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Notion of non-absolute family of spaces

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Abstract

The scenario of this article is to introduce the space $\mathfrak{R}_s^t(p,\Delta)$ based on a general Riesz sequence space. Its completeness property is derived and its linear isomorphism property with $\ell(p)$ is proved. The Köthe-dual property of the space $\mathfrak{R}_s^t(p,\Delta)$ is also derived. Furthermore, its basis is constructed and some characterization of infinite matrices are given.

Keywords: Sequence spaces, Köthe-duals, Infinite matrix
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1 Introduction

By $\Psi = \mathbb{C}^{\mathbb{N}_0}$, we denote the set of all real or complex-valued sequences, where $\mathbb{C}$ represents the complex field and

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\}, \qquad (\mathbb{N} := \{1, 2, 3, \cdots\}).$$

Each linear subspace of Ψ is known as a sequence space as can be seen in [11], [17], [24] and many others. Also, ℓ_∞, c, c_0 and $\ell(p)$denotes all bounded sequences, all convergent sequences and null sequences and p-absolutely convergent series, respectively.

For an infinite matrix $\mathcal{B} = (b_{ij})$ and $\varrho = (\varrho_j) \in \Psi$, then as in [3], [22], the $\mathcal{B}$-transform of ϱ is defined by

$$\mathcal{B}\varrho = \{(\mathcal{B}\varrho)_i\},$$

provided it exists $\forall\ i \in \mathbb{N}_0$, where

$$(\mathcal{B}\varrho)_i = \sum_{j=0}^{\infty} b_{ij}\varrho_j.$$

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The matrix domain for an infinite matrix $\mathcal{B} = (b_{ij})$ is defined as

$$\mathcal{G}_{\mathcal{B}} = \{\varrho = (\varrho_j) \in \Psi \ : \ \mathcal{B}\varrho \in \mathcal{G}\}. \tag{1.1}$$

In [14], the author has given the new techniques and introduced the spaces $U(\triangle)$ as follows

$$U(\triangle) = \{\mho = (\mho_j) \in \Omega : (\triangle\mho_j) \in U\}$$

for $U \in \{\ell_\infty, c, c_0\}$ and $\triangle\mho_j = \mho_j - \mho_{j-1}$.

Let (t_j) be sequence of positive numbers with $T_i = \sum_{j=0}^{i} t_j$ for $i \in \mathbb{N}_0$, then, from [2], we have

$$\mathfrak{R}^t(p, \Delta) = \left\{\varsigma = (\varsigma_j) \in \Psi : (\Delta\varsigma_j) \in r^t(p)\right\},$$

where $r^t(p)$ is given in [1] for $0 \leqq p_j \leqq \mathcal{H} < \infty$.

As in [19], we define the following:

$$\mathfrak{R}^t(p, s) = \left\{\rho = (\rho_j) \in \Psi : \sum_j \left|\frac{1}{T_j^{s+1}} \sum_{i=0}^{j} t_i \rho_i\right|^{p_j} < \infty\right\}.$$

2 The Riesz Sequence Space $\mathfrak{R}_s^t(p, \Delta)$

This section introduces the new space $\mathfrak{R}_s^t(p, \Delta)$, and prove that this space is a complete paranormed space. Also, we show it is linearly isomorphic to the space $\ell(p)$.

Following the investigations made by Dowlath and Hamid [4]-[5], Ganie *et al.* [6]-[9], Grosse-Erdmann [10], Jalal *et al.* [12]-[13], Lascarides [15], Naik and Tarry [18], Sheikh and Ganie [20]-[21], Talebi[23], Yeşilkayagil [25] we introduce the space $\mathfrak{R}_s^t(p, \Delta)$ as follows.

Definition 2.1. We define the space $\mathfrak{R}_s^t(p, \Delta)$ as

$$\mathfrak{R}_s^t(p, \Delta) = \left\{\rho = (\rho_j) \in \Psi : \sum_j \left|\frac{1}{T_j^{s+1}} \sum_{k=0}^{j} t_k \Delta\rho_k\right|^{p_j} < \infty\right\},$$

where

$$0 < p_j \leqq \mathcal{H} < \infty \qquad \text{and} \qquad s \geqq 0.$$

Using (1.1), the given space can be written as:

$$\mathfrak{R}_s^t(p, \Delta) = \{\ell(p)\}_{\mathfrak{R}_s^t(\Delta)}.$$

Define sequence $y = (y_j)$ as $\mathfrak{R}_s^t(\Delta)$-transform of a sequence $\rho = (\rho_j)$, that is,

$$y_j = \frac{1}{T_j^{s+1}} \sum_{k=0}^{j} t_k \Delta\rho_k. \tag{2.1}$$

Remark 2.2. Choosing $s = 0$ will yields what has been given in [2].

Theorem 2.3. The space $\mathfrak{R}_s^t(p, \Delta)$ is a complete linear metric space paranormed by h_Δ given by:

$$h_\Delta(\varsigma) = \left(\sum_k \left|\frac{1}{T_k^{s+1}} \sum_{j=0}^{k-1} (t_j - t_{j+1})\varsigma_j + \frac{t_k}{T_k^{s+1}}\varsigma_k\right|^{p_k}\right)^{\frac{1}{M}}.$$

Proof . The linearity of $\mathfrak{R}_s^t(p,\Delta)$ with respect to the co-ordinatewise addition and scalar multiplication follows from the following inequalities which are satisfied for $\varsigma,\tau \in \mathfrak{R}_s^t(p,\Delta)$:

$$\left(\sum_k \left|\frac{1}{T_k^{s+1}}\sum_{j=0}^{k-1}(t_j-t_{j+1})(\varsigma_j+\tau_j)+\frac{t_k}{t_k}(\varsigma_k+\tau_k)\right|^{p_k}\right)^{\frac{1}{M}}$$
$$\leqq \left(\sum_k \left|\frac{1}{T_k^{s+1}}\sum_{j=0}^{k-1}(t_j-t_{j+1})\varsigma_j+\frac{t_k}{T_k^{s+1}}\varsigma_k\right|^{p_k}\right)^{\frac{1}{M}}$$
$$+\left(\sum_k \left|\frac{1}{T_k^{s+1}}\sum_{j=0}^{k-1}(t_j-t_{j+1})\tau_j+\frac{t_k}{T_k^{s+1}}\tau_k\right|^{p_k}\right)^{\frac{1}{M}} \tag{2.2}$$

and, for any $\alpha \in \mathbb{R}$ (see [16]), we have

$$|\alpha|^{p_k} \leqq \max(1,|\alpha|^M). \tag{2.3}$$

For $\theta=(0,0,0,\cdots)$, we have

$$h_\Delta(\theta)=0 \qquad \text{and} \qquad h_\Delta(\varsigma)=h_\Delta(-\varsigma)$$

for all $\varsigma \in \mathfrak{R}_s^t(p,\Delta)$). Also, the inequalities (3) and (4) give the subadditivity of h_Δ and

$$h_\Delta(\alpha\varsigma) \leqq \max(1,|\alpha|)h_\Delta(\varsigma).$$

Let $\{\varsigma^n\}$ be sequence of points of $\mathfrak{R}_s^t(p,\Delta)$ such that $h_\Delta(\varsigma^n-\varsigma)\to 0$ and let (α_n) be sequence of scalars such that $\alpha_n\to\alpha$. Then $\{h_\Delta(\varsigma^n)\}$ is bounded, since, by subadditivity, the following inequality:

$$h_\Delta(\varsigma^n) \le h_\Delta(\varsigma)+h_\Delta(\varsigma^n-\varsigma)$$

holds. Thus,

$$h_\Delta(\alpha_n\varsigma^n-\alpha\varsigma)=\left(\sum_k \left|\frac{1}{T_k^{s+1}}\sum_{j=0}^{k}(t_j-t_{j+1})(\alpha_n\varsigma_j^n-\alpha\varsigma_j)\right|^{p_k}\right)^{\frac{1}{M}}$$
$$\leqq |\alpha_n-\alpha|^{\frac{1}{M}}\, h_\Delta(\varsigma^n)+|\alpha|^{\frac{1}{M}}\, h_\Delta(\varsigma^n-\varsigma)$$

which tends to zero as $n\to\infty$. Hence, the continuity of scalar multiplication is established, so h_Δ is a paranorm on $\mathfrak{R}_s^t(p,\Delta)$.

Now, to prove its completeness property, choose $\{\varsigma^j\}$ as a Cauchy sequence in $\mathfrak{R}_s^t(p,\Delta)$, where

$$\varsigma^i=\{\varsigma_0^i,\varsigma_1^i,\cdots\}.$$

Hence, for a given $\epsilon>0$, there exists a positive integer $n_0(\epsilon)$ such that

$$h_\Delta(\varsigma^i-\varsigma^j)<\epsilon \tag{2.4}$$

for all $i,j \geqq n_0(\epsilon)$. Definition of h_Δ for each fixed $k\in\mathbb{N}_0$ yields

$$\left|(\mathfrak{R}_s^q\Delta\varsigma^i)_k-(\mathfrak{R}_s^q\Delta\varsigma^j)_k\right| \leqq \left(\sum_k \left|(\mathfrak{R}_s^q\Delta\varsigma^i)_k-(\mathfrak{R}_s^t\Delta\varsigma^j)_k\right|^{p_k}\right)^{\frac{1}{M}}<\epsilon$$

for $i,j\geqq n_0(\epsilon)$. Consequently, $\{(\mathfrak{R}_s^q\Delta\varsigma^0)_k,(\mathfrak{R}_s^q\Delta\varsigma^1)_k,\cdots\}$ is a Cauchy sequence of real numbers for every fixed $k\in\mathbb{N}_0$. But $\mathbb{R}$ being complete, it converges as follows:

$$(\mathfrak{R}_s^t\Delta\varsigma^i)_k \to (\mathfrak{R}_s^q\Delta\varsigma)_k \qquad (i\to\infty).$$

Using these infinitely many limits $(\mathfrak{R}_s^q\Delta\varsigma)_0, (\mathfrak{R}_s^q\Delta\varsigma)_1, \cdots$, we define the sequences $\{(\mathfrak{R}_s^q\Delta\varsigma)_0\}$ and $(\mathfrak{R}_s^q\Delta\varsigma)_1, \cdots\}$. From (2.3), then for each $m \in \mathbb{N}_0$ and $i, j \geqq n_0(\epsilon)$ that

$$\sum_{k=0}^{m} \left|(\mathfrak{R}_s^q\Delta\varsigma^i)_k - (\mathfrak{R}_s^q\Delta\varsigma^j)_k\right|^{p_k} \leqq h_\Delta(\varsigma^i - \varsigma^j)^M < \epsilon^M. \tag{2.5}$$

Take any $i, j \geqq n_0(\epsilon)$. First, if we let $j \to \infty$ in (2.5) and then $m \to \infty$, we obtain

$$h_\Delta(\varsigma^i - \varsigma) \leqq \epsilon.$$

Finally, taking $\epsilon = 1$ in (2.5) and letting $i \geqq n_0(1)$, we see, for each $m \in \mathbb{N}_0$ and by using Minkowski's inequality, that

$$\left(\sum_{k=0}^{m} \left|(\mathfrak{R}^t\varsigma)_k\right|^{p_k}\right)^{\frac{1}{M}} \leqq h_\Delta(\varsigma^i - \varsigma) + h_\Delta(\varsigma^i) \leq 1 + h_\Delta(\varsigma^i),$$

which shows that $\varsigma \in \mathfrak{R}_s^t(p,\Delta)$. Since $h_\Delta(\varsigma - \varsigma^i) \leqq \epsilon \ \forall i \geqq n_0(\epsilon)$, it follows that $\varsigma^i \to \varsigma$ as $i \to \infty$. Hence $\mathfrak{R}_s^t(p,\Delta)$ is complete. □

Clearly, the absoluteness property is not satisfied on the spaces $\mathfrak{R}_s^t(p,\Delta))$, which means $h_\Delta(x) \neq h_\Delta(|x|)$ for at least one sequence in $\mathfrak{R}_s^t(p,\Delta)$ and hence $\mathfrak{R}_s^t(p,\Delta)$ is a sequence space of non-absolute type.

Theorem 2.3. *The space $\mathfrak{R}_s^t(p,\Delta)$ is linearly isomorphic to the space $\ell(p)$, where $0 < p_k \leqq \mathcal{H} < \infty$.*

Proof . In order to establish the result, we must show the existence of a linear bijection between the spaces $\mathfrak{R}_s^t(p,\Delta))$ and $l(p)$, where $0 < p_k \leq \mathcal{H} < \infty$. With the notation of (2.1), define the transformation $\mathcal{G}$ from $\mathfrak{R}_s^t(p,\Delta))$ to $\ell(p)$ by $x \to y = \mathcal{G}x$. The linearity of $\mathcal{G}$ is trivial. Further, it is obvious that $x = \theta$ whenever $\mathcal{G}x = \theta$ and hence that $\mathcal{G}$ is injective.

Let $\xi \in \ell(p)$ and define the sequence $\zeta = (\zeta_k)$ by

$$\varrho_k = \sum_{n=0}^{k-1} \left(\frac{1}{t_n} - \frac{1}{t_{n+1}}\right) T_k^{s+1}\xi_k + \frac{T_k^{s+1}}{t_k}\xi_k,$$

for $k \in \mathbb{N}_0$. Then

$$\begin{aligned} h_\Delta(\zeta) &= \left(\sum_k \left|\frac{1}{T_k^{s+1}}\sum_{j=0}^{k-1}(t_j - t_{j+1})\zeta_j + \frac{t_k}{T_k^{s+1}}\zeta_k\right|^{p_k}\right)^{\frac{1}{M}} \\ &= \left(\sum_k \left|\sum_{j=0}^{k}\delta_{kj}\xi_j\right|^{p_k}\right)^{\frac{1}{M}} \\ &= \left(\sum_k |\xi_k|^{p_k}\right)^{\frac{1}{M}} = \mathfrak{H}_1(\xi) < \infty, \end{aligned}$$

where

$$\delta_{kj} = \begin{cases} 1 & (k = j) \\ 0 & (k \neq j). \end{cases}$$

Thus, we have $x \in \mathfrak{R}_s^t(p,\Delta)$. Consequently, $\mathcal{G}$ is surjective and is paranorm-preserving. Hence, clearly, $\mathcal{G}$ is a linear bijection. Consequently, the spaces $\mathfrak{R}_s^t(p,\Delta)$ and $\ell(p)$ are linearly isomorphic. □

3 Basis and α-, β- and γ-Duals of the Space $\mathfrak{R}_s^t(p,\Delta)$

In this section, we compute α , β- and γ-duals of the space $\mathfrak{R}_s^t(p,\Delta)$ and determine its basis..

For the sequence spaces Υ and Φ, define the following set:

$$\mathfrak{S}(\Upsilon,\Phi)=\{\nu=(\nu_k):\zeta\nu=(\zeta_k\nu_k)\in\Phi\ \forall\ \zeta\in\Upsilon\}. \tag{3.1}$$

By the representation of (3.1), we may define the α-, β- and γ- duals of a sequence space Υ, respectively, denoted by Υ^α, Υ^β and Υ^γ, and are defined by

$$\Upsilon^\alpha=\mathfrak{S}(\Upsilon,l_1),\ \ \Upsilon^\beta=\mathfrak{S}(\Upsilon,cs)\quad\text{and}\quad\Upsilon^\gamma=\mathfrak{S}(\Upsilon,bs).$$

If a sequence space Λ paranormed by h contains a sequence (b_n) with the property that, for every $\zeta\in\Lambda$, there is a unique sequence of scalars (α_n) such that

$$\lim_n h(\zeta-\sum_{k=0}^{n}\alpha_k b_k)=0,$$

then (b_n) is called a Schauder basis (or, briefly, basis) for Λ. The series $\sum\alpha_k b_k$ which has the sum ζ is then called the expansion of ζ with respect to (b_n) and written as $\zeta=\sum\alpha_k b_k$.

We now state the following lemmas which are needed in proving our theorems.

Lemma 3.1. (see [10])
(i) *Let $1<p_k\leqq\mathcal{H}<\infty$. Then $A\in(\ell(p),\ell_1)$ if and only if there exists an integer $B>1$ such that*

$$\sup_{K\in\mathfrak{F}}\sum_k\left|\sum_{n\in K}a_{nk}B^{-1}\right|^{p_k'}<\infty.$$

(ii) *Let $0<p_k\leqq 1$. Then $A\in(\ell(p),\ell_1)$ if and only if*

$$\sup_{K\in\mathfrak{F}}\sup_k\left|\sum_{n\in K}a_{nk}B^{-1}\right|^{p_k}<\infty.$$

Lemma 3.2. (see [16])
(i) *Let $1<p_k\leqq\mathcal{H}<\infty$. Then $A\in(\ell(p),\ell_\infty)$ if and only if there exists an integer $B>1$ such that*

$$\sup_n\sum_k|a_{nk}B^{-1}|^{p_k'}<\infty. \tag{3.2}$$

(ii) *Let $0<p_k\leqq 1$ for every $k\in\mathbb{N}_0$. Then $A\in(\ell(p):\ell_\infty)$ if and only if*

$$\sup_{n,k}|a_{nk}|^{p_k}<\infty. \tag{3.3}$$

Lemma 3.3. (see [16]) *Let $0<p_k\leqq\mathcal{H}<\infty$ for every $k\in\mathbb{N}_0$. Then $A\in(\ell(p):c)$ if and only if (3.2) and (3.3) holds true and*

$$\lim_n a_{nk}=\beta_k\ for\ k\in\mathbb{N}_0 \tag{3.4}$$

also holds true.

Theorem 3.4. *Let $1 < p_k \leqq \mathcal{H} < \infty$ for every $k \in \mathbb{N}_0$. Define the sets $D_1^s(p)$ and $D_2^s(p)$ as follows*:

$$D_1^s(p) = \bigcup_{B>1} \left\{ a = (a_k) : \sup_{K\in\mathfrak{F}} \sum_k \left| \sum_{n\in K} \left(\frac{1}{t_k} - \frac{1}{t_{k+1}}\right) a_n T_k^{s+1} + \frac{a_n}{T_n} Q_n^{s+1} B^{-1} \right|^{p_k'} < \infty \right\}$$

and

$$D_2^s(p) = \bigcup_{B>1} \left\{ a = (a_k) : \sum_k \left| \left[\left(\frac{a_k}{t_k} + \left(\frac{1}{t_k} - \frac{1}{t_{k+1}}\right) \sum_{i=k+1}^{n} a_i \right) T_k^{s+1} \right] B^{-1} \right|^{p_k'} < \infty \right\}.$$

Then

$$\left[\mathfrak{R}_s^t(p,\Delta)\right]^\alpha = D_1^s(p) \quad \text{and} \quad \left[\mathfrak{R}_s^t(p,\Delta)\right]^\beta = D_2^s(p) \cap \; cs = \left[\mathfrak{R}_s^t(p,\Delta)\right]^\gamma.$$

Proof . Choose $a = (a_k) \in \Psi$. We can easily derive with (2.1) that

$$a_n \varsigma_n = \sum_{k=0}^{n-1} \left(\frac{1}{t_k} - \frac{1}{t_{k+1}}\right) a_n T_k^{s+1} y_k + \frac{a_n}{T_n} Q_n^{s+1} y_n = (\mathcal{C}y)_n \tag{3.5}$$

where $\mathcal{C} = \{c_{nk}\}$ is defined as follows:

$$c_{nk} = \begin{cases} \left(\frac{1}{t_k} - \frac{1}{t_{k+1}}\right) a_n T_k^{s+1} & (0 \leqq k \leqq n-1) \\ \frac{a_n}{T_n} Q_n^{s+1} & (k = n) \\ 0 & (k > n) \end{cases}$$

for all $n, k \in \mathbb{N}_0$. Thus we observe by combining (3.5) with Part (i) of Lemma 3.1 that

$$a\varsigma = (a_n \varsigma_n) \in \ell_1$$

whenever $\varsigma = (\varsigma_n) \in \mathfrak{R}_s^t(p,\Delta)$ if and only if $\mathcal{C}y \in \ell_1$ whenever $y \in \ell(p)$. This yields

$$\left[\mathfrak{R}_s^t(p,\Delta)\right]^\alpha = D_1^s(p).$$

We further consider the following equation:

$$\sum_{k=0}^{n} a_k \varsigma_k = \sum_{k=0}^{n} \left[\left(\frac{a_k}{t_k} + \left(\frac{1}{t_k} - \frac{1}{t_{k+1}}\right) \sum_{i=k+1}^{n} a_i \right) T_k^{s+1} \right] y_k = (\mathcal{D}y)_n, \tag{3.6}$$

where $\mathcal{D} = (d_{nk})$ is defined as follows:

$$d_{nk} = \begin{cases} \frac{a_k}{t_k} + \left(\frac{1}{t_k} - \frac{1}{t_{k+1}}\right) \sum_{i=k+1}^{n} a_i) T_k^{s+1} & (0 \leqq k \leqq n) \\ 0 & (k > n). \end{cases}$$

Thus we deduce from Lemma 3.3 with (3.6) that $ax = (a_n x_n) \in cs$ whenever $x = (x_n) \in \mathfrak{R}_s^t(p,\Delta)$ if and only if $\mathcal{D}y \in c$ whenever $y \in \ell(p)$. Therefore, we find from (8) that

$$\sum_k \left| \left[\left(\frac{a_k}{t_k} + \left(\frac{1}{t_k} - \frac{1}{t_{k+1}}\right) \sum_{i=k+1}^{n} a_i \right) T_k^{s+1} \right] B^{-1} \right|^{p_k'} < \infty \tag{3.7}$$

and $\lim_n d_{nk}$ exists and hence shows that

$$\left[\mathfrak{R}_s^t(p,\Delta)\right]^{\beta} = D_2^s(p) \cap\ cs.$$

As this, from Lemma 3.2 together with (3.6) that $a\varsigma = (a_k\varsigma_k) \in cs$ whenever $\varsigma = (\varsigma_n) \in \mathfrak{R}_s^t(p,\Delta)$ if and only if $\mathcal{D}y \in \ell_\infty$ whenever $y = (y_k) \in \ell(p)$. Therefore, we again obtain the condition (3.7) which means that

$$\left[\mathfrak{R}_s^t(p,\Delta)\right]^{\gamma} = D_2^s(p) \cap\ cs.$$

□

Theorem 3.5. *Let $0 < p_k \leqq 1$ for every $k \in \mathbb{N}_0$. Define the sets $D_3^s(p)$ and $D_4^s(p)$ as follows:*

$$D_3^s(p) = \left\{a \in \Psi : \sup_{K \in \mathfrak{F}} \sup_k \left|\sum_{n\in K} \left[\left(\frac{1}{t_k} - \frac{1}{t_{k+1}}\right) a_n T_k^{s+1} + \frac{a_n Q_n^{s+1}}{T_n}\right] B^{-1}\right|^{p_k} < \infty\right\}$$

and

$$D_4^s(p) = \left\{a \in \Psi : \sup_k \left|\left[\left(\frac{a_k}{t_k} + \left(\frac{1}{t_k} - \frac{1}{t_{k+1}}\right) \sum_{i=k+1}^{n} a_i\right) T_k^{s+1}\right] B^{-1}\right|^{p_k} < \infty\right\}.$$

Then

$$\left[\mathfrak{R}_s^t(p,\Delta)\right]^{\alpha} = D_3^s(p) \quad \text{and} \quad \left[\mathfrak{R}_s^t(p,\Delta)\right]^{\beta} = \left[\mathfrak{R}_s^t(p,\Delta)\right]^{\gamma} = D_4(p) \cap\ cs.$$

Proof . The proof follows by the similar technique as in the proof of Theorem 2.7 above by using second parts of Lemmas 3.1 and 3.2 instead of the first parts. So, we omit the details. □

Theorem 3.6. *Define the sequence $b^{(k)}(t) = \{b_n^{(k)}(t)\}$ of the elements of the space $\mathfrak{R}_s^t(p,\Delta))$ for every fixed $k \in \mathbb{N}_0$ by*

$$b_n^{(k)}(t) = \begin{cases} \left(\frac{1}{T_n} - \frac{1}{t_{n+1}}\right) T_n^{s+1} + \frac{T_k^{s+1}}{t_k}, & (0 \leqq n \leqq k-1) \\ 0 & (n > k-1). \end{cases}$$

Then the sequence $\{b^{(k)}(t)\}$ is a basis for the space $\mathfrak{R}_s^t(p,\Delta))$ and any $\varsigma \in \mathfrak{R}_s^t(p,\Delta))$ has a unique representation given by

$$\varsigma = \sum_k \lambda_k(t) b^{(k)}(t) \tag{3.8}$$

where $\lambda_k(t) = ((\mathfrak{R}_s^t \Delta\varsigma)_k$ for all $k \in \mathbb{N}_0$ and $0 < p_k \leqq \mathcal{H} < \infty$.

Proof . It is obvious that $b^{(k)}(t) \subset \mathfrak{R}_s^t(p,\Delta)$, since

$$\mathfrak{R}_s^t \Delta b^{(k)}(t) = e^{(k)} \in \ell(p) \quad \text{for} \quad k \in \mathbb{N}_0 \tag{3.9}$$

and

$$0 < p_k \leqq \mathcal{H} < \infty,$$

where $e^{(k)}$ is the sequence whose only non-zero term is 1 at *kth* place for each $k \in \mathbb{N}_0$.

Let $\varsigma \in \mathfrak{R}_s^t(p,\Delta)$ be given. For every non-negative integer r, we put

$$\varsigma^{[r]} = \sum_{k=0}^{r} \lambda_k(t) b^{(k)}(t) \tag{3.10}$$

We then obtain by applying $\mathfrak{R}_s^t\Delta$ to (3.10) with (3.9) that

$$\mathfrak{R}_s^t\Delta\varsigma^{[r]} = \sum_{k=0}^{r}\lambda_k(t)(\mathfrak{R}_s^t(\Delta)b^{(k)}(t)) = \sum_{k=0}^{r}\lambda_k(t)e^{(k)}$$

and

$$\left(\mathfrak{R}_s^t\Delta\left(\varsigma - \varsigma^{[r]}\right)\right)_i = \begin{cases} 0 & (0 \leqq i \leqq r) \\ (\mathfrak{R}_s^t\Delta\varsigma)_i & (i > r), \end{cases}$$

where $i, r \in \mathbb{N}_0$. Given $\varepsilon > 0$, there exists an integer r_0 such that

$$\left(\sum_{i=r}^{\infty}|(\mathfrak{R}_s^t\Delta\varsigma)_i|^{p_k}\right)^{\frac{1}{M}} < \frac{\varepsilon}{2}$$

for all $r \geqq r_0$. Hence we have

$$\begin{aligned}\mathfrak{H}_\Delta\left(\varsigma - \varsigma^{[r]}\right) &= \left(\sum_{i=r}^{\infty}|(\mathfrak{R}_s^t\Delta\varsigma)_i|^{p_k}\right)^{\frac{1}{M}} \\ &\leqq \left(\sum_{i=r_0}^{\infty}|(\mathfrak{R}_s^t\Delta\varsigma)_i|^{p_k}\right)^{\frac{1}{M}} < \frac{\varepsilon}{2} < \varepsilon\end{aligned}$$

for all $r \geqq r_0$, which proves that $\varsigma \in \mathfrak{R}_s^t(p, \Delta)$ is represented as (3.8).

To prove this representation for $\varsigma \in \mathfrak{R}_s^t(p, \Delta)$, given by (3.7), is unique, we assume that there exists a representation in the following form:

$$\varsigma = \sum_k \mu_k(t)b^k(t).$$

Since the linear transformation $\mathcal{G}$ from $\mathfrak{R}_s^t(p, \Delta))$ to $\ell(p)$, used in Theorem 2.2, is continuous, we have

$$\begin{aligned}(\mathfrak{R}_s^t\Delta\varsigma)_n &= \sum_k \mu_k(t)\left(\mathfrak{R}_s^t\Delta b^k(t)\right)_n \\ &= \sum_k \mu_k(t)e_n^{(k)} = \mu_n(t)\end{aligned}$$

for $n \in \mathbb{N}_0$, which contradicts the fact that $(\mathfrak{R}_s^t\Delta\varsigma)_n = \lambda_n(t)$ for all $n \in \mathbb{N}_0$. Hence the representation (3.8) is unique.
□

4 Matrix Mappings on the Space $\mathfrak{R}_s^t(p, \Delta)$

In this section, we characterize the matrix mappings from the space $\mathfrak{R}_s^t(p, \Delta)$ to the space ℓ_∞.

Theorem 4.1.
(i) *Let* $1 < p_k \leqq \mathcal{H} < \infty$ *for every* $k \in \mathbb{N}_0$. *Then* $A \in (\mathfrak{R}_s^t(p, \Delta), \ell_\infty)$ *if and only if, for a natural number* $B > 1$,

$$C(B) = \sup_n \sum_k \left|\left[\frac{a_{nk}}{t_k} + \left(\frac{1}{t_k} - \frac{1}{t_{k+1}}\right)\sum_{i=k+1}^{n} a_{ni}\right] B^{-1}T_k^{s+1}\right|^{p_k'} \tag{4.1}$$

and $\{a_{nk}\}_{k\in\mathbb{N}_0} \in cs$ *for each* $n \in \mathbb{N}_0$.

(ii) *Let* $0 < p_k \leqq 1$ *for every* $k \in \mathbb{N}_0$. *Then* $A \in (\mathfrak{R}_s^t(p, \Delta), \ell_\infty)$ *if and only if*

$$\sup_{n,k}\left|\left[\frac{a_{nk}}{t_k} + \left(\frac{1}{t_k} - \frac{1}{t_{k+1}}\right)\sum_{i=k+1}^{n} a_{ni}\right] T_k^{s+1}\right|^{p_k} \tag{4.2}$$

and $\{a_{nk}\}_{k\in\mathbb{N}_0} \in cs$ *for each* $n \in \mathbb{N}_0$.

Proof . Here we only prove Part (i), since Part (ii) may be established in a similar fashion. So, let $A \in (\mathfrak{R}_s^t(p,\Delta), \ell_\infty)$ and $1 < p_k \leqq \mathcal{H} < \infty$ for every $k \in \mathbb{N}_0$. Then $A\zeta$ exists for $\zeta \in \mathfrak{R}_s^t(p,\Delta)$ and implies that $\{a_{nk}\}_{k\in\mathbb{N}_0} \in \{\mathfrak{R}_s^t(p,\Delta)\}^\beta$ for each $n \in \mathbb{N}_0$. Hence the necessity part of (4.1) holds true.

Conversely, suppose that the necessity parts (4.1) hold true and $\zeta \in \mathfrak{R}_s^t(p,\Delta)$.

Since $\{a_{nk}\}_{k\in\mathbb{N}_0} \in \{\mathfrak{R}_s^t(p,\Delta)\}^\beta$ for every fixed $n \in \mathbb{N}_0$, the A-transform of ζ exists. Consider the following equality obtained by using the relation (11):

$$\sum_{k=0}^{m} a_{nk}\zeta_k = \sum_{k=0}^{m} \left[\frac{a_{nk}}{t_k} + \left(\frac{1}{t_k} - \frac{1}{t_{k+1}}\right)\sum_{i=k+1}^{m} a_{ni}\right] T_k^{s+1} y_k. \tag{4.3}$$

Now, using the hypothesis of Theorem 4.1, we derive from (4.3) as $m \to \infty$ that

$$\sum_{k} a_{nk}\zeta_k = \sum_{k} \left[\frac{a_{nk}}{t_k} + \left(\frac{1}{t_k} - \frac{1}{t_{k+1}}\right)\sum_{i=k+1}^{\infty} a_{ni}\right] T_k^{s+1} y_k. \tag{4.4}$$

As in the earlier work [3], for $B > 0$ and for any complex numbers a and b, we have

$$|ab| \leqq B\left(\left|aB^{-1}\right|^{p'} + |b|^p\right)$$

with $p^{-1} + {p'}^{-1} = 1$. Therefore, by using this inequality, we see from (4.4) that

$$\begin{aligned}\sup_{n\in\mathbb{N}_0}\left|\sum_{k} a_{nk}\zeta_k\right| &\leqq \sup_{n\in\mathbb{N}_0}\sum_{k}\left|\left[\frac{a_{nk}}{t_k} + \left(\frac{1}{t_k} - \frac{1}{t_{k+1}}\right)\sum_{i=k+1}^{\infty} a_{ni}\right] T_k^{s+1}\right| |y_k| \\ &\leqq B\left[C(B) + \mathfrak{H}_1^B(y)\right] < \infty.\end{aligned}$$

This shows that $A\zeta \in \ell_\infty$ whenever $\zeta \in \mathfrak{R}_s^t(p,\Delta)$. □

5 Conclusion

In this manuscript, we have introduced the space $\mathfrak{R}_s^t(p,\Delta)$ based on general sequences of Riesz form and the operator Δ. We have shown it to be complete paranormed space and its linear isomorphism property with $\ell(p)$ have been determined. The basis and Köthe-duals property of the concerned space has been determined. Also, some characterization of infinite matrices concerning it are given. The consequences of the results obtained in this manuscript are more general and extensive than the pre-existing known results.

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