Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 2479–2491
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.26448.3317

# Periodic solutions for a class of perturbed fifth-order autonomous differential equations via averaging theory

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*(Communicated by Haydar Akca)*

**Abstract**

In this work, we use the averaging theory of first order to study the periodic solutions of the perturbed fifth-order autonomous differential equation

$$x^{(5)} - \lambda \ddddot{x} + (p^2+1)\dddot{x} - \lambda(p^2+1)\ddot{x} + p^2\dot{x} - \lambda p^2 x = \varepsilon F(x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}),$$

where $\lambda$, and $\varepsilon$ are real parameters, $p$ is rational number different from $-1$, $0$, $1$, $\varepsilon$ is a small enough and $F \in C^2$ is a nonlinear autonomous function. we present some applications to illustrate our main results.

Keywords: Periodic orbit, Fifth-order differential equation, Averaging theory
2020 MSC: 34C25, 34C29, 37G15

## 1 Introduction and statement of the main results

When studying of the dynamics of differential systems following the analysis of their equilibrium points, we should study the existence or not of their periodic orbits. Differential equations (DEs) are one of the most important tools in mathematical modeling. For examples, the phenomena of physics, fluid and heat flow, motion of objects, vibrations, chemical reactions and nuclear reactions have been modeled by systems of DEs. Many applications of ODEs of different orders can be found in the mathematical modeling of real-life problems. Second and third order differential equations can be found in [1, 4, 6, 8, 16, 20], fourth-order DEs often arise in many fields of applied science such as mechanics, quantum chemistry, electronic and control engineering. Also, beam theory [9], fluid dynamics [2] and [10], ship dynamics [29] and neural networks [17]. Numerically and analytically numerous approximations to solve such differential equations of various orders have is studied in the literature. Most solutions of the mathematical models of these applications must be approximated.

The averaging theory is a classical tool for the study of the dynamics of nonlinear differential systems with periodic forcing. The averaging theory has a long history that begins with the classical work of Lagrange and Laplace. Details of the averaging theory can be found in the books of Verhulst [28] and Sanders and Verhulst [23]. The averaging theory is used to the study of periodic solutions for second and higher order differential equations (see [11, 12, 13, 14, 15]).

In [24], the authors studied the periodic solution of the following fifth-order differential equation

$$x^{(5)} - e\ddddot{x} - d\dddot{x} - c\ddot{x} - b\dot{x} - ax = \varepsilon F(t, x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}), \tag{1.1}$$

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*Received:* February 2022 *Accepted:* July 2022

where $a = \lambda\mu\delta$, $b = -(\lambda\mu + \lambda\delta + \mu\delta)$, $c = \lambda + \mu + \delta + \lambda\mu\delta$, $d = -(1 + \lambda\mu + \lambda\delta + \mu\delta)$, $e = \lambda + \mu + \delta$, $\varepsilon$ is a small parameter and $F \in C^2$ is $2\pi-$ periodic in $t$. Here, the variable $x$ and the parameters $\lambda$, $\mu$, $\delta$ and $\varepsilon$ are real.

In [25], the authors studied equation (1.1) with $F = F(x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x})$ which is autonomous. They studied five cases.

The objective of this work is to study the periodic solutions for a class of fifth-order autonomous ordinary differential equations

$$x^{(5)} - \lambda \ddddot{x} + (p^2 + 1) \dddot{x} - \lambda(p^2 + 1)\ddot{x} + p^2\dot{x} - \lambda p^2 x = \varepsilon F(x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}), \tag{1.2}$$

where $\lambda$, and $\varepsilon$ are real parameters, $p$ is rational number different from $-1$, $0$, $1$, and $\varepsilon$ is small enough, $F \in C^2$ is a nonlinear autonomous function.

In general to obtain analytically periodic solutions of a differential system is a very difficult task, usually impossible. Recently the study of the periodic solutions of fifth-order of differential equations has been considered by several authors (see [5, 7, 19, 21, 26, 27]). Here, using the averaging theory we reduce this difficult problem for the differential equation (1.2) to find the zeros of a nonlinear system of three or four equations. For more information and details about the averaging theory see section (2) and the references quoted there.

Now, we present the principal results for the periodic solutions of equation (1.2). For the different value of the parameter $\lambda$, we distinguish the two following cases.

**Case 1:** $\lambda \neq 0, \quad p \in \{-1, 0, 1\}$.

**Case 2:** $\lambda = 0, \quad p \in \{-1, 0, 1\}$.

For each one of these cases, we will give a theorem which provides sufficient conditions for the existence of periodic solutions of equation (1.2) and we provide two application of our results.

**Theorem 1.1.** Assume that $p$ is a rational different from $-1, 0, 1$, $\lambda \neq 0$ in differential equation (1.2). For all positive simple zero $(r_0^*, Z_0^*, U_0^*)$ solution of the system

$$\mathcal{F}_i(r_0, Z_0, U_0) = 0, \ i = 1, ..., 3, \tag{1.3}$$

satisfying

$$\det\left(\frac{\partial(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)}{\partial(r_0, Z_0, U_0)}\right|_{(r_0, Z_0, U_0) = (r_0^*, Z_0^*, U_0^*)} \neq 0, \tag{1.4}$$

where

$$\begin{aligned}
\mathcal{F}_1(r_0, Z_0, U_0) &= \frac{1}{2\pi p_2}\int_0^{2\pi p_2} \cos\theta F(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5) d\theta, \\
\mathcal{F}_2(r_0, Z_0, U_0) &= \frac{1}{2\pi p_2}\int_0^{2\pi p_2} \frac{-pU_0 \sin\theta + r_0\cos(p\theta)}{r_0} \ F(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5) d\theta, \\
\mathcal{F}_3(r_0, Z_0, U_0) &= \frac{1}{2\pi p_2}\int_0^{2\pi p_2} \frac{pZ_0 \sin\theta - r_0\sin(p\theta)}{r_0} \ F(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5) d\theta,
\end{aligned} \tag{1.5}$$

be with $p = p_1/p_2$, where $p_1$, $p_2$ are positive integers, and

$$\begin{aligned}
\mathcal{A}_1 &= -\frac{r_0\cos\theta + \lambda r_0 \sin\theta}{(p^2-1)(\lambda^2+1)} + \frac{(U_0\lambda + Z_0 p)\cos(p\theta) - (U_0 p - Z_0\lambda)\sin(p\theta)}{p(p^2+\lambda^2)(p^2-1)}, \\
\mathcal{A}_2 &= \frac{-r_0\lambda\cos\theta + r_0 \sin\theta}{(p^2-1)(\lambda^2+1)} - \frac{(U_0 p - Z_0\lambda)\cos(p\theta) + (U_0\lambda + Z_0 p)\sin(p\theta)}{(p^2+\lambda^2)(p^2-1)}, \\
\mathcal{A}_3 &= \frac{r_0\cos\theta + \lambda r_0 \sin\theta}{(p^2-1)(\lambda^2+1)} + \frac{-(U_0\lambda + Z_0 p)p\cos(p\theta) + (U_0 p - Z_0\lambda)p\sin(p\theta)}{(p^2+\lambda^2)(p^2-1)}, \\
\mathcal{A}_4 &= \frac{r_0\lambda\cos\theta - r_0 \sin\theta}{(p^2-1)(\lambda^2+1)} + \frac{(U_0 p - Z_0\lambda)p^2\cos(p\theta) + (U_0\lambda + Z_0 p)p^2\sin(p\theta)}{(p^2+\lambda^2)(p^2-1)}, \\
\mathcal{A}_5 &= -\frac{r_0\cos\theta + \lambda r_0 \sin\theta}{(p^2-1)(\lambda^2+1)} + \frac{(U_0\lambda + Z_0 p)p^3\cos(p\theta) - (U_0 p - Z_0\lambda)p^3\sin(p\theta)}{(p^2+\lambda^2)(p^2-1)},
\end{aligned} \tag{1.6}$$

there is a periodic solution $x\,(t,\varepsilon)$ of equation (1.2) tending to the periodic solution

$$x(t) = -\frac{r_0^*(\cos t + \lambda \sin t)}{(p^2-1)(\lambda^2+1)} + \frac{(U_0^*\lambda + Z_0^* p)\cos(pt) - (U_0^* p - Z_0^*\lambda)\sin(pt)}{p(p^2+\lambda^2)(p^2-1)}, \tag{1.7}$$

of $x^{(5)} - \lambda\,\ddddot{x} + (p^2+1)\,\dddot{x} - \lambda(p^2+1)\ddot{x} + p^2\dot{x} - \lambda p^2 x = 0$, when $\varepsilon \longrightarrow 0$. Note that this solution is periodic of period $2\pi p_2$.

Theorem 1.1 will be proved in section 3. An applications of Theorem 1.1 is the following.

**Corollary 1.2.** If $F(x,\dot{x},\ddot{x},\dddot{x},\ddddot{x}) = \dot{x}^2 - \dot{x} - \dddot{x}$ then the differential equation (1.2) with $p=2$, $\lambda=3$ has two periodic solutions $x_i(t,\varepsilon)$ for $i=1,...,2$ tending to the periodic solutions

$$x_1(t) = -\cos t - 3\sin t - \frac{3}{4}\cos(2t) + \frac{1}{4}\sin(2t),$$

$$x_2(t) = \cos t + 3\sin t - \frac{3}{4}\cos(2t) + \frac{1}{4}\sin(2t),$$

of $x^{(5)} - 3\,\ddddot{x} + 5\,\dddot{x} - 15\ddot{x} + 4\dot{x} - 12x = 0$ when $\varepsilon \to 0$.

Corollary 1.2 is proved in Section 4.

**Theorem 1.3.** Assume that $p$ is a rational different from $-1, 0, 1$, $\lambda = 0$ in differential equation (1.2). For all positive simple zero $(r_0^*, Z_0^*, U_0^*, V_0^*)$ solution of the system

$$\mathcal{F}_i(r_0, Z_0, U_0, V_0^*) = 0,\ \ i = 1,...,4, \tag{1.8}$$

satisfying

$$\det\Big(\frac{\partial(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3,\mathcal{F}_4)}{\partial(r_0,Z_0,U_0,V_0)}\Big|_{(r_0,Z_0,U_0,V_0)=(r_0^*,Z_0^*,U_0^*,V_0^*)} \neq 0, \tag{1.9}$$

where

$$\begin{aligned}
\mathcal{F}_1(r_0,Z_0,U_0,V_0) &= \frac{1}{2\pi p_2}\int_0^{2\pi p_2} \cos\theta\ F(\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5)d\theta,\\
\mathcal{F}_2(r_0,Z_0,U_0,V_0) &= \frac{1}{2\pi p_2}\int_0^{2\pi p_2} \frac{-pU_0\sin\theta + r_0\cos(p\theta)}{r_0}\ F(\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5)d\theta,\\
\mathcal{F}_3(r_0,Z_0,U_0,V_0) &= \frac{1}{2\pi p_2}\int_0^{2\pi p_2} \frac{pZ_0\sin\theta - r_0\sin(p\theta)}{r_0}\ F(\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5)d\theta,\\
\mathcal{F}_4(r_0,Z_0,U_0,V_0) &= \frac{1}{2\pi p_2}\int_0^{2\pi p_2} F(\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5)d\theta,
\end{aligned} \tag{1.10}$$

be with $p = p_1/p_2$, where $p_1$, $p_2$ are positive integers, and

$$\begin{aligned}
\mathcal{B}_1 &= \frac{-r_0p^2\cos\theta + Z_0\cos(p\theta) - U_0\sin(p\theta) + (p^2-1)V_0}{p^2(p^2-1)},\\
\mathcal{B}_2 &= \frac{r_0p\sin\theta - U_0\cos(p\theta) - Z_0\sin(p\theta)}{p(p^2-1)},\\
\mathcal{B}_3 &= \frac{r_0\cos\theta - Z_0\cos(p\theta) + U_0\sin(p\theta)}{p^2-1},\\
\mathcal{B}_4 &= \frac{-r_0\sin\theta + U_0p\cos(p\theta) + Z_0p\sin(p\theta)}{p^2-1},\\
\mathcal{B}_5 &= \frac{-r_0\cos\theta + Z_0p^2\cos(p\theta) - U_0p^2\sin(p\theta)}{p^2-1},
\end{aligned} \tag{1.11}$$

there is a periodic solution $x\,(t,\varepsilon)$ of equation (1.2) tending to the periodic solution

$$x(t) = \frac{-r_0^* p^2 \cos t + Z_0^* \cos(pt) - U_0^* \sin(pt) + (p^2-1)V_0^*}{p^2(p^2-1)}, \tag{1.12}$$

of $x^{(5)} - \lambda\,\ddot{\ddot{x}} + (p^2+1)\,\dddot{x} - \lambda(p^2+1)\ddot{x} + p^2\dot{x} - \lambda p^2 x = 0$, when $\varepsilon \longrightarrow 0$. Note that this solution is periodic of period $2\pi p_2$.

Theorem 1.3 Will be proved in section 5. An application of Theorem 1.3 is given in the following corollary.

**Corollary 1.4.** If $F(x, \dot{x}, \ddot{x}, \dddot{x}, \ddot{\ddot{x}}) = x^2 - 2\dot{x}^2 - \dot{x}$ then the differential equation (1.2) with $p = \dfrac{1}{2}$, $\lambda = 0$ has four periodic solutions $x_i(t,\varepsilon)$ for $i = 1, ..., 4$ tending to the periodic solutions

$$x_1(t) = \frac{10}{3}(\cos t - \cos(\frac{1}{2}t) + \sin(\frac{1}{2}t)),\ x_2(t) = \frac{10}{3}(-\cos t + \cos(\frac{1}{2}t) + \sin(\frac{1}{2}t)),$$
$$x_3(t) = \frac{10}{3}(-\cos t - \cos(\frac{1}{2}t) - \sin(\frac{1}{2}t)),\ x_4(t) = \frac{10}{3}(\cos t + \cos(\frac{1}{2}t) - \sin(\frac{1}{2}t)),$$

of $x^{(5)} + \dfrac{5}{4}\,\dddot{x} + \dfrac{1}{4}\dot{x} = 0$ when $\varepsilon \to 0$.

Corollary 1.4 is proved in section 6.

## 2 Averaging theory

In this section, we recall the basic results from the averaging theory that we need for proving the main results of this work.

We consider the problem of the bifurcation of $T$–periodic solutions from differential systems of the form

$$\dot{\mathbf{x}} = F_0(\mathbf{x}, t) + \varepsilon F_1(\mathbf{x}, t) + \varepsilon^2 F_2(\mathbf{x}, t, \varepsilon), \tag{2.1}$$

with $\varepsilon > 0$ small enough. The functions $F_0, F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions, $T$–periodic in the variable $t$, and $\Omega$ is an open subset of $\mathbb{R}^n$.The principal hypothesis is that the unperturbed system

$$\dot{\mathbf{x}} = F_0(\mathbf{x}, t), \tag{2.2}$$

has a $k$-dimensional submanifold $\mathcal{Z}$ of $T$-periodic solutions.

Let $\mathbf{x}(\mathbf{z}, t)$ be the solution of the unperturbed system (2.2) such that $\mathbf{x}(0, \mathbf{z}) = \mathbf{z}$. The linearization of the system along the periodic solution $\mathbf{x}(\mathbf{z}, t)$ is written as

$$\dot{\mathbf{y}} = D_{\mathbf{x}} F_0(\mathbf{x}(\mathbf{z}, t), t)\mathbf{y}. \tag{2.3}$$

We define by $M_{\mathbf{z}}(t)$ as the fundamental matrix of the linear differential system (2.3), and by $\xi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first $k$ coordinates; i.e. $\xi(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$.

The periodic solutions contained in $\mathcal{Z}$ for system (2.1) is given in the following result.

**Theorem 2.1.** Let $W \subset \mathbb{R}^k$, be open and bounded and $\beta_1$: $\mathrm{Cl}(W) \to \mathbb{R}^{n-k}$ be a $\mathcal{C}^2$ function. We assume that

(i) $\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta_1(\alpha)),\ \ \alpha \in \mathrm{Cl}(W)\} \subset \Omega$ and that for each $\mathbf{z}_\alpha \in \mathcal{Z}$ the solution $\mathbf{x}(\mathbf{z}_\alpha, t)$ of (2.2) is $T$–periodic;

(ii) for each $\mathbf{z}_\alpha \in \mathcal{Z}$ there is a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of (2.3) such that the matrix $M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T)$ has in the upper right corner the $k \times (n-k)$ zero matrix, and in the lower right corner a $(n-k) \times (n-k)$ matrix $\Delta_\alpha$ with $\det(\Delta_\alpha) \neq 0$.

We consider the function $\mathcal{F} : \mathrm{Cl}(W) \to \mathbb{R}^k$

$$\mathcal{F}(\alpha) = \xi \left( \frac{1}{T} \int_0^T M_{\mathbf{z}_\alpha}^{-1}(t) F_1(\mathbf{x}(\mathbf{z}_\alpha, t), t) dt \right). \tag{2.4}$$

If there exists $a \in W$ with $\mathcal{F}(a) = 0$ and $\det\left(\left(d\mathcal{F}/d\alpha\right)(a)\right) \neq 0$, then there is a $T$–periodic solution $\varphi(t, \varepsilon)$ of system (2.1) such that $\varphi(0, \varepsilon) \to \mathbf{z}_a$ as $\varepsilon \to 0$.

Theorem 2.1 goes back to [18] and [22], for a shorter proof see [3].

We assume that the system (2.1) has an open set $V$ with $\mathrm{Cl}(V) \subset \Omega$ such that for each $\mathbf{z} \in \mathrm{Cl}(V)$. $\mathrm{Cl}(V)$ is a set formed only by periodic orbits; i.e. it is *isochronous* for the system (2.1). If $k = n$, we have the following theorem.

**Theorem 2.2.** Consider the function $\mathcal{F} : \mathrm{Cl}(V) \to \mathbb{R}^n$

$$\mathcal{F}(\mathbf{z}) = \int_0^T M_{\mathbf{z}}^{-1}(\mathbf{z}, t) F_1(\mathbf{x}(\mathbf{z}, t), t) dt. \tag{2.5}$$

If there exists $a \in V$ with $\mathcal{F}(a) = 0$ and $\det\left(\left(d\mathcal{F}/d\mathbf{z}\right)(a)\right) \neq 0$, then there exists a $T$–periodic solution of system (2.1) such that when $\varepsilon \to 0$ we have that $\mathbf{x}(0, \varepsilon) \to a$.

For a proof, see [3].

## 3 Proof of Theorem 1.1

If $y = \dot{x}$, $z = \ddot{x}$, $u = \dddot{x}$, $v = \ddddot{x}$, then equation (1.2) can be written as

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = u, \\ \dot{u} = v, \\ \dot{v} = \lambda p^2 x - p^2 y + \lambda(p^2+1) z - (p^2+1) u + \lambda v + \varepsilon F(x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}), \end{cases} \tag{3.1}$$

with $\varepsilon = 0$, system (3.1) has a unique singular point at the origin. The eigenvalues of the linear part of this system are $\pm i$, $\pm pi$ and $\lambda$. By the linear inversible transformation

$$\begin{pmatrix} X \\ Y \\ Z \\ U \\ V \end{pmatrix} = \begin{pmatrix} 0 & -\lambda p^2 & p^2 & -\lambda & 1 \\ -\lambda p^2 & p^2 & -\lambda & 1 & 0 \\ 0 & -\lambda & 1 & -\lambda & 1 \\ -\lambda p & p & -\lambda p & p & 0 \\ p^2 & 0 & p^2+1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix}, \tag{3.2}$$

we obtain the transformation of the system (3.1) as follows

$$\begin{cases} \dot{X} = -Y + \varepsilon H\left(X, Y, Z, U, V\right), \\ \dot{Y} = X, \\ \dot{Z} = -pU + \varepsilon H\left(X, Y, Z, U, V\right), \\ \dot{U} = pZ, \\ \dot{V} = \lambda V + \varepsilon H\left(X, Y, Z, U, V\right), \end{cases} \tag{3.3}$$

where $H(X,Y,Z,U,V) = F(A,B,C,D,J)$ with

$$A = \frac{\lambda(\lambda^2+1)U + p(p^2-1)V - p(\lambda^2+p^2)X - \lambda p(\lambda^2+p^2)Y + p(\lambda^2+1)Z}{p(p^2-1)(\lambda^2+p^2)(\lambda^2+1)},$$

$$B = \frac{-p(\lambda^2+1)U + \lambda(p^2-1)V - \lambda(\lambda^2+p^2)X + (\lambda^2+p^2)Y + \lambda(\lambda^2+1)Z}{(p^2-1)(\lambda^2+p^2)(\lambda^2+1)},$$

$$C = \frac{-\lambda p(\lambda^2+1)U + \lambda^2(p^2-1)V + (\lambda^2+p^2)X + \lambda(\lambda^2+p^2)Y - p^2(\lambda^2+1)Z}{(p^2-1)(\lambda^2+p^2)(\lambda^2+1)},$$

$$D = \frac{p^3(\lambda^2+1)U + \lambda^3(p^2-1)V + \lambda(\lambda^2+p^2)X - (\lambda^2+p^2)Y - \lambda p^2(\lambda^2+1)Z}{(p^2-1)(\lambda^2+p^2)(\lambda^2+1)},$$

$$J = \frac{\lambda p^3(\lambda^2+1)U + \lambda^4(p^2-1)V - (\lambda^2+p^2)X - \lambda(\lambda^2+p^2)Y + p^4(\lambda^2+1)Z}{(p^2-1)(\lambda^2+p^2)(\lambda^2+1)}.$$

The linear part of the system (3.3) at the origin is in the real Jordan normal from and that the change of variables (3.2) is defined when $p$ is a rational different from $-1$, $0, 1$, because the determinant of the matrix of the change is $p(p^2-1)^2(\lambda^2+1)(\lambda^2+p^2)$. We switch now from the cartesian variables $(X,Y,Z,U,V)$ to the cylindrical variables $(r,\theta,Z,U,V)$ of $\mathbb{R}^5$, with $X = r\cos\theta$, $Y = r\sin\theta$, and we find

$$\begin{cases} \dot{r} = \varepsilon\cos\theta G(r,\theta,Z,U,V), \\ \dot{\theta} = 1 - \dfrac{\varepsilon}{r}\sin\theta G(r,\theta,Z,U,V), \\ \dot{Z} = -pU + \varepsilon G(r,\theta,Z,U,V), \\ \dot{U} = pZ, \\ \dot{V} = \lambda V + \varepsilon G(r,\theta,Z,U,V), \end{cases} \tag{3.4}$$

where $G(r,\theta,Z,U,V) = F(a,b,c,d,j)$ with

$$a = \frac{\lambda(\lambda^2+1)U + p(p^2-1)V - p(\lambda^2+p^2)r\cos\theta - \lambda p(\lambda^2+p^2)r\sin\theta + p(\lambda^2+1)Z}{p(p^2-1)(\lambda^2+p^2)(\lambda^2+1)},$$

$$b = \frac{-p(\lambda^2+1)U + \lambda(p^2-1)V - \lambda(\lambda^2+p^2)r\cos\theta + (\lambda^2+p^2)r\sin\theta + \lambda(\lambda^2+1)Z}{(p^2-1)(\lambda^2+p^2)(\lambda^2+1)},$$

$$c = \frac{-\lambda p(\lambda^2+1)U + \lambda^2(p^2-1)V + (\lambda^2+p^2)r\cos\theta + \lambda(\lambda^2+p^2)r\sin\theta - p^2(\lambda^2+1)Z}{(p^2-1)(\lambda^2+p^2)(\lambda^2+1)},$$

$$d = \frac{p^3(\lambda^2+1)U + \lambda^3(p^2-1)V + \lambda(\lambda^2+p^2)r\cos\theta - (\lambda^2+p^2)r\sin\theta - \lambda p^2(\lambda^2+1)Z}{(p^2-1)(\lambda^2+p^2)(\lambda^2+1)},$$

$$j = \frac{\lambda p^3(\lambda^2+1)U + \lambda^4(p^2-1)V - (\lambda^2+p^2)r\cos\theta - \lambda(\lambda^2+p^2)r\sin\theta + p^4(\lambda^2+1)Z}{(p^2-1)(\lambda^2+p^2)(\lambda^2+1)}.$$

Dividing by $\dot{\theta}$, the system (3.4) becomes

$$\begin{cases} \dfrac{dr}{d\theta} = \varepsilon \cos\theta G + O(\varepsilon^2), \\ \dfrac{dZ}{d\theta} = -pU + \varepsilon(1 - \dfrac{pU}{r}\sin\theta)G + O(\varepsilon^2), \\ \dfrac{dU}{d\theta} = pZ + \varepsilon\dfrac{pZ}{r}\sin\theta G + O(\varepsilon^2), \\ \dfrac{dV}{d\theta} = \lambda V + \varepsilon(1 + \dfrac{\lambda V}{r}\sin\theta)G + O(\varepsilon^2), \end{cases} \tag{3.5}$$

where $G = G(r, \theta, Z, U, V)$.

The system 3.5 can be written as system (2.1) with

$$x = \begin{pmatrix} r \\ Z \\ U \\ V \end{pmatrix}, \quad t = \theta, \ F_0(\theta, x) = \begin{pmatrix} 0 \\ -pU \\ pZ \\ \lambda V \end{pmatrix},$$

$$F_1(\theta, x) = \begin{pmatrix} \cos\theta G \\ (1 - \dfrac{pU}{r}\sin\theta)G \\ \dfrac{pZ\sin\theta}{r}G \\ (1 + \dfrac{\lambda V}{r}\sin\theta)G \end{pmatrix}.$$

We will now apply the Theorem 2.1 to system (3.5). System (3.5) with $\varepsilon = 0$ has the $2\pi p_2$ -periodic solutions

$$\begin{pmatrix} r(\theta) \\ Z(\theta) \\ U(\theta) \\ V(\theta) \end{pmatrix} = \begin{pmatrix} r_0 \\ Z_0\cos(p\theta) - U_0\sin(p\theta) \\ U_0\cos(p\theta) + Z_0\sin(p\theta) \\ 0 \end{pmatrix}, \tag{3.6}$$

for $(r_0, Z_0, U_0, V_0) \in \mathbb{R}$ with $r_0 > 0$ and $p = p_1/p_2$, where $p_1$, $p_2$ are positive integers.

By the statement of Theorem 2.1, We have that $k = 3$ and $n = 4$. We take

$$W = \{(r_0, Z, U) : 0 < r_0^2 + Z^2 + U^2 < R^2\} \subset \mathbb{R}^3,$$

with $R > 0$, $\alpha = (r_0, Z_0, U_0)$ and $\beta : W \to \mathbb{R}$, $\beta(r_0, Z_0, U_0) = 0$. The set $\mathcal{Z}$ is

$$\mathcal{Z} = \{\mathbf{z}_\alpha = (r_0, Z_0, U_0, 0), \quad (r_0, Z_0, U_0) \in W\}.$$

For eache $\mathbf{z}_\alpha \in \mathcal{Z}$, the solution $x(\theta, \mathbf{z}_\alpha)$ is $2\pi p_2$ periodic. The fundamental matrix $M_{\mathbf{z}_\alpha}(\theta)$ of the linear system (3.5) with $\varepsilon = 0$ related to the $2\pi p_2$-periodic solution $\mathbf{z}_\alpha = (r_0, Z_0, U_0, 0)$ such that $M_{\mathbf{z}_\alpha}(0)$ is the identity of $R^4$, we obtain

$$M(\theta) = M_{z_\alpha}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(p\theta) & -\sin(p\theta) & 0 \\ 0 & \sin(p\theta) & \cos(p\theta) & 0 \\ 0 & 0 & 0 & e^{\lambda\theta} \end{pmatrix}.$$

The matrix

$$M^{-1}(0) - M^{-1}(2\pi n) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 - e^{-2\pi n\lambda} \end{pmatrix}, \tag{3.7}$$

satisfy the assumptions of statement $(ii)$ of Theorem 2.1, for $\lambda \neq 0$, we can apply it to system (3.5).

Now, by taking $\xi : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ is $\xi(r, Z, U, V) = (r, Z, U)$, we calculate $\mathcal{F}(\alpha)$ given by (2.4), we obtain

$$\mathcal{F}(\alpha) = \mathcal{F}(r_0, Z_0, U_0) = \begin{pmatrix} \frac{1}{2\pi p_2} \int_0^{2\pi p_2} \cos\theta F(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5) d\theta \\ \frac{1}{2\pi p_2} \int_0^{2\pi p_2} \frac{-pU_0 \sin\theta + r_0 \cos(p\theta)}{r_0} F(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5) d\theta \\ \frac{1}{2\pi p_2} \int_0^{2\pi p_2} \frac{pZ_0 \sin\theta - r_0 \sin(p\theta)}{r_0} F(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5) d\theta \end{pmatrix} = \begin{pmatrix} \mathcal{F}_1(r_0, Z_0, U_0) \\ \mathcal{F}_2(r_0, Z_0, U_0) \\ \mathcal{F}_3(r_0, Z_0, U_0) \end{pmatrix},$$

where $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ and $\mathcal{A}_5$ are given in the Theorem 1.1. Then, by Theorem 2.1 we have that for every simple zero $(r_0^*, Z_0^*, U_0^*) \in W$ of the function $\mathcal{F}(r_0, Z_0, U_0)$ we have a periodic solution $(r, Z, U, V)(\theta, \varepsilon)$ of system (3.5) such that

$$(r, Z, U, V)(0, \varepsilon) \longrightarrow (r_0^*, Z_0^*, U_0^*, 0) \quad \text{as} \quad \varepsilon \longrightarrow 0.$$

By going back to the changes of coordinates, we obtain a periodic solution $(r, \theta, Z, U, V)(t, \varepsilon)$ of system (3.4) such that

$$(r, \theta, Z, U, V)(0, \varepsilon) \longrightarrow (r_0^*, 0, Z_0^*, U_0^*, 0) \quad \text{as} \quad \varepsilon \longrightarrow 0.$$

There exists a periodic solution $(X, Y, Z, U, V)(t, \varepsilon)$ of system (3.3) such that

$$(X, Y, Z, U, V)(0, \varepsilon) \longrightarrow (r_0^*, 0, Z_0^*, U_0^*, 0) \quad \text{as} \quad \varepsilon \longrightarrow 0. \tag{3.8}$$

At last, we have a periodic solution $x(t, \varepsilon)$ of equation (1.2) such that

$$x(t, \varepsilon) \longrightarrow -\frac{r_0^*(\cos t + \lambda \sin t)}{(p^2 - 1)(\lambda^2 + 1)} + \frac{(U_0^*\lambda + Z_0^* p)\cos(pt) - (U_0^* p - Z_0^*\lambda)\sin(pt)}{p(p^2 + \lambda^2)(p^2 - 1)},$$

of equation $x^{(5)} - \lambda \ddot{\ddot{x}} + (p^2 + 1)\dddot{x} - \lambda(p^2 + 1)\ddot{x} + p^2\dot{x} - \lambda p^2 x = 0$, when $\varepsilon \longrightarrow 0$. Theorem 1.1 is proved.

## 4 Proof of Corollary 1.2

Consider the function

$$F(x, \dot{x}, \ddot{x}, \dddot{x}, \ddot{\ddot{x}}) = \dot{x}^2 - \dot{x} - \dddot{x},$$

which corresponds to the case $p = 2$ and $\lambda = 3$. The functions $\mathcal{F}_i$=$\mathcal{F}_i(r_0, Z_0, U_0)$ for $i = 1, ..., 3$ of Theorem 1.1 are

$$\begin{aligned} \mathcal{F}_1 &= \frac{r}{2340}(3U_0 - 11Z_0), \\ \mathcal{F}_2 &= -\frac{1}{13}U_0 + \frac{3}{26}Z_0 - \frac{11}{1170}U_0^2 + \frac{1}{450}r_0 - \frac{1}{390}U_0 Z_0, \\ \mathcal{F}_3 &= \frac{3}{26}U_0 + \frac{1}{13}Z_0 + \frac{1}{390}Z_0^2 + \frac{1}{600}r_0^2 + \frac{11}{1170}U_0 Z_0, \end{aligned}$$

System $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = 0$ has the tow solutions

$$(r_0^*, Z_0^*, U_0^*) = (30, 0, 0, -\frac{9}{2}, -\frac{33}{2}),$$
$$(r_0^*, Z_0^*, U_0^*) = (-30, 0, 0, -\frac{9}{2}, -\frac{33}{2}).$$

Since the Jacobian

$$\det(\frac{\partial(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)}{\partial(r_0, Z_0, U_0)}|_{(r_0, Z_0, U_0) = (r_0^*, Z_0^*, U_0^*)} = -\frac{1}{312} \neq 0,$$

by Theorem 1.1 equation (1.2) has the tow periodic solution of the statement of the corollary 1.2.

## 5 Proof of Theorem 1.3

If $\lambda = 0$, and $p$ is rational number different from $-1$, 0, 1, equation

$$x^{(5)} + (p^2+1)\dddot{x} + p^2\dot{x} = \varepsilon F(x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}), \tag{5.1}$$

can be written as

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = u, \\ \dot{u} = v, \\ \dot{v} = -p^2 y - (p^2+1)u + \varepsilon F(x, \dot{x}, \ddot{x}, \dddot{x}, \ddddot{x}), \end{cases} \tag{5.2}$$

with $\varepsilon = 0$, system (5.2) has a unique singular point at the origin. The eigenvalues of the linear part of this system are $\pm i$, $\pm pi$ and 0. By the change of variable

$$\begin{pmatrix} X \\ Y \\ Z \\ U \\ V \end{pmatrix} = \begin{pmatrix} 0 & 0 & p^2 & 0 & 1 \\ 0 & p^2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & p & 0 & p & 0 \\ p^2 & 0 & p^2+1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix}, \tag{5.3}$$

we obtain the transformation of the system (5.2) as follows

$$\begin{cases} \dot{X} = -Y + \varepsilon G\left(t, X, Y, Z, U, V\right), \\ \dot{Y} = X, \\ \dot{Z} = -pU + \varepsilon G\left(t, X, Y, Z, U, V\right), \\ \dot{U} = pZ, \\ \dot{V} = \varepsilon G\left(t, X, Y, Z, U, V\right), \end{cases} \tag{5.4}$$

where

$$F(A, B, C, D, J) = G\left(X, Y, Z, U, V\right),$$

with

$$A = \frac{(p^2-1)V - p^2X + Z}{p^2(p^2-1)}, \ B = \frac{-U+pY}{(p^2-1)p}, \ C = \frac{X-Z}{p^2-1}, \ D = \frac{pU-Y}{p^2-1},$$
$$J = \frac{-X+p^2Z}{p^2-1},$$

The linear part of the system (5.4) at the origin is in the real Jordan normal from and that the change of variables (5.3) is defined when $p$ is a rational different from $-1$, $0, 1$, because the determinant of the matrix of the change is $p^3(p^2-1)^2$. We switch now from the Cartesian variables $(X, Y, Z, U, V)$ to the cylindrical coordinates $(r, \theta, Z, U, V)$ of $\mathbb{R}^5$, with $X = r\cos\theta$, $Y = r\sin\theta$, and we find

$$\begin{cases} \dot{r} = \varepsilon\cos\theta H(r, \theta, Z, U, V), \\ \dot{\theta} = 1 - \varepsilon\dfrac{\sin\theta}{r}H(r, \theta, Z, U, V), \\ \dot{Z} = -pU + \varepsilon H(r, \theta, Z, U, V), \\ \dot{U} = pZ, \\ \dot{V} = \varepsilon H(r, \theta, Z, U, V), \end{cases} \tag{5.5}$$

where $H(r, \theta, Z, U, V) = F(a, b, c, d, j)$ with

$$a = \frac{(p^2-1)V - p^2 r\cos\theta + Z}{p^2(p^2-1)},\ b = \frac{-U + pr\sin\theta}{(p^2-1)p},\ c = \frac{r\cos\theta - Z}{(p^2-1)},\ d = \frac{pU - r\sin\theta}{(p^2-1)},$$

$$j = \frac{-r\cos\theta + p^2 Z}{p^2-1},$$

Dividing by $\dot{\theta}$, the system (5.5) becomes

$$\begin{cases} \dfrac{dr}{d\theta} = \varepsilon \cos\theta H + O(\varepsilon^2), \\ \dfrac{dZ}{d\theta} = -pU + \varepsilon \dfrac{r - pU\sin\theta}{r} H + O(\varepsilon^2), \\ \dfrac{dU}{d\theta} = pZ + \varepsilon \dfrac{pZ\sin\theta}{r} H + O(\varepsilon^2), \\ \dfrac{dV}{d\theta} = \varepsilon H + O(\varepsilon^2), \end{cases} \tag{5.6}$$

where $H = H(r, \theta, Z, U, V)$.

We will now apply Theorem 2.2 to the system (5.6). System (5.6) can be written as system (2.1) taking

$$x = \begin{pmatrix} r \\ Z \\ U \\ V \end{pmatrix}, t = \theta,\ F_0(\theta, x) = \begin{pmatrix} 0 \\ -pU \\ pZ \\ 0 \end{pmatrix},$$

$$F_1(\theta, x) = \begin{pmatrix} cos\theta H \\ \dfrac{r - pU\sin\theta}{r} H \\ \dfrac{pZ\sin\theta}{r} H \\ H \end{pmatrix}.$$

System (5.6) with $\varepsilon = 0$ has the $2\pi p_2$ -periodic solutions

$$\begin{pmatrix} r(\theta) \\ Z(\theta) \\ U(\theta) \\ V(\theta) \end{pmatrix} = \begin{pmatrix} r_0 \\ Z_0\cos(p\theta) - U_0\sin(p\theta) \\ U_0\cos(p\theta) + Z_0\sin(p\theta) \\ V_0 \end{pmatrix}, \tag{5.7}$$

for $(r_0, Z_0, U_0, V_0) \in \mathbb{R}$ with $r_0 > 0$ and $p = p_1/p_2$, where $p_1$, $p_2$ are positive integers. To look for the periodic solutions of our equation (5.1) we must calculate the zeros $\alpha = (r_0, Z_0, U_0, V_0)$ of the system $\mathcal{F}(\alpha) = 0$, Where $\mathcal{F}(\alpha)$ is given (2.5). The fundamental matrix $M(\theta)$ of the system (5.6) with $\varepsilon = 0$ along any periodic solution is

$$M(\theta) = M_{z_\alpha}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(p\theta) & -\sin(p\theta) & 0 \\ 0 & \sin(p\theta) & \cos(p\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We calculate the function $\mathcal{F}(\alpha)$ given by (2.5), we have obtained that the system $\mathcal{F}(\alpha) = 0$ can be written as

$$\begin{pmatrix} \mathcal{F}_1(r, Z, U, V, W) \\ \mathcal{F}_2(r, Z, U, V, W) \\ \mathcal{F}_3(r, Z, U, V, W) \\ \mathcal{F}_4(r, Z, U, V, W) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{5.8}$$

where

$$\mathcal{F}_1(r,Z,U,V) = \frac{1}{2\pi p_2}\int_0^{2\pi p_2} \cos\theta\; F(\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5)d\theta,$$

$$\mathcal{F}_2(r,Z,U,V) = \frac{1}{2\pi p_2}\int_0^{2\pi p_2} \frac{-pU_0\sin\theta + r_0\cos(p\theta)}{r_0}\; F(\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5)d\theta,$$

$$\mathcal{F}_3(r,Z,U,V) = \frac{1}{2\pi p_2}\int_0^{2\pi p_2} \frac{pZ_0\sin\theta - r_0\sin(p\theta)}{r_0}\; F(\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5)d\theta,$$

$$\mathcal{F}_4(r,Z,U,V) = \frac{1}{2\pi p_2}\int_0^{2\pi p_2} F(\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5)d\theta,$$

with $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{B}_4$, and $\mathcal{B}_5$ as in the statement of Theorem 1.3.

If determinant (1.9) is nonzero, the zeros $(r^*, Z^*, U^*, V^*)$ of system (5.8) with respect to the variable $r$, $Z$, $U$, and $V$, providing periodic orbits of system (5.6) with $\varepsilon > 0$ small enough if they are simple. By going back to the changes of coordinates, for all simple zero $(r^*, Z^*, U^*, V^*)$ of system (5.8), we exist a $2\pi p_2$-periodic solution $x(t)$ of equation (5.1) for $\varepsilon > 0$ small enough such that

$$x(t,\varepsilon) \longrightarrow \frac{-r_0^* p^2\cos t + Z_0^*\cos(pt) - U_0^*\sin(pt) + (p^2-1)V_0^*}{p^2(p^2-1)}, \tag{5.9}$$

of $x^{(5)} + (p^2+1)\dddot{x} + p^2\dot{x} = 0$, when $\varepsilon \longrightarrow 0$. Theorem 1.3 is proved.

## 6 Proof of Corollary 1.4

Consider the function

$$F(x,\dot{x},\ddot{x},\dddot{x},\ddddot{x}) = x^2 - 2\dot{x}^2 - \dot{x},$$

which corresponds to the case $p = \dfrac{1}{2}$ and $\lambda = 0$. The functions $\mathcal{F}_i$=$\mathcal{F}_i(r_0, Z_0, U_0, V_0)$ for $i = 1, ..., 4$ of Theorem 1.3 are

$$\begin{aligned}
\mathcal{F}_1 &= -\frac{32}{3}U_0^2 + \frac{16}{3}r_0Z_0 + \frac{32}{3}Z_0^2,\\
\mathcal{F}_2 &= \frac{32U_0^2Z_0 - 64V_0Z_0r_0 - 5U_0r_0}{3r},\\
\mathcal{F}_3 &= \frac{-64U_0V_0r_0 - 32U_0Z_0^2 + 5Z_0r_0}{3r},\\
\mathcal{F}_4 &= \frac{64}{9}U_0^2 + 16V_0^2 + \frac{64}{9}Z_0^2 - \frac{8}{9}r_0^2,
\end{aligned}$$

System $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \mathcal{F}_4 = 0$ has the four solutions

$$\begin{aligned}
(r_0^*, Z_0^*, U_0^*, V_0^*) &= (\frac{5}{2}, \frac{5}{8}, \frac{5}{8}, 0),\\
(r_0^*, Z_0^*, U_0^*, V_0^*) &= (-\frac{5}{2}, -\frac{5}{8}, \frac{5}{8}, 0),\\
(r_0^*, Z_0^*, U_0^*, V_0^*) &= (-\frac{5}{2}, \frac{5}{8}, -\frac{5}{8}, 0),\\
(r_0^*, Z_0^*, U_0^*, V_0^*) &= (\frac{5}{2}, -\frac{5}{8}, -\frac{5}{8}, 0).
\end{aligned}$$

Since the Jacobian (1.9) for theses four solutions $(r_0^*, Z_0^*, U_0^*, V_0^*)$ is

$$-\frac{640000}{243},$$

respectively, we obtain using Theorem 1.3, the four solutions given in statement of the corollary 1.4

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