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Janowski-type mappings associated with the conic like domains

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Abstract

In geometry, a conic is a plane curve whose coordinates satisfy a quadratic equation in two variables and can be expressed in matrix form. This equation allows deducing and expressing geometric properties of conic sections. In this article, we define certain subclasses $\mathcal{U}_k\mathcal{S}(\lambda,\gamma,\tau,\rho)$ and $\mathcal{U}_k^{\Im}\mathcal{S}(\lambda,\gamma,\tau,\rho)$ of holomorphic mappings associated with the Janowski-type mappings. These functions are actually generalizations of some basic families of starlike and convex mappings. We study sufficient conditions for $f \in \mathcal{U}_k\mathcal{S}(\lambda,\gamma,\tau,\rho)$ along with the characterization, coefficient bounds and other problems. Using certain conditions for functions in the class $\mathcal{U}_k\mathcal{S}(\lambda,\gamma,\varrho,\sigma)$, we also define another class and study some subordination related result.

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1 Basic Introduction

In geometry, a conic is a plane curve whose coordinates satisfy a quadratic equation in two variables and which can be expressed in matrix form. The quadratic equation representation allows us deducing and expressing geometric properties of conic sections. The simplest form of the conic domain Δ_k, $k \geqslant 0$ is given in the following:

$$\Delta_k = \left\{ w = (u,v) : u^2 > k^2 (u-1)^2 + v^2 \right\}.$$

Let $\mathbb{U}_1^0 := \{z \in \mathbb{C} \text{ and } |z| < 1\}$ and let $\mathcal{H} = \mathcal{H}(\mathbb{U}_1^0)$ denote the family of holomorphic mappings f in $\mathbb{U}_1^0$. Suppose that

$$f(z) = \alpha + \sum_{n=1}^{\infty} \alpha_n z^n \in \mathcal{H}[\alpha, n] \subset \mathcal{H}.$$

Let $q \in \mathcal{Q}$ be the Carathéodory map such that $\Re(q(z)) > 0$ and $q(z) = 1 + \sum_{n=1}^{\infty} q_n z^n, z \in \mathbb{U}_1^0$. The Möbius transformation $l_0(z) = \frac{1+z}{1-z} \in \mathcal{Q}$, is the known extremal function. Let the notation $\prec$ be phrased as subordination. For $f, \ell \in \mathcal{H}$, $f(z) \prec \ell(z)$, if for $w \in \mathcal{H}(\mathbb{U}_1^0) : w(0) = 0$ and $|w(z)| < 1$, we write $f(z) = \ell\big(w(z)\big)$. For detail, see [10]. Using this idea, Janowski [8] introduced the family $\mathcal{Q}[\tau,\rho]$ for $-1 \leqslant \rho < \tau \leqslant 1$. A mapping $q \in \mathcal{Q}[\tau,\rho]$, if

$$q(z) \prec \frac{1+\tau z}{1+\rho z} \text{ or } q(z) = \frac{1+\tau w(z)}{1+\rho w(z)}, w(0) = 0 \text{ and } |w(z)| < 1, z \in \mathbb{U}_1^0.$$

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For some related work, we refer, [2, 3, 4, 5, 6, 8, 10]. Clearly, $\mathcal{Q}[\tau,\rho]$ is contained in $\mathcal{Q}\left(\frac{1-\tau}{1-\rho}\right)$. This family $\mathcal{Q}[\tau,\rho]$ is related with the class $\mathcal{Q}$ as given below. A mappings $q \in \mathcal{Q}$ iff

$$\frac{(\tau+1)\,q(z)-(\tau-1)}{(\rho+1)\,q(z)-(\rho-1)} \in \mathcal{Q}[\tau,\rho].$$

Let $f \in \mathcal{A} \subset \mathcal{H}[\alpha,n]$ has the following series form

$$f(z) = z + \sum_{n=2}^{\infty} \alpha_n z^n,\ z \in \mathbb{U}_1^0. \tag{1.1}$$

A mapping $f \in \mathcal{U}_k\mathcal{S}(\beta)$, if the following inequality holds:

$$\Re\left\{\frac{zf'(z)}{f(z)} - \beta\right\} > k\left|\frac{zf'(z)}{f(z)} - 1\right|,\ z \in \mathbb{U}_1^0, \tag{1.2}$$

where $-1 \leqslant \beta < 1$ and $k \geqslant 0$. A mapping $f \in \mathcal{U}_k\mathcal{C}(\beta)$ iff $zf \in \mathcal{U}_k\mathcal{S}(\beta)$. The above families are studied by Goodman [7] and Rönning [13]. In 2011, Noor and Malik [11] introduced the family $\mathcal{U}_k\mathcal{S}(\tau,\rho)$ which is defined as:

Definition 1.1. A mapping $f \in \mathcal{A}$ given by (1.1), is in the family $\mathcal{U}_k\mathcal{S}(\tau,\rho)$ provided that $f(z) \neq 0$ and

$$\Re\left\{\frac{(\rho-1)\frac{zf'(z)}{f(z)}-(\tau-1)}{(\rho+1)\frac{zf'(z)}{f(z)}+(\tau-1)}\right\} > k\left|\frac{(\rho-1)\frac{zf'(z)}{f(z)}-(\tau-1)}{(\rho+1)\frac{zf'(z)}{f(z)}+(\tau-1)} - 1\right|,\ z \in \mathbb{U}_1^0,$$

where $-1 \leqslant \rho < \tau \leqslant 1$ and $k \geqslant 0$.

This family consists of mappings f associated with uniformly k-starlike mappings in $\mathbb{U}_1^0$. Subsequently, we have the following:

Definition 1.2. Let $f \in \mathcal{A}$. Then $f \in \mathcal{U}_k\mathcal{S}(\gamma,\tau,\rho)$, if

$$\Re\left\{\frac{(\rho-1)\frac{zF_\gamma'(z)}{F_\gamma(z)}-(\tau-1)}{(\rho+1)\frac{zF_\gamma'(z)}{F_\gamma(z)}+(\tau-1)}\right\} > k\left|\frac{(\rho-1)\frac{zF_\gamma'(z)}{F_\gamma(z)}-(\tau-1)}{(\rho+1)\frac{zF_\gamma'(z)}{F_\gamma(z)}+(\tau-1)}\right|, z\ \in \mathbb{U}_1^0,$$

where F_γ is a convex combination of f and zf', $0 \leqslant \lambda, \gamma \leqslant 1, -1 \leqslant \rho < \tau < 1$,and $k \geqslant 0$.

Remark 1.3. The mapping F_γ is convergent as a holomorphic mappings in $\mathbb{U}_1^0$.

Extending these ideas, we define a new class $\mathcal{U}_k\mathcal{S}(\lambda,\gamma,\tau,\rho)$ of holomorphic mappings.

Definition 1.4. Let $f \in \mathcal{A}$. Then $f \in \mathcal{U}_k\mathcal{S}(\lambda,\gamma,\tau,\rho)$, if

$$\Re\left\{\frac{(\rho-1)\frac{zF_{\gamma,\lambda}'(z)}{F_{\gamma,\lambda}(z)}-(\tau-1)}{(\rho+1)\frac{zF_{\gamma,\lambda}'(z)}{F_{\gamma,\lambda}(z)}+(\tau-1)}\right\} > k\left|\frac{(\rho-1)\frac{zF_{\gamma,\lambda}'(z)}{F_{\gamma,\lambda}(z)}-(\tau-1)}{(\rho+1)\frac{zF_{\gamma,\lambda}'(z)}{F_{\gamma,\lambda}(z)}+(\tau-1)}\right|,\ z \in \mathbb{U}, \tag{1.3}$$

where

$$F_{\gamma,\lambda}(z) = \sum_{n=2}^{\infty}\left[1+(n-1)(\lambda)+(n-1)^2\lambda\gamma\right]\alpha_n z^n,\ z \in \mathbb{U}, \tag{1.4}$$

$0 \leqslant \lambda, \gamma \leqslant 1, -1 \leqslant \rho < \tau < 1$ and $k \geqslant 0$.

Remark 1.5. It is quite simple to see that $F_{\gamma,\lambda}$ converges as a convex combination two convergent series in the unit disc. Also $F_{\gamma,\lambda} \in \mathcal{U}_k\mathcal{S}(\beta)$ in $\mathbb{U}$.

Let $\Im$ be the class of holomorphic mapping f of positive coefficients and having the series of the form:

$$f(z) = z - \sum_{n=2}^{\infty} \alpha_n z^n, \quad \alpha_n \geqslant 0, z \in \mathbb{U}_1^0. \tag{1.5}$$

For details of the class $\Im$, we refer [15].

Definition 1.6. Let f be given by (1.1). Then $f \in \mathcal{U}_k^{\Im}\mathcal{S}(\lambda, \gamma, \tau, \rho)$, if and only if

$$f \in \mathcal{U}_k\mathcal{S}(\lambda, \gamma, \tau, \rho) \cap \Im$$

where $-1 \leqslant \rho < \tau < 1, 0 \leqslant \lambda, \gamma \leqslant 1, k \geqslant 0$ and $\Im$ is given by (1.5).

For some special choices, we obtain the following known classes:

i. $\mathcal{U}_k\mathcal{S}(0,0,\tau,\rho) = \mathcal{U}_k\mathcal{S}(\tau,\rho)$ and $\mathcal{U}_k\mathcal{S}(1,1,\tau,\rho) = \mathcal{U}_k\mathcal{C}(\tau,\rho)$.

ii. $\mathcal{U}_k\mathcal{S}(0,0,1,-1) = \mathcal{U}_k\mathcal{S}$ and $\mathcal{U}_k\mathcal{S}(1,1,1,-1) = \mathcal{U}_k\mathcal{C}$.

iii. $\mathcal{U}_k\mathcal{S}(0,1-2\beta,-1) = \mathcal{U}_k\mathcal{S}(\beta)$ and $\mathcal{U}_k\mathcal{S}(1,1,1-2\beta,-1) = \mathcal{U}_k\mathcal{C}(\beta)$.

iv. $\mathcal{U}_0\mathcal{S}(0,0,\tau,\rho) = \mathcal{S}^*(\tau,\rho)$ and $\mathcal{U}_0\mathcal{S}(1,1,\tau,\rho) = \mathcal{C}(\tau,\rho)$.

The class $\mathcal{U}_k\mathcal{S}(\lambda, \gamma, \tau, \rho)$ also reduces to the families mentioned in (1.2), see [13]. For detail of the above classes and various other cases related to the earlier contributions, see [3, 8, 9, 11, 14, 16, 17, 18, 19] with references therein.

2 Preliminaries

Subsequently, we define the subordinating factor sequence.

Definition 2.1. A sequence $\langle c_n : n = 1, 2, 3, ...\rangle$ is termed as a subordinating factor sequence for some mappings in $\mathcal{C}$, if for each $f \in \mathcal{C}$, we have

$$\sum_{n=1}^{\infty} \alpha_n c_n z^n \prec f(z), \;\; \alpha_1 = 1, z \in \mathbb{U}_1^0. \tag{2.1}$$

Lemma 2.2. The sequence $\langle c_n : n = 1, 2, 3, ...\rangle$ is a subordinating factor sequence, iff

$$\Re\left\{1 - 2\sum_{n=2}^{\infty} c_n z^n\right\} > 0.$$

For detail, see [9, 20]. Throughout, we assume $0 \leqslant \lambda, \gamma \leqslant 1, k \geqslant 0$ and $-1 \leqslant \rho < \tau \leqslant 1$.

3 Main Discussion

Theorem 3.1. If a function f given by (1.1) satisfies the condition

$$\sum_{n=2}^{\infty} \left[\{3 + 2k + \rho\}(n-1) + \rho - \tau\right]\left[1 + (n-1)(\lambda + \gamma) + (n-1)^2 \lambda\gamma\right] |\alpha_n| \leqslant \tau - \rho, \tag{3.1}$$

then the function $f \in \mathcal{U}_k\mathcal{S}(\lambda, \gamma, \tau, \rho)$ defined by (1.3), where $n \geqslant \frac{1+\tau}{1+\rho}$ for $-1 \leqslant \rho < \tau \leqslant 1, 0 \leqslant \lambda, \gamma \leqslant 1$ and $k \geqslant 0$.

Proof . To have the desired proof, we only show that

$$k\left|\frac{(\rho-1)\frac{zF'_{\gamma,\lambda}(z)}{F_{\gamma,\lambda}(z)} - (\tau-1)}{(\rho+1)\frac{zF'_{\gamma,\lambda}(z)}{F_{\gamma,\lambda}(z)} + (\tau-1)} - 1\right| - \Re\left\{\frac{(\rho-1)\frac{zF'_{\gamma,\lambda}(z)}{F_{\gamma,\lambda}(z)} - (\tau-1)}{(\rho+1)\frac{zF'_{\gamma,\lambda}(z)}{F_{\gamma,\lambda}(z)} + (\tau-1)} - 1\right\} \leqslant 1$$

where $F_{\gamma,\lambda}$ is given by (1.4), $-1 \leqslant \rho < \tau < 1, 0 \leqslant \lambda, \gamma \leqslant 1$ and $k \geqslant 0$. For $F_{\gamma,\lambda}$ given by (1.4) and

$$zF'_{\lambda,m}(z) = z + \sum_{n=2}^{\infty} n\left[1 + (n-1)\lambda\right]^m \alpha_n z^n,$$

consider that

$$\begin{aligned}
&k\left|\frac{(\rho-1)\frac{zF'_{\gamma,\lambda}(z)}{F_{\gamma,\lambda}(z)} - (\tau-1)}{(\rho+1)\frac{zF'_{\gamma,\lambda}(z)}{F_{\gamma,\lambda}(z)} - (\tau+1)} - 1\right| - \Re\left\{\frac{(\rho-1)\frac{zF'_{\gamma,\lambda}(z)}{F_{\gamma,\lambda}(z)} - (\tau-1)}{(\rho+1)\frac{zF'_{\gamma,\lambda}(z)}{F_{\gamma,\lambda}(z)} - (\tau+1)} - 1\right\}\\
&\leqslant (1+k)\left|\frac{(\rho-1)\,zF'_{\gamma,\lambda}(z) - (\tau-1)\,F_{\gamma,\lambda}(z)}{(\rho+1)\,zF'_{\gamma,\lambda}(z) - (\tau+1)\,F_{\gamma,\lambda}(z)} - 1\right|\\
&= 2(1+k)\left|\frac{zF'_{\gamma,\lambda}(z) - F_{\gamma,\lambda}(z)}{(\rho+1)\,zF'_{\gamma,\lambda}(z) - (\tau+1)\,F_{\gamma,\lambda}(z)}\right|, \quad \left(n \geqslant \frac{1+\tau}{1+\rho}\right)\\
&\leqslant \frac{2\sum_{n=2}^{\infty}(1+k)(n-1)\left[1 + (n-1)(\lambda+\gamma) + (n-1)^2\lambda\gamma\right]|\alpha_n|}{\tau - \rho - \sum_{n=2}^{\infty}\{n\rho - \tau + n - 1\}\left[1 + (n-1)(\lambda+\gamma) + (n-1)^2\lambda\gamma\right]|\alpha_n|}.
\end{aligned}$$

The last expression is bounded above by 1 if

$$\sum_{n=2}^{\infty}\left[(3+2k+\rho)(n-1) + \rho - \tau\right]\left[1 + (n-1)(\lambda+\gamma) + (n-1)^2\lambda\gamma\right]|\alpha_n| \leqslant \tau - \rho.$$

□

We next prove the characterization of the mapping f as below.

Theorem 3.2. A mapping f given by (1.5) belongs to the family $\mathcal{U}_k^{\Im}\mathcal{S}(\lambda, \gamma, \tau, \rho)$ if and only if

$$\sum_{n=2}^{\infty}\{(n-1)(1+2k-\rho) + \tau - \rho\}(1 - \lambda + n\lambda)(1 - \gamma + n\gamma)\,\alpha_n \leqslant \tau - \rho, \tag{3.2}$$

where $-1 \leqslant \rho < \tau \leqslant 1, 0 \leqslant \lambda, \gamma \leqslant 1$ and $k \geqslant 0$.

Proof . Suppose that $f \in \mathcal{U}_k^{\Im}\mathcal{S}(\lambda, \gamma, \tau, \rho)$. Then, making use of the fact that

$$\Re(w) > k|w-1| \Leftrightarrow \Re\left\{w(1 + ke^{i\theta}) - ke^{i\theta}\right\} > 0 \tag{3.3}$$

and taking

$$w(z) = \frac{(\rho-1)\frac{zF'_{\gamma,\lambda}(z)}{F_{\gamma,\lambda}(z)} - (\tau-1)}{(\rho+1)\frac{zF'_{\gamma,\lambda}(z)}{F_{\gamma,\lambda}(z)} - (\tau+1)},$$

where $F_{\gamma,\lambda}$ is given by (1.4), $-1 \leqslant \rho < \tau < 1, 0 \leqslant \lambda, \gamma \leqslant 1, k \geqslant 0$ in (1.3), we obtain

$$\Re\left\{(1 + ke^{i\theta})\frac{(\rho-1)\frac{zF'_{\gamma,\lambda}(z)}{F_{\gamma,\lambda}(z)} - (\tau-1)}{(\rho+1)\frac{zF'_{\gamma,\lambda}(z)}{F_{\gamma,\lambda}(z)} - (\tau+1)} - ke^{i\theta}\right\} > 0,$$

or equivalently

$$\Re\left\{(1 + ke^{i\theta})\frac{(\rho-1)\,zF'_{\gamma,\lambda}(z) - (\tau-1)\,F_{\gamma,\lambda}(z)}{(\rho+1)\,zF'_{\gamma,\lambda}(z) - (\tau+1)\,F_{\gamma,\lambda}(z)} - ke^{i\theta}\right\} > 0,$$

which on simple manipulation yields

$$\Re\frac{\tau - \rho + \sum_{n=2}^{\infty}\left\{n\left(\rho - 1 - 2ke^{i\theta}\right) + 1 - \tau + 2ke^{i\theta}\right\}(1 - \lambda + n\lambda)(1 - \gamma + n\gamma)\,\alpha_n z^{n-1}}{(\tau - \rho) + \sum_{n=2}^{\infty}\{n(\rho+1) - 1 - \tau\}\left[1 + (n-1)(\lambda+\gamma) + (n-1)^2\lambda\gamma\right]\alpha_n z^{n-1}} > 0.$$

This result holds true for all $z \in \mathbb{U}_1^0$. Taking the limit $z \to 1^-$ through real values, we thus obtain that

$$\Re\left(\frac{\tau-\rho+\sum_{n=2}^{\infty}\left\{n\rho-n-2kne^{i\theta}+1-\tau+2ke^{i\theta}\right\}(1-\lambda+n\lambda)(1-\gamma+n\gamma)\alpha_n}{(\tau-\rho)+\sum_{n=2}^{\infty}\{n(\rho+1)-1-\tau\}\left[1+(n-1)(\lambda+\gamma)+(n-1)^2\lambda\gamma\right]\alpha_n}\right)>0,$$

which further implies that

$$\left\{\tau-\rho-\sum_{n=2}^{\infty}\{(1+2k-\rho)(n-1)+\tau-\rho\}(1-\lambda+n\lambda)(1-\gamma+n\gamma)\alpha_n\right\}>0,$$

so we have

$$\sum_{n=2}^{\infty}\{(1+2k-\rho)(n-1)+\tau-\rho\}(1-\lambda+n\lambda)(1-\gamma+n\gamma)\alpha_n<\tau-\rho.$$

Conversely, we let the inequality (3.2) hold true. Then, in view of the fact that $\Re(w(z))>0$ if and only if $|w(z)-1|<|w(z)+1|$, for

$$w(z)=\frac{(\rho-1)\frac{zF'_{\gamma,\lambda}(z)}{F_{\gamma,\lambda}(z)}-(\tau-1)}{(\rho+1)\frac{zF'_{\gamma,\lambda}(z)}{F_{\gamma,\lambda}(z)}-(\tau+1)}-k\left|\frac{(\rho-1)\frac{zF'_{\gamma,\lambda}(z)}{F_{\gamma,\lambda}(z)}-(\tau-1)}{(\rho+1)\frac{zF'_{\gamma,\lambda}(z)}{F_{\gamma,\lambda}(z)}-(\tau+1)}-1\right|. \tag{3.4}$$

we consider

$$\begin{aligned}
&|w(z)+1|\\
&=\left|\frac{(\rho-1)zF'_{\gamma,\lambda}(z)-(\tau-1)F_{\gamma,\lambda}(z)}{(\rho+1)zF'_{\gamma,\lambda}(z)-(\tau+1)F_{\gamma,\lambda}(z)}-k\left|\frac{(\rho-1)zF'_{\gamma,\lambda}(z)-(\tau-1)F_{\gamma,\lambda}(z)}{(\rho+1)zF'_{\gamma,\lambda}(z)-(\tau+1)F_{\gamma,\lambda}(z)}-1\right|+1\right|\\
&=\frac{2|z|}{|G(\tau,\rho,\lambda,\gamma,z)|}\left|\tau-\rho+\sum_{n=2}^{\infty}\{n\rho-\tau+kn-k\}(1-\lambda+n\lambda)(1-\gamma+n\gamma)\alpha_n z^{n-1}\right|\\
&>\frac{2}{|G(\tau,\rho,\gamma,z)|}\left[\tau-\rho-\sum_{n=2}^{\infty}\{n\rho-\tau+kn-k\}(1-\gamma+\gamma n)\alpha_n\right],
\end{aligned}\tag{3.5}$$

where $G(\tau,\rho,\lambda,\gamma,z)=(\rho+1)zF'_{\gamma,\lambda}(z)-(\tau+1)F_{\gamma,\lambda}(z)$. Also for $|w(z)-1|=c$, we have

$$\begin{aligned}
c&=\left|\frac{(\rho-1)zF'_{\gamma,\lambda}(z)-(\tau-1)F_{\gamma,\lambda}(z)}{(\rho+1)zF'_{\gamma,\lambda}(z)-(\tau+1)F_{\gamma,\lambda}(z)}-1-k\left|\frac{(\rho-1)zF'_{\gamma,\lambda}(z)-(\tau-1)F_{\gamma,\lambda}(z)}{(\rho+1)zF'_{\gamma,\lambda}(z)-(\tau+1)F_{\gamma,\lambda}(z)}-1\right|\right|\\
&=2\left|\frac{-zF'_{\gamma,\lambda}(z)+F_{\gamma,\lambda}(z)}{(\rho+1)zF'_{\gamma,\lambda}(z)-(\tau+1)F_{\gamma,\lambda}(z)}-k\left|\frac{-zF'_{\gamma,\lambda}(z)+F_{\gamma,\lambda}(z)}{(\rho+1)zF'_{\gamma,\lambda}(z)-(\tau+1)F_{\gamma,\lambda}(z)}\right|\right|\\
&<\frac{2|z|}{|G(\tau,\rho,\lambda,\gamma,z)|}\sum_{n=2}^{\infty}(n+nk-1-k)(1-\lambda+n\lambda)(1-\gamma+n\gamma)\alpha_n.
\end{aligned}\tag{3.6}$$

where $G(\tau,\rho,\lambda,\gamma,z)=(\rho+1)zF'_{\gamma,\lambda}(z)-(\tau+1)F_{\gamma,\lambda}(z)$. From the condition (3.2) and the inequalities (3.5) and (3.6), we deduce that

$$|w(z)+1|-|w(z)-1|>0,$$

where w is defined by (3.4). This completes the proof of Theorem 3.2. □

We next provide coefficient bound for a given mapping f to belong to the family $\mathcal{U}_k^{\Im}\mathcal{S}(\lambda,\gamma,\tau,\rho)$.

Corollary 3.3. A mapping f belongs to the family $\mathcal{U}_k^{\Im}\mathcal{S}(\lambda,\gamma,\tau,\rho)$ if

$$\sum_{n=2}^{\infty}\alpha_n<\frac{\tau-\rho}{\{1+2k-2\rho+\tau\}(1+\lambda)(1+\gamma)}.$$

where $-1\leqslant\rho<\tau<1, 0\leqslant\lambda,\gamma\leqslant 1$ and $k\geqslant 0$.

Corollary 3.4. For a mapping f belonging to the family $\mathcal{U}_k^{\Im}\mathcal{S}(\lambda,\gamma,\tau,\rho)$, we have

$$\alpha_n < \frac{\tau-\rho}{\{1+2k-2\rho+\tau\}(1+\lambda)(1+\gamma)}.$$

where $-1 \leqslant \rho < \tau < 1, 0 \leqslant \lambda, \gamma \leqslant 1$ and $k \geqslant 0$.

The subsequent theorem deals with the integral representation for a given mapping $f \in \mathcal{U}_k^{\Im}\mathcal{S}(\lambda,\gamma,\tau,\rho)$.

Theorem 3.5. If a mapping f given by (1.5) belongs to the family $\mathcal{U}_k^{\Im}\mathcal{S}(\lambda,\gamma,\tau,\rho)$, then f has the following representation:

$$f(z) = z^{\frac{1}{\lambda}-1}\left[\int_0^z \frac{1}{\gamma}u^{1-\frac{1}{\lambda}}\int_0^u \left\{\frac{1}{\lambda}v^{\frac{1}{\lambda}-1}\exp\left(\int_0^v \frac{2k\tau-Q(t)(\tau-1)}{t\{2k+Q(t)(\rho-1)\}}dt\right)\mathrm{d}v\right\}\mathrm{d}u\right],$$

where $-1 \leqslant \rho < \tau < 1, 0 < \lambda, \gamma \leqslant 1$ and $k \geqslant 0$.

Proof . For $k = 0$, the proof is obvious. Let $k > 0$. Then, for $f \in \mathcal{U}_k^{\Im}\mathcal{S}(\lambda,\gamma,\tau,\rho)$ and

$$w(z) = \frac{(\rho-1)\frac{zF'_{\gamma,\lambda}(z)}{F_{\gamma,\lambda}(z)}-(\tau-1)}{(\rho+1)\frac{zF'_{\gamma,\lambda}(z)}{F_{\gamma,\lambda}(z)}+(\tau-1)}$$

we take

$$\Re(w) > k|w-1|,$$

which leads

$$|\frac{w-1}{w}| < \frac{1}{k}.$$

We suppose that

$$\frac{w-1}{w} = \frac{Q(z)}{k}$$

and

$$w(z) = \frac{k}{k-Q(z)},$$

which yields

$$\frac{(\rho-1)\frac{zF'_{\gamma,\lambda}(z)}{F_{\gamma,\lambda}(z)}-(\tau-1)}{(\rho-1)\frac{zF'_{\gamma,\lambda}(z)}{F_{\gamma,\lambda}(z)}-(\tau+1)} = \frac{k}{k-Q(z)}.$$

Thus on simplification, we have

$$\frac{zF'_{\gamma,\lambda}(z)}{F_{\gamma,\lambda}(z)} = \frac{2k\tau-Q(z)(\tau-1)}{2k+Q(z)(\rho-1)}.$$

which proves that

$$F_{\gamma,\lambda}(z) = \exp\left(\int_0^z \frac{2k\tau-Q(t)(\tau-1)}{t\{2k+Q(t)(\rho-1)\}}\mathrm{d}t\right)$$

or

$$(1-\lambda)F_\gamma(z)+\lambda zF'_\gamma(z) = \frac{1}{\lambda}u^{\frac{1}{\lambda}-1}\exp\left(\int_0^z \frac{2k\tau-Q(t)(\tau-1)}{t\{2k+Q(t)(\rho-1)\}}\mathrm{d}t\right),$$

which on simple manipulation yields

$$f(z) = z^{\frac{1}{\lambda}-1}\left[\int_0^z \frac{1}{\gamma}u^{1-\frac{1}{\lambda}}\int_0^u \left\{\frac{1}{\lambda}v^{\frac{1}{\lambda}-1}\exp\left(\int_0^v \frac{2k\tau-Q(t)(\tau-1)}{t\{2k+Q(t)(\rho-1)\}}dt\right)\mathrm{d}v\right\}\mathrm{d}u\right].$$

This finishes the proof of Theorem 3.5. □

Theorem 3.6. If f_i is such that

$$f_i(z) = z - \sum_{n=2}^{\infty}\alpha_{n,i}z^n \in \mathcal{U}_k^{\Im}\mathcal{S}(\lambda,\gamma,\tau,\rho),\ i=1,2, z\in\mathbb{U}_1^0,$$

then

$$f(z) = (1-\delta)f_1(z)+\delta f_2(z) \in \mathcal{U}_k^{\Im}\mathcal{S}(\lambda,\gamma,\tau,\rho),\ 0\leqslant\delta\leqslant 1, z\in\mathbb{U}_1^0.$$

Proof . For the mappings f_i such that

$$f_i(z) = z - \sum_{n=2}^{\infty} \alpha_{n,i} z^n \in \mathcal{U}_k^{\Im}\mathcal{S}(\lambda, \gamma, \tau, \rho)$$

and by using Theorem 3.2, we write

$$\sum_{n=2}^{\infty} \{(1+2k-\rho)(n-1) + \tau - \rho\}(1-\lambda+n\lambda)(1-\gamma+n\gamma)\alpha_{n,1} \leqslant \tau - \rho \tag{3.7}$$

and

$$\sum_{n=2}^{\infty} \{(1+2k-\rho)(n-1) + \tau - \rho\}(1-\lambda+n\lambda)(1-\gamma+n\gamma)\alpha_{n,2} \leqslant \tau - \rho. \tag{3.8}$$

By considering (3.7) and (3.8), we write

$$\begin{aligned}
&(1-\delta)\sum_{n=2}^{\infty} \{(1+2k-\rho)(n-1) + \tau - \rho\}(1-\lambda+n\lambda)(1-\gamma+n\gamma)\alpha_{n,1} \\
&+ \delta \sum_{n=2}^{\infty} \{(1+2k-\rho)(n-1) + \tau - \rho\}(1-\lambda+n\lambda)(1-\gamma+n\gamma)\alpha_{n,2} \\
&\leqslant (1-\delta)(\tau-\rho) + \delta(\tau-\rho) = \tau - \rho.
\end{aligned}$$

Again by using Theorem 3.2, we reach the conclusion. □

In the following, we define the family $\mathcal{U}_k\mathcal{S}^*(\lambda, \gamma, \tau, \rho)$ of holomorphic mappings f satisfying the coefficient conditions (3.1). Assume that

$$f(z) = z + \sum_{n=2}^{\infty} \alpha_n z^n \in \mathcal{A}.$$

Then $f \in \mathcal{U}_k\mathcal{S}^*(\lambda, \gamma, \tau, \rho)$, if we have

$$\sum_{n=2}^{\infty} [(3+2k+\rho)(n-1) + \rho - \tau](1-\lambda+n\lambda)(1-\gamma+n\gamma)|\alpha_n| \leqslant \tau - \rho,$$

for some $k \geqslant 0, 0 \leqslant \lambda, \gamma \leqslant 1$ and $-1 \leqslant \rho < \tau \leqslant 1$.

For special choices of $\lambda, \gamma, \tau, \rho, k$ and the mapping h, we mention the study of Aouf and Mostafa [2] and others. Clearly

$$\mathcal{U}_k\mathcal{S}^*(\lambda, \gamma, \tau, \rho) \subset \mathcal{U}_k\mathcal{S}(\lambda, \gamma, \tau, \rho)$$

Adopting the procedure found in [2, 3, 16], we have:

Theorem 3.7. If $f \in \mathcal{U}_k\mathcal{S}^*(\lambda, \gamma, \tau, \rho)$ and

$$\Re(f(z)) > -\frac{\tau - \rho + (3+2k+2\rho-\tau)(1+\gamma)(1+\lambda)}{(3+2k+2\rho-\tau)(1+\gamma)(1+\lambda)}, \ z \in \mathbb{U}_1^0, \tag{3.9}$$

then

$$\frac{(1+2k-2\rho+\tau)(1+\gamma)(1+\lambda)}{2[\tau-\rho+(1+2k-2\rho+\tau)(1+\gamma)(1+\lambda)]}(f*h)(z) \prec h(z), \ z \in \mathbb{U}_1^0, \tag{3.10}$$

where

$$f(z) * h(z) = z + \sum_{n=2}^{\infty} \alpha_n c_n z^n = h(z) * f(z), z \in \mathbb{U}_1^0,$$

for the mapping f given by (1.1) and

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z \in \mathbb{U}_1^0.$$

for all $h \in \mathcal{C}$. The constant factor

$$\frac{(1+2k-2\rho+\tau)(1+\gamma)(1+\lambda)}{2[\tau-\rho+(1+2k-2\rho+\tau)(1+\gamma)(1+\lambda)]}$$

in (3.10) cannot be replaced by a larger one.

Proof . Let $f \in \mathcal{U}_k\mathcal{S}^*(\lambda,\gamma,\tau,\rho)$ and let $h(z)=z+\sum\limits_{n=2}^{\infty}c_nz^n$. Then

$$\frac{(3+2k+2\rho-\tau)(1+\gamma)\,(1+\lambda)\,(f*h)(z)}{2\,[\tau-\rho+(3+2k+2\rho-\tau)(1+\gamma)\,(1+\lambda)]} = \frac{(3+2k+2\rho-\tau)(1+\gamma)\,(1+\lambda)\,[z+\sum\limits_{n=2}^{\infty}\alpha_nc_nz^n]}{2\,[\tau-\rho+(3+2k+2\rho-\tau)(1+\gamma)\,(1+\lambda)]}$$

In view of Definition 2.1 and Lemma 2.2, (3.10) will hold if

$$\left\{\frac{(3+2k+2\rho-\tau)(1+\gamma)\,(1+\lambda)\,\alpha_n}{2\,[\tau-\rho+(3+2k+2\rho-\tau)(1+\gamma)\,(1+\lambda)]}\right\}_{n=1}^{\infty},a_1=1 \tag{3.11}$$

is a subordinating factor sequence. Using Lemma 2.2, we see that (3.11) is equivalent to

$$\Re\left\{1+\sum_{n=1}^{\infty}\frac{(3+2k+2\rho-\tau)(1+\gamma)\,(1+\lambda)\,\alpha_nz^n}{\tau-\rho+(3+2k+2\rho-\tau)(1+\gamma)\,(1+\lambda)}\right\}>0. \tag{3.12}$$

The mapping

$$\varphi\,(n)=\{(3+2k+\rho)\,(n-1)+\rho-\tau\}\,(1-\lambda+n\lambda)\,(1-\gamma+n\gamma)\,.$$

is an increasing mapping for $n\geqslant 2$. Taking this fact into account along with (3.12), we can write

$$\begin{aligned}
&\Re\left\{1+\sum_{n=1}^{\infty}\frac{(3+2k+2\rho-\tau)(1+\gamma)\,(1+\lambda)\,\alpha_nz^n}{\tau-\rho+(3+2k+2\rho-\tau)(1+\gamma)\,(1+\lambda)}\right\},K=3+2k\\
&=1+\Re[\frac{(K+2\rho-\tau)(1+\gamma)\,(1+\lambda)\,z}{\tau-\rho+(K+2\rho-\tau)(1+\gamma)\,(1+\lambda)}+\sum_{n=2}^{\infty}\frac{(K+2\rho-\tau)(1+\gamma)\,(1+\lambda)\,\alpha_nz^n}{\tau-\rho+(K+2\rho-\tau)(1+\gamma)\,(1+\lambda)}]\\
&\geqslant 1-\frac{(K+2\rho-\tau)(1+\gamma)\,(1+\lambda)\,|z|}{\tau-\rho+(K+2\rho-\tau)(1+\gamma)\,(1+\lambda)}-\frac{\sum\limits_{n=2}^{\infty}(K+2\rho-\tau)(1+\gamma)\,(1+\lambda)\,|\alpha_n|\,|z|^n}{\tau-\rho+(K+2\rho-\tau)(1+\gamma)\,(1+\lambda)}\\
&\geqslant 1-\frac{(K+2\rho-\tau)(1+\gamma)\,(1+\lambda)\,r}{\tau-\rho+(K+2\rho-\tau)(1+\gamma)\,(1+\lambda)}-\frac{\sum\limits_{n=2}^{\infty}(K+2\rho-\tau)(1+\gamma)\,(1+\lambda)\,|\alpha_n|\,r^n}{\tau-\rho+(K+2\rho-\tau)(1+\gamma)\,(1+\lambda)}\\
&\geqslant 1-\frac{(K+2\rho-\tau)(1+\gamma)\,(1+\lambda)\,r}{\tau-\rho+(K+2\rho-\tau)\tau)(1+\gamma)\,(1+\lambda)}-\frac{\sum\limits_{n=2}^{\infty}(K+2\rho-\tau)(1+\gamma)\,(1+\lambda)\,|\alpha_n|\,r}{\tau-\rho+(K+2\rho-\tau)(1+\gamma)\,(1+\lambda)}
\end{aligned}$$

On using (3.1), we see that

$$\begin{aligned}
&\Re\left\{1+\sum_{n=1}^{\infty}\frac{(3+2k+2\rho-\tau)(1+\gamma)\,(1+\lambda)\,|\alpha_n|\,z^n}{[\tau-\rho+(3+2k+2\rho-\tau)(1+\gamma)\,(1+\lambda)]}\right\}\\
&\geqslant 1-\frac{(K+2\rho-\tau)(1+\gamma)\,(1+\lambda)\,r}{\tau-\rho+(K+2\rho-\tau)(1+\gamma)\,(1+\lambda)}-\frac{(\tau-\rho)\,r}{\tau-\rho+(K+2\rho-\tau)(1+\gamma)\,(1+\lambda)\,.}\\
&=1-r>0,r\to 1,K=3+2k
\end{aligned}$$

This proves the (3.12). Thus we get (3.10). Also (3.9) is obtained from (3.10) for the mapping

$$h\,(z)=\frac{z}{1-z},\qquad (z\in\mathbb{U}_1^0).$$

For sharpness of

$$\frac{(3+2k+2\rho-\tau)(1+\gamma)\,(1+\lambda)}{2[(\tau-\rho)+(3+2k+2\rho-\tau)(1+\gamma)\,(1+\lambda)]},$$

we propose the function f_0 such that

$$f_0(z)=z-\frac{(\tau-\rho)}{(3+2k+2\rho-\tau)(1+\gamma)\,(1+\lambda)}z^2. \tag{3.13}$$

Using (3.10) and (3.13), we can have

$$\frac{(3+2k+2\rho-\tau)(1+\gamma)\,(1+\lambda)}{2[\tau-\rho+(3+2k+2\rho-\tau)(1+\gamma)\,(1+\lambda)]}f_0(z) \prec \frac{z}{1-z}, z \in \mathbb{U}_1^0.$$

Assume that

$$\begin{aligned}&\Re\left\{\frac{(3+2k+2\rho-\tau)(1+\gamma)\,(1+\lambda)}{2[\tau-\rho+(3+2k+2\rho-\tau)(1+\gamma)\,(1+\lambda)]}f_0(z)\right\}\\ &=\frac{(3+2k+2\rho-\tau)(1+\gamma)\,(1+\lambda)}{2[\tau-\rho+(3+2k+2\rho-\tau)(1+\gamma)\,(1+\lambda)]}\Re\left(f_0(z)\right), K=3+2k\\ &\geqslant -\frac{(K+2\rho-\tau)(1+\gamma)\,(1+\lambda))}{2[(\tau-\rho)+(K+2\rho-\tau)(1+\gamma)\,(1+\lambda)]}\frac{\tau-\rho+(K+2\rho-\tau))(1+\gamma)\,(1+\lambda)}{(K+2\rho-\tau)(1+\gamma)\,(1+\lambda)}.\end{aligned}$$

Thus, we see that

$$\min_{|z|\leqslant r}\Re\left\{\frac{(3+2k+2\rho-\tau)(1+\gamma)\,(1+\lambda)}{2[\tau-\rho+(3+2k+2\rho-\tau)(1+\gamma)\,(1+\lambda)]}f_0(z)\right\}=-\frac{1}{2}.$$

This establishes that the constant $\frac{(3+2k+2\rho-\tau)(1+\gamma)(1+\lambda))}{2[\tau-\rho+(3+2k+2\rho-\tau)(1+\gamma)(1+\lambda)]}$ is best possible. $\square$

4 Concluding Remarks

In this study, we have used idea of convex combination in defining some subfamilies $\mathcal{U}_k\mathcal{S}(\lambda,\gamma,\tau,\rho)$ and $\mathcal{U}_k^{\Im}\mathcal{S}(\lambda,\gamma,\tau,\rho)$ of holomorphic mappings involving starlike and convex mappings and associated with the conic domains. We derived sufficient conditions along with the characterization, coefficient bounds, integral representation and other related properties. Using the sufficient conditions for mappings belonging to the family $\mathcal{U}_k\mathcal{S}(\lambda,\gamma,\tau,\rho)$, we also defined a family $\mathcal{U}_k\mathcal{S}^*(\lambda,\gamma,\tau,\rho)$ and then making use of a particular sequence, we discussed some subordination result. Our findings can be related with the existing known results.

5 Declarations

The authors declare that the research is original and have not submitted anywhere in any form. All authors have equally contributed and have no competing interest to disclose. The authors accept any sort of ethical responsibility regarding the manuscript publication.

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