 0 \\ &\qquad \Rightarrow \delta_n(\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : x_{i_1,i_2,...,i_n} \in U\}) \neq 0 \\ &\qquad \Rightarrow L \in \Gamma_n(x). \end{aligned}$$

This proved the theorem. □

Theorem 2.12. Let X be a topological space and $x = (x_{i_1,i_2,...,i_n}),\ i_1, i_2, ..., i_n \in \mathbb{N}$, be a sequence in X. Then the set $\Gamma_n(x)$ is a closed set.

Proof . Let L be a limit point of $\Gamma_n(x)$. Then from Definition 2.1, for a given neighbourhood U of L, U contains a point of $\Gamma_n(x)$, other than l, say L. Choose a neighbourhood V of L, so that $V \subseteq U$. Also, $L \in \Gamma_n(x)$. Therefore

$$\delta_n(\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : x_{i_1,i_2,...,i_n} \in V\}) \neq 0.$$

As $V \subseteq U$,

$$\delta_n(\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : x_{i_1,i_2,...,i_n} \in U\}) \neq 0.$$

Hence,

$$l \in \Gamma_n(x).$$

□

Theorem 2.13. Let X be a topological space and $x = (x_{i_1,i_2,...,i_n})$ and $y = (y_{i_1,i_2,...,i_n})$ are sequences in X such that $x_{i_1,i_2,...,i_n} = y_{i_1,i_2,...,i_n}$, for almost all $(i_1, i_2, ..., i_n) \in \mathbb{N}^n$. Then $\wedge_n(x) = \wedge_n(y)$ and $\Gamma_n(x) = \Gamma_n(y)$.

Proof . Let $L \in \wedge_n(x)$. Then by Definition 2.6, $\exists$ a subsequence K; $\delta_n(K) \neq 0$ such that

$$\lim_{(i_1,i_2,...,i_n)\in K} x_{i_1,i_2,...,i_n} = L.$$

Also, it is given that,

$$x_{i_1,i_2,...,i_n} = \ y_{i_1,i_2,...,i_n} \text{ for almost all } (i_1, i_2, ..., i_n) \in \mathbb{N}^n.$$

Therefore,

$$\delta_n(\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : x_{i_1,i_2,...,i_n} \neq \ y_{i_1,i_2,...,i_n}\}) = 0.$$

Clearly,

$$\delta_n(\{(i_1, i_2, ..., i_n) \in K : x_{i_1,i_2,...,i_n} \neq \ y_{i_1,i_2,...,i_n}\}) = 0,$$

or we can say that

$$\delta_n(\{(i_1, i_2, ..., i_n) \in K : x_{i_1,i_2,...,i_n} = \ y_{i_1,i_2,...,i_n}\}) \neq 0.$$

Since x is convergent to L in K, we get a subset K' of K such that $\delta_n(K') \neq 0$ and y is convergent to L in K'. Thus, $L \in \wedge_n(y)$. Hence, $\wedge_n(x) \subseteq \wedge_n(y)$. By symmetry, $\wedge_n(y) \subseteq \wedge_n(x)$. Therefore, $\wedge_n(x) = \wedge_n(y)$. In a similar way, one can show that $\Gamma_n(x) = \Gamma_n(y)$. □

Theorem 2.14. Let X be a hereditarily Lindelöf space and $x = (x_{i_1,i_2,...,i_n})$, $i_1, i_2, ..., i_n \in \mathbb{N}$, be a sequences in X, then there exists a sequence $y = (y_{i_1,i_2,...,i_n})$ in X such that $L_n(y) = \Gamma_n(x)$ and $x_{i_1,i_2,...,i_n} = \ y_{i_1,i_2,...,i_n}$, for almost all $(i_1, i_2, ..., i_n) \in \mathbb{N}^n$, also range of y is a subset of range of x.

Proof . We know that $\Gamma_n(x) \subseteq L_n(x)$, if equality holds then by previous theorem it is trivial that $L_n(y) = \Gamma_n(x)$.

Suppose $\Gamma_n(x)$ is a proper subset of $L_n(x)$. Then for each $\alpha \in L_n(x)\backslash\Gamma_n(x)$, we can choose a neighbourhood U_α of α such that

$$\delta_n(\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : x_{i_1,i_2,...,i_n} \in U_\alpha\}) = 0.$$

We collect all such U_α's and make an open cover $\{U_\alpha : \alpha \in L_n(x)\backslash\Gamma_n(x)\}$ of the set $L_n(x)\backslash\Gamma_n(x)$. Since X is a hereditarily Lindelöf space, we get a countable subcover $\{U_{\alpha_j} : j \in \mathbb{N}\}$ of the open cover. As $\alpha_j \in L_n(x)\backslash\Gamma_n(x)$, $j \in \mathbb{N}$, so for each neighbourhood U_{α_j}, we have a subsequence $(x_{i_1,i_2,...,i_n})_{(i_1,i_2,...,i_n)\in K^j}$ of $(x_{i_1,i_2,...,i_n})$ with $\delta_n(K^j) = 0$. Corollary 9 of [13] ensures that with the help of countable sets K^j, we can make a single set K in $\mathbb{N}^n$ such that $\delta_n(K) = 0$ and for each $j \in \mathbb{N}$, $\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : x_{i_1,i_2,...,i_n} \in U_{\alpha_j}\}\backslash K$ is a finite set.

Let $\mathbb{N}^n\backslash K = \{((m_1)_{i_1}, (m_2)_{i_2}, ..., (m_n)_{i_n}) : (i_1, i_2, ..., i_n) \in \mathbb{N}^n\}$. We define

$$y_{i_1,i_2,...,i_n} = \begin{cases} x_{((m_1)_{i_1},(m_2)_{i_2},...,(m_n)_{i_n}):(i_1,i_2,...,i_n)\in\mathbb{N}^n}, & \text{if } (i_1, i_2, ..., i_n) \in K; \\ x_{i_1,i_2,...,i_n}, & \text{if } (i_1, i_2, ..., i_n) \in \mathbb{N}^n\backslash K. \end{cases} \tag{2.1}$$

It is clear that

$$\delta_n(\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : y_{i_1,i_2,...,i_n} \neq x_{i_1,i_2,...,i_n}\}) = 0. \tag{2.2}$$

So by Theorem 2.13,

$$\Gamma_n(x) = \Gamma_n(y). \tag{2.3}$$

Since $\delta_n(K) = 0$, therefore the subsequence $(x_{((m_1)_{i_1},(m_2)_{i_2},...,(m_n)_{i_n})})$ of $(y_{i_1,i_2,...,i_n})$ has no accumulation point in $L_n(x)\backslash\Gamma_n(x)$ and thus no statistical limit point of $(y_{i_1,i_2,...,i_n})$. Therefore, $L_n(y) = \Gamma_n(y)$ and thus, $L_n(y) = \Gamma_n(x)$.

Equation (2.2) and (2.1) gives proof for the later part of the theorem.

□

Theorem 2.15. Let X be a first countable space and $x = (x_{i_1,i_2,...,i_n})$, $i_1, i_2, ..., i_n \in \mathbb{N}$, be a sequences in X. Then the set $\wedge_n(x)$ is an F_σ set.

Proof . For any $m \in \mathbb{N}$,

$$F_m = \left\{ l \in X : \exists K \subseteq \mathbb{N}^n \lim_{i_1,i_2,...,i_n \in K} x_{i_1,i_2,...,i_n} = l \ \& \ \delta_n(K) \geq \frac{1}{m} \right\},$$

where

$$K = \{((m_1)_{i_1}, (m_2)_{i_2}, ..., (m_n)_{i_n}) : (i_1, i_2, ..., i_n) \in \mathbb{N}^n\}.$$

It is clear that $\wedge_n(x) = \cup_{m=1}^{\infty} F_m$, so if we show that F_m is a closed set of X, then our work is done.

Let L be a limit point of F_m. Then for a given neighbourhood U of L, U contains a point of F_m, other than L.

Since X is a first countable space, so for each L in X, there exists sequence $N_1, N_2, ...$ of neighbourhood of L so that for any neighbourhood U of L, $\exists\ j$ such that $N_j \subset U$. So we get a sequence l_j in F_m converging to L.

As $l_j \in F_m$, so for every l_j we can choose a subsequence $A^{(j)} \subset \mathbb{N}^n$ so that

$$\lim_{i_1,i_2,...,i_n \in A^{(j)}} x_{i_1,i_2,...,i_n} = l_j \& \bar{\delta_n}(A^{(j)}) \geq \frac{1}{m}.$$

Now, choose a sequence ϵ_j of the real number converging to zero, then there is a sequence $(i_1, i_2, \ldots, i_n)_1 < (i_1, i_2, \ldots, i_n)_2 < \cdots$ in $\mathbb{N}^n$ such that

$$\frac{\left|A^{(j)} \cap \left\{\left((i_1)_{j-1},(i_1)_j\right] \times \left((i_2)_{j-1},(i_2)_j\right] \times ... \times \left((i_n)_{j-1},(i_n)_j\right]\right\}\right|}{\prod\limits_{k=1}^{n}(i_k)_j} \geq \frac{1}{m} - \epsilon_j,\ j \in \mathbb{N}.$$

If we take $A = \cup_{j=1}^{\infty}\{A^{(j)} \cap \{\left((i_1)_{j-1},(i_1)_j\right] \times \left((i_2)_{j-1},(i_2)_j\right] \times ... \times \left((i_n)_{j-1},(i_n)_j\right]\}\}$, then $\bar{\delta_n}(A) \geq \dfrac{1}{m}$.

We arrange A as increasing ordered set, let

$$A = \{(a_1, a_2, ..., a_n)_1 < (a_1, a_2, ..., a_n)_2 < ... < (a_1, a_2, ..., a_n)_r < ...\}.$$

As

$$\lim_{i_1,i_2,...,i_n} (x_{i_1,i_2,...,i_n})_{i_1,i_2,...,i_n \in A^{(j)}} = L_j,$$

so there are only finite number of L_j, say $L_1, L_2, ..., L_t$ which are not in U,

Also,

$$A \subset \cup\{A^j : j \geq t\} \subset \{i_1, i_2, ..., i_n \in \mathbb{N}^n : x_{i_1,i_2,...,i_n} \in U\} \backslash \{\text{finite set}\}.$$

So,

$$\lim_{r\to\infty} x_{(a_1,a_2,...,a_n)_r} = L.$$

Therefore, $L \in F_m$, that is F_m is closed. Hence, $\wedge_n(x)$ is an F_σ set in X. $\square$

Note 1. $hcld(X) = \omega$ denotes the fact that all closed subset of the space X are separable.[5]

Lemma 2.16. Let X be a topological space such that $hcld(X) = \omega$. Then for each closed set $F \subset X$, there exists a sequence $x = (x_{i_1,i_2,...,i_n})$, $i_1, i_2, ..., i_n \in \mathbb{N}$, in X such that $F = L_n(X)$.

Proof . As F is a closed subset of X, it is separable, so it has a countable dense subset of F, say, $M = \{m_j : j \in \mathbb{N}\}$. We decompose $\mathbb{N}^n$ into pairwise disjoint infinite sets $\mathbb{N}^n = \cup_{j\in\mathbb{N}} A^j$. Define $x = (x_{i_1,i_2,...,i_n})$ as:

$$x_{i_1,i_2,...,i_n} = m_j \text{ for each } (i_1, i_2, ..., i_n) \in A^j.$$

Let $L \in L_n(x)$ and U be a neighbourhood of L. So there are infinite $(i_1, i_2, ..., i_n) \in \mathbb{N}^n$ such that $x_{i_1,i_2,...,i_n} \in U$, thus infinite number of m_j in U. Therefore, L is a limit point of M. But $M \subset F$, so L is a limit point of F. Since F is closed, $L \in F$. We get $L_n(x) \subset F$.

Now suppose $L \in F$ and V be a neighbourhood of L. Take an m_k in V. From the definition of the sequence x, it is clear that there are infinite number of $(i_1, i_2, ..., i_n)$ in $\mathbb{N}^n$, so that $x_{i_1,i_2,...,i_n} \in V$. Thus, $L \in L_n(x)$, hence $F \subset L_n(x)$. $\square$

Theorem 2.17. Let X be a topological space such that $hcld(X) = \omega$ and $M \subset X$ is an arbitrary F_σ set. Then there exists a sequence $(x_{i_1,i_2,...,i_n})$, $i_1, i_2, ..., i_n \in \mathbb{N}$, in X such that $M = \wedge_n(x)$.

Proof . Since M is an F_σ set, we can write $M = \cup_{j\in\mathbb{N}} M_j$, where each M_j is a closed subset of X. From Lemma 2.16, each M_j contains a sequence say $(x_{(i_1,i_2,...,i_n)_j})$ so that $L_n(x_{(i_1,i_2,...,i_n)_j}) = M_j$. Decompose $\mathbb{N}^n = \cup_{j\in\mathbb{N}} A^j$ so that $\delta_n(A^j) = \frac{1}{2^j}$ and $\delta_n(\mathbb{N}^n \backslash \cup_{k=1}^{j} A^k) \to 0,\ as\ k \to \infty$ [see Example 2.10]. Further, we decompose each A^j into pairwise disjoint sets as $A^j = \cup_{k\in\mathbb{N}} B_k^j$, where $\bar{\delta}_n(B_k^j) = \frac{1}{2^j};\ j \in \mathbb{N}$. Now we define $x = (x_{i_1,i_2,...,i_n})$ as:

$x_{i_1,i_2,...,i_n} = x_{(i_1,i_2,...,i_n)^j_k}$ for each $i_1, i_2, ..., i_n \in B^j_k$. We show $M = \wedge_n(x)$.

Suppose there is a subsequence $(x_{(i_1,i_2,...,i_n)_i})$ so that $\lim\limits_{i\in\mathbb{N}} x_{(i_1,i_2,...,i_n)_i} = L$, $L \notin M$.

Then L's neighbourhood contains only finitely many members of subsequence in M. Therefore for every j, $\cup_{k=1}^{j} M_k$ also contains finitely many $(i_1, i_2, ..., i_n)_1, (i_1, i_2, ..., i_n)_2, ..., (i_1, i_2, ..., i_n)_m$. Thus, $L \notin \wedge_n(x)$.

Further suppose $l \in M$, then there exists $j \in \mathbb{N}$ such that $l \in M_j$. If subsequence $\left(x_{(i_1,i_2,...,i_n)^j_{i_k}}\right)$ of $\left(x_{(i_1,i_2,...,i_n)^j_k}\right)$ converges to l, then for any $0 < \epsilon < \frac{1}{2^j}$, there exists a sequence $(s_1, s_2, ..., s_n)_1 < (s_1, s_2, ..., s_n)_2 < ...$ in $\mathbb{N}^n$ such that

$$\frac{\left|B^j_{i,k} \cap \left\{\left((s_1)_{i-1}, (s_1)_i\right] \times \left((s_2)_{i-1}, (s_2)_i\right] \times ... \times \left((s_n)_{i-1}, (s_n)_i\right]\right\}\right|}{\prod\limits_{m=1}^{n} (s_m)_i} \geq \frac{1}{2^j} - \epsilon.$$

Put $B = \cup_{i=1}^{\infty} \left\{B^j_{i,k} \cap \left\{\left((s_1)_{i-1}, (s_1)_i\right] \times \left((s_2)_{i-1}, (s_2)_i\right] \times ... \times \left((s_n)_{i-1}, (s_n)_i\right]\right\}\right\}$.

Then, $\delta_n(B) \geq \frac{1}{2^j} - \epsilon$. Therefore, $\lim\limits_{i_1,i_2,...,i_n} (x_{i_1,i_2,...,i_n})_{(i_1,i_2,...,i_n)\in B} = l$. Hence, $l \in \wedge_n(x)$. □

Theorem 2.18. Let X be a topological space such that $hcld(X) = \omega$. Then for each closed set $F \subset X$, there exists a sequence $x = (x_{i_1,i_2,...,i_n})$, $i_1, i_2, ..., i_n \in \mathbb{N}$, in X such that $F = \Gamma_n(X)$.

Proof: The proof can be done in a similar way of Lemma 2.16 and Theorem 2.17.

3 s^*_n convergence and its basic Properties

In this section, we define some new type of n-convergence in topological space and discuss their basic properties.

Definition 3.1. Let X be a topological space. An n-sequence $x = (x_{i_1,i_2,...,i_n})$, $i_1, i_2, ..., i_n \in \mathbb{N}$, in X converges to $L \in X$ in Pringsheim sense, if for every open set $U \subseteq X$ containing L, there exists an $n_0 \in \mathbb{N}$ such that

$$x_{i_1,i_2,...,i_n} \in U, \ \forall i_1, i_2, ..., i_n \geq n_0.$$

Then we write $\lim\limits_{i_1,i_2,...,i_n} x_{i_1,i_2,...,i_n} = L$.

By limit of n-sequence we mean limit in Pringsheim sense. For the sake of simplicity we will just call it lim instead of P-lim.

Definition 3.2. Let $x = (x_{i_1,i_2,...,i_n})$, $i_1, i_2, ..., i_n \in \mathbb{N}$, be an n-sequence in X. Then x is said to converge statistically or s_n-converge to $L \in X$, if for every neighbourhood U of L,

$$\delta_n(\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : x_{i_1,i_2,...,i_n} \notin U\}) = 0.$$

We denote it as s_n-$\lim x = L$.

Remark 3.3. If we take $X = \mathbb{R}$ and extend the result of [10] mentioned above in the Definition 1.2, then we can say that above definition is equivalent to the following statement:

$(x_{i_1,i_2,...,i_n})$ is s_n-convergent to $L \in X$, if there exists a subset A of $\mathbb{N}^n$ with $\delta_n(A) = 1$ such that the sequence $(x_{i_1,i_2,...,i_n})$ in A is convergent to L, i.e., for every neighbourhood V of L there is a number $n_0 \in \mathbb{N}$, such that $x_{i_1,i_2,...,i_n} \in V$ for every $i_1, i_2, ..., i_n \geq n_0$.

We use this phenomenon to extend Definition 1.3 to n-sequences.

Definition 3.4. Let $x = (x_{i_1,i_2,...,i_n}) \in X$; $i_1, i_2, ..., i_n \in \mathbb{N}$. Then we say that x is s^*_n-convergent to $L \in X$, if there exists $A \subset \mathbb{N}^n$, $\delta_n(A) = 1$ such that

$$\lim_{i_1,i_2,...,i_n} x = L, \ for \ (i_1, i_2, ..., i_n) \in A.$$

We denote it as s^*_n-$\lim x = L$.

Theorem 3.5. Let $x = (x_{i_1,i_2,...,i_n})$, $i_1, i_2, ..., i_n \in \mathbb{N}$, be a statistically convergent n-sequence in X. Then it has a unique s_n-limit.

Proof . Suppose $x = (x_{i_1,i_2,...,i_n})$ has two s_n-limit, say L and l. Then there exists two neighbourhood U and V of L and l, respectively, such that $U \cap V = \phi$ and

$$\delta_n(\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : x_{i_1,i_2,...,i_n} \in U^c\}) = 0;$$
$$\delta_n(\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : x_{i_1,i_2,...,i_n} \in V^c\}) = 0.$$

Now,

$$\begin{aligned}\delta_n(\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : x_{i_1,i_2,...,i_n} \in (U \cap V)^c\})&\\ =&\delta_n(\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : x_{i_1,i_2,...,i_n} \in (U^c \cup V^c)\})\\ \leq&\delta_n(\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : x_{i_1,i_2,...,i_n} \in U^c\})\\ &+ \delta_n(\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : x_{i_1,i_2,...,i_n} \in V^c\})\\ =&0.\end{aligned}$$

Therefore,

$$\delta_n(\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : x_{i_1,i_2,...,i_n} \in (U \cap V)\}) = 1,$$

which is a contradiction as $U \cap V = \phi$. □

Theorem 3.6. Let $x = (x_{i_1,i_2,...,i_n})$, $i_1, i_2, ..., i_n \in \mathbb{N}$, be an n-sequence in X. If $s_n^*\text{-}\lim\limits_{i_1,i_2,...,i_n} x = L$, then $s_n\text{-}\lim\limits_{i_1,i_2,...,i_n} x = L$.

Proof . Let U be a neighbourhood of L and $s_n^*\text{-}\lim x = L$, then there exists $A \subset \mathbb{N}^n$, $\delta_n(A) = 1$ and $n_0 = n_0(U)$ such that $(x_{i_1,i_2,...,i_n}) \in U$, for $i_1, i_2, ..., i_n \geq n_0$ and $(i_1, i_2, ..., i_n)$ in A.

Then,

$$\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : x_{i_1,i_2,...,i_n} \notin U\} \subset \{1, 2, ..., n_0\} \cup \{\mathbb{N}^n \backslash A\}.$$

Since $\delta_n(\{1, 2, ..., n_0\} \cup \{\mathbb{N}^n \backslash A\}) = 0$, $s_n\text{-}\lim\limits_{i_1,i_2,...,i_n} x = L$. □

Corollary 3.7. If $s_n^*\text{-}\lim x = L$, then L is unique.

The converse of Theorem 3.6 holds for first countable space.

Theorem 3.8. Let X be a first countable space. If a sequence $x = (x_{i_1,i_2,...,i_n})$, $i_1, i_2, ..., i_n \in \mathbb{N}$, in X is statistically convergent to L, then it is also s_n^*-convergent to L.

Proof . Fix a countable decreasing local base $U_1 \supset U_2 \supset ...$ at L and define the set

$$A_j = \{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : x_{i_1,i_2,...,i_n} \in U_j\}; j \in \mathbb{N}.$$

It is clear that $A_1 \supset A_2 \supset ...$ as $U_1 \supset U_2 \supset ...$.

Now we assert that $\delta_n(A_j) = 1$.

Since $(x_{i_1,i_2,...,i_n})$ is statistically convergent to L, for each $j \in \mathbb{N}$,

$$\delta_n(\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : x_{i_1,i_2,...,i_n} \notin U_j\}) = 0.$$

Hence, $\delta_n(A_j) = 1$.

Take an element $(i_1, i_2, ..., i_n) \in A_1$, corresponding to which we have a $k_1 = \max(i_1, i_2, ..., i_n)$ in $\mathbb{N}$. Now $\delta_n(A_2) = 1$, therefore we can choose an $(i_1, i_2, ..., i_n)$ in A_2, corresponding to which we have a $k_2 = \max(i_1, i_2, ..., i_n)$ so that $k_2 > k_1$ and for every $m \geq k_2$,

$$\frac{|A_2(m)|}{m} = \frac{|\{(i_1, i_2, ..., i_n) \in A_2 : \max(i_1, i_2, ..., i_n) \leq m\}|}{m} > \frac{1}{2}.$$

Similarly, we get $k_j's$ for every $A_j's$ such that $k_1 < k_2 < ...$ and for each $m \geq k_j$,

$$\frac{|A_j(m)|}{m} = \frac{|\{(i_1, i_2, ..., i_n) \in A_j : \max(i_1, i_2, ..., i_n) \leq m\}|}{m} > 1 - \frac{1}{j}.$$

Now we define a set $A \subset \mathbb{N}^n$ as follows:

If $\max(i_1, i_2, ..., i_n) \leq k_1$, then $(i_1, i_2, ..., i_n) \in A$; if $j \geq 1$ and $k_j < \max(i_1, i_2, ..., i_n) \leq k_{j+1}$, then $(i_1, i_2, ..., i_n) \in A$ if and only if $(i_1, i_2, ..., i_n) \in A_j$. Now if $m \in \mathbb{N}$ such that $k_j \leq m \leq k_{j+1}$, then

$$\frac{|A(m)|}{m} \geq \frac{|A_j(m)|}{m} > 1 - \frac{1}{j}.$$

Therefore, $\delta_n(A) = 1$.

To show $(x_{i_1,i_2,...,i_n})_{i_1,i_2,...,i_n \in A}$ is convergent to L. Let V be a neighbourhood of L, since X is a first countable space; $U_j \subset V$.

Suppose $(i_1, i_2, ..., i_n) \in A$, set $\max(i_1, i_2, ..., i_n) = m$. If $m \geq k_j$, then $\exists\ p \geq j$ with $k_p \leq m \leq k_{p+1}$; from the definition of A, $(i_1, i_2, ..., i_n) \in A_p$.

Hence $x_{i_1,i_2,...,i_n} \in U_p \subset U_j \subset V$, i.e. for $(i_1, i_2, ..., i_n) \in A$, $,x = (x_{i_1,i_2,...,i_n})$ is convergent to L.

Therefore, $(x_{i_1,i_2,...,i_n})$ is s_n^*-convergent to L. □

We now state some important and basic facts without proof, since these can be established using standard techniques. We are not giving proof as it is simple or direct from the single sequences.

1. A subsequence of a statistically convergent sequence need not be statistically convergent.

2. Let $A \subseteq \mathbb{N}^n$, then A is statistical dense if $\delta_n(A) = 1$.

3. Union and intersection of two statistically dense subsets in $\mathbb{N}^n$ are also statistically dense.

4. Let $(x_{i_1,i_2,...,i_n})$, $i_1, i_2, ..., i_n \in \mathbb{N}$, be an n-sequence and K is statistically dense in $\mathbb{N}^n$. Then subsequence $(x_{i_1,i_2,...,i_n})_{(i_1,i_2,...,i_n) \in K}$ is statistically dense in $(x_{i_1,i_2,...,i_n})$.

Theorem 3.9. An n-sequence $x = (x_{i_1,i_2,...,i_n})$, $i_1, i_2, ..., i_n \in \mathbb{N}$, is statistically convergent if and only if it has a statistically dense subsequence which is statistically convergent.

Proof . Suppose s_n^*- $\lim x = L$ and there exists $K \subseteq \mathbb{N}^n$, such that $(x_{i_1,i_2,...,i_n})_{(i_1,i_2,...,i_n) \in K}$ is statistically dense subsequence of x. If possible, suppose the subsequence is statistically divergent. Then for all $y \in X$, there exists a neighbourhood V of y such that $\delta_n(\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : x_{i_1,i_2,...,i_n} \notin V\}) \neq 0$.

Now,

$$\delta_n(\{(i_1, i_2, ..., i_n) \in \mathbb{N}^n : x_{i_1,i_2,...,i_n} \notin V\}) \geq \delta_n(\{(i_1, i_2, ..., i_n) \in K : x_{i_1,i_2,...,i_n} \notin V\}).$$

Then, $(x_{i_1,i_2,...,i_n})_{(i_1,i_2,...,i_n \in \mathbb{N}^n)}$ is statistically divergent, which is a contradiction. □

Theorem 3.10. Let s_n^*- $\lim x = L$. Then there exists a subsequence of x convergent to L.

Proof . The proof is simple and direct from the definition of s_n^*-convergent (Definition 3.4), so we skip it. □

4 Conclusion

As mentioned earlier the motivation of the paper was the importance and lack of research in n-dimensional spaces. This paper includes some basic yet important result of topological spaces in $\mathbb{N}^n$. The authors believe that there is immense scope for more sophisticated research in this area, and need further research. The statistical approach of the paper can also be applied in hyperspaces, open covers and selection principle, the interested reader can see [5].

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