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APPROXIMATELY GENERALIZED ADDITIVE FUNCTIONS IN SEVERAL VARIABLES

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ABSTRACT. The goal of this paper is to investigate the solution and stability in random normed spaces, in non–Archimedean spaces and also in p–Banach spaces and finally the stability using the alternative fixed point of generalized additive functions in several variables.

1. Introduction and preliminaries

In 1940, Ulam gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. The stability problem of functional equations originated from a question of Ulam [74] concerning the stability of group homomorphisms.

In 1941, Hyers [32] considered the case of approximately additive functions $f : X \longrightarrow Y$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in X$ and for some $\varepsilon > 0$, where X and Y are Banach spaces. Then there exists a unique additive function $A : X \longrightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \varepsilon$$

for all $x \in X$.

Aoki [5] and Rassias [56] provided a generalization of the Hyers theorem for additive and linear functions, respectively, by allowing the Cauchy difference to be unbounded.

Theorem 1.1. (*Th.M. Rassias*). *Let $f : X \to Y$ be a function from a normed vector space X into a Banach space Y subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in X$, where ε and p are constants with $\varepsilon > 0$ and $p < 1$. Then there exists a unique additive function $A : X \to Y$ satisfying

$$\|f(x) - A(x)\| \leq \varepsilon\|x\|^p/(1 - 2^{p-1}) \tag{1.2}$$

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for all $x \in X$. If $p < 0$ then inequality (1.1) holds for $x, y \neq 0$ and (1.2) for $x \neq 0$. Also, if for each fixed $x \in X$ the function $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then A is linear.

The above Theorem has provided a lot of influence during the last three decades in the development of a generalization of the Hyers–Ulam stability concept. This new concept is known as generalized Hyers–Ulam stability or Hyers–Ulam–Rassias stability of functional equations (see [14, 33]). Furthermore, a generalization of Rassias theorem was obtained by Găvruta, who replaced $\varepsilon(\| x \|^p + \| y \|^p)$ by a general control function $\varphi(x, y)$; cf. [21]–[27].

It was shown by Rassias [57] that the norm defined over a real vector space X is induced by an inner product if and only if for a fixed integer $n \geq 2$

$$n\|\frac{1}{n}\sum_{i=1}^{n} x_i\|^2 + \sum_{i=1}^{n} \|x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\|^2 = \sum_{i=1}^{n} \|x_i\|^2$$

for all $x_1, ..., x_n \in X$ (see also [4, 37]). During the last three decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers–Ulam stability to a number of functional equations and functions (see [13]–[28], [34, 36, 39, 41] and [59]–[66]). We also refer the readers to the books [1, 14, 33, 38, 58].

Now, we consider the general n-dimensional additive functional equation for $n \geq 2$ and then investigate the stability in random normed spaces and in non-Archimedean spaces, moreover, the stability for functions from quasi-normed spaces into p–Banach spaces and finally the stability by using the alternative fixed point, of an n-dimensional additive functional equation as follows:

$$\begin{aligned}&\sum_{k=2}^{n}\Big(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1} \cdots \sum_{i_{n-k+1}=i_{n-k}+1}^{n}\Big) f\Big(\sum_{i=1, i\neq i_1,\ldots,i_{n-k+1}}^{n} a_i x_i - \sum_{r=1}^{n-k+1} a_{i_r} x_{i_r}\Big) \\ &\quad + f\Big(\sum_{i=1}^{n} a_i x_i\Big) = 2^{n-1} a_1 f(x_1)\end{aligned} \tag{1.3}$$

where $a_1, ..., a_n \in \mathbb{Z} - \{0\}$ with $a_1 \neq \pm 1$. As a special case, if $n = 2$ in (1.3), then the functional equation (1.3) reduces to

$$f(a_1x_1 - a_2x_2) + f(a_1x_1 + a_2x_2) = 2a_1f(x_1)$$

also by putting $n = 3$ in (1.3), we obtain

$$\begin{aligned}&\sum_{i_1=2}^{2}\sum_{i_2=i_1+1}^{3} f\Big(\sum_{i=1, i\neq i_1, i_2}^{3} a_i x_i - \sum_{r=1}^{2} a_{i_r} x_{i_r}\Big) + \sum_{i_1=2}^{3} f\Big(\sum_{i=1, i\neq i_1}^{3} a_i x_i - a_{i_1} x_{i_1}\Big) \\ &\quad + f\Big(\sum_{i=1}^{3} a_i x_i\Big) = 2^2 a_1 f(x_1)\end{aligned}$$

that is,

$$\begin{aligned}&f(a_1x_1 - a_2x_2 - a_3x_3) + f(a_1x_1 - a_2x_2 + a_3x_3) + f(a_1x_1 + a_2x_2 - a_3x_3) \\ &\quad + f(a_1x_1 + a_2x_2 + a_3x_3) = 2^2a_1f(x_1)\end{aligned}$$

Throughout this paper, assume that $a_1, ..., a_n$ are nonzero fixed integers with $a_1 \neq \pm 1$.

2. Generalized additive functions in several variables

Let both X and Y be real vector spaces. We here present the solution of (1.3).

Theorem 2.1. *A function $f : X \to Y$ satisfies the functional equation* (1.3) *if and only if $f : X \to Y$ is additive.*

Proof. Let f satisfies (1.3). Setting $x_i = 0$ $(i = 1, ..., n)$ in (1.3), we have

$$\sum_{k=2}^{n}(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n})f(0)+f(0)=2^{n-1}a_1f(0)$$

that is,

$$\sum_{i_1=2}^{2}\sum_{i_2=i_1+1}^{3}\cdots\sum_{i_{n-1}=i_{n-2}+1}^{n}f(0)+\sum_{i_1=2}^{3}\sum_{i_2=i_1+1}^{4}\cdots\sum_{i_{n-2}=i_{n-3}+1}^{n}f(0)+\ldots+\sum_{i_1=2}^{n}f(0)$$
$$+f(0)=2^{n-1}a_1f(0)$$

that is,

$$(\binom{n-1}{n-1}+\binom{n-1}{n-2}+\ldots+\binom{n-1}{1}+1)f(0)=2^{n-1}a_1f(0) \tag{2.1}$$

on the other hand, we have the relation

$$1+\sum_{i=1}^{n-1}\binom{n-1}{i}=\sum_{i=0}^{n-1}\binom{n-1}{i}=2^{n-1}$$

hence, it follows from (2.1) that $2^{n-1}(a_1-1)f(0)=0$ and since $a_1 \neq \pm 1$, so $f(0)=0$.
Putting $x_i = 0$ $(i = 3, ..., n)$ in (1.3) and then using $f(0) = 0$, we get

$$f(a_1x_1-a_2x_2)+(\binom{n-2}{1}f(a_1x_1-a_2x_2)+\binom{n-2}{n-2}f(a_1x_1+a_2x_2))$$
$$+\ldots+(\binom{n-2}{n-3}f(a_1x_1-a_2x_2)+\binom{n-2}{2}f(a_1x_1+a_2x_2))$$
$$+(\binom{n-2}{n-2}f(a_1x_1-a_2x_2)+\binom{n-2}{1}f(a_1x_1+a_2x_2))$$
$$+f(a_1x_1+a_2x_2)=2^{n-1}a_1f(x_1)$$

that is,

$$(1+\sum_{i=1}^{n-2}\binom{n-2}{i})(f(a_1x_1+a_2x_2)+f(a_1x_1-a_2x_2))=2^{n-1}a_1f(x_1) \tag{2.2}$$

for all $x_1, x_2 \in X$. It follows from (2.2) and $\sum_{i=0}^{n-2}\binom{n-2}{i}=2^{n-2}$ that

$$f(a_1x_1+a_2x_2)+f(a_1x_1-a_2x_2)=2a_1f(x_1) \tag{2.3}$$

for all$x_1, x_2 \in X$. Setting $x_2 = 0$ in (2.3), gives $f(a_1x_1) = a_1f(x_1)$ for all $x_1 \in X$. Replacing x_2 by $\frac{a_1}{a_2}x_2$ in (2.3) and then using $f(a_1x_1) = a_1f(x_1)$, we get

$$f(x_1 + x_2) + f(x_1 - x_2) = 2f(x_1) \tag{2.4}$$

for all $x_1, x_2 \in X$. Putting $x_2 = x_1$ in (2.4) to get $f(2x_1) = 2f(x_1)$ for all $x_1 \in X$. Replacing x_1 and x_2 by $x_1 + x_2$ and $x_1 - x_2$ in (2.4), respectively, and then using $f(2x_1) = 2f(x_1)$, we obtain that

$$f(x_1 + x_2) = f(x_1) + f(x_2) \tag{2.5}$$

for all $x_1, x_2 \in X$, which implies that f is additive.

Conversely, suppose that f is additive, thus f satisfies (2.5). Putting $x_1 = x_2 = 0$ in (2.5), we get $f(0) = 0$. Setting $x_2 = x_1$ in (2.5), we have $f(2x_1) = 2f(x_1)$ for all $x_1 \in X$. Putting $x_2 = -2x_1$ in (2.5) and then using $f(2x_1) = 2f(x_1)$, we obtain $f(-x_1) = -f(x_1)$. Letting $x_2 = x_1$ and $x_2 = 2x_1$ in (2.5), respectively, we obtain that $f(2x_1) = 2f(x_1)$ and $f(3x_1) = 3f(x_1)$ for all $x_1 \in X$. So, $f(mx_1) = mf(x_1)$ for any integer m. Replacing x_2 by $-x_2$ in (2.5) and using the oddness of f, we have

$$f(x_1 + x_2) + f(x_1 - x_2) = 2f(x_1) \tag{2.6}$$

for all $x_1, x_2 \in X$. Replacing x_1 and x_2 by a_1x_1 and a_2x_2 in (2.6), respectively, then by using the identity $f(mx_1) = mf(x_1)$, we obtain

$$f(a_1x_1 + a_2x_2) + f(a_1x_1 - a_2x_2) = 2a_1f(x_1) \tag{2.7}$$

for all $x_1, x_2 \in X$. Now, we are going to prove our assumption by induction on $n \geq 2$. It holds on $n = 2$; see equation (2.7). Assume that it holds on the case where $n = p$; that is, we have

$$\sum_{k=2}^{p}\Big(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{p-k+1}=i_{p-k}+1}^{p}\Big)f\Big(\sum_{i=1, i\neq i_1,\dots,i_{p-k+1}}^{p} a_ix_i - \sum_{r=1}^{p-k+1} a_{i_r}x_{i_r}\Big) + f\Big(\sum_{i=1}^{p} a_ix_i\Big) = 2^{p-1}a_1f(x_1)$$

for all $x_1, ..., x_p \in X$. It follows from condition (2.5) that

$$f\Big(\sum_{i=1}^{p} a_ix_i + a_{p+1}x_{p+1}\Big) + f\Big(\sum_{i=1}^{p} a_ix_i - a_{p+1}x_{p+1}\Big) = 2f\Big(\sum_{i=1}^{p} a_ix_i\Big) \tag{2.8}$$

for all $x_1, ..., x_{p+1} \in X$. Replacing x_p by $-x_p$ in (2.8), we obtain

$$\begin{aligned} &f\Big(\sum_{i=1}^{p-1} a_ix_i - a_px_p + a_{p+1}x_{p+1}\Big) + f\Big(\sum_{i=1}^{p-1} a_ix_i - a_px_p - a_{p+1}x_{p+1}\Big) \\ &\quad = 2f\Big(\sum_{i=1}^{p-1} a_ix_i - a_px_p\Big) \end{aligned} \tag{2.9}$$

for all $x_1, ..., x_{p+1} \in X$. Adding (2.8) to (2.9), we have

$$f(\sum_{i=1}^{p-1} a_i x_i - a_p x_p - a_{p+1}x_{p+1}) + f(\sum_{i=1}^{p-1} a_i x_i - a_p x_p + a_{p+1}x_{p+1})$$
$$+ f(\sum_{i=1}^{p-1} a_i x_i + a_p x_p - a_{p+1}x_{p+1}) + f(\sum_{i=1}^{p-1} a_i x_i + a_p x_p + a_{p+1}x_{p+1})$$
$$= 2[f(\sum_{i=1}^{p-1} a_i x_i + a_p x_p) + f(\sum_{i=1}^{p-1} a_i x_i - a_p x_p)]$$

for all $x_1, ..., x_{p+1} \in X$. By using the above method, for x_{p-1} until x_2, we infer that

$$\sum_{k=2}^{p+1}(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1} ... \sum_{i_{p-k+2}=i_{p-k+1}+1}^{p+1})f(\sum_{i=1, i\neq i_1,...,i_{p-k+2}}^{p+1} a_i x_i - \sum_{r=1}^{p-k+2} a_{i_r}x_{i_r}) + f(\sum_{i=1}^{p+1} a_i x_i)$$
$$= 2[\sum_{k=2}^{p}(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1} ... \sum_{i_{p-k+1}=i_{p-k}+1}^{p})f(\sum_{i=1, i\neq i_1,...,i_{p-k+1}}^{p} a_i x_i - \sum_{r=1}^{p-k+1} a_{i_r}x_{i_r}) + f(\sum_{i=1}^{p} a_i x_i)]$$

for all $x_1, ..., x_{p+1} \in X$. Now, by the case $n = p$, we lead to

$$\sum_{k=2}^{p+1}(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1} ... \sum_{i_{p-k+2}=i_{p-k+1}+1}^{p+1})f(\sum_{i=1, i\neq i_1,...,i_{p-k+2}}^{p+1} a_i x_i - \sum_{r=1}^{p-k+2} a_{i_r}x_{i_r})$$
$$+ f(\sum_{i=1}^{p+1} a_i x_i) = 2[2^{p-1}a_1 f(x_1)]$$

for all $x_1, ..., x_{p+1} \in X$, so (1.3) holds for $n = p + 1$. This complete the proof of the theorem. □

3. Approximately additive functions in random normed spaces

The aim of this section is to investigate the stability of the given general n-dimensional additive functional equation (1.3), in random normed spaces.

In the sequel we adopt the usual terminology, notations and conventions of the theory of random normed spaces, as in [10, 47, 44, 71, 72]. Throughout this paper, let Δ^+ is the space of distribution functions that is,

$$\Delta^+ := \{F : \mathbb{R} \cup \{-\infty, \infty\} \to [0, 1] \ : \ F \ is \ left-continuous,$$
$$non-decreasing \ on \ \mathbb{R}, \ F(0) = 0 \ and \ F(+\infty) = 1\}$$

and the subset $D^+ \subseteq \Delta^+$ is the set,

$$D^+ = \{F \in \Delta^+ : l^- F(+\infty) = 1\}$$

where, $l^- f(x)$ denotes the left limit of the function f at the point x. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 3.1. ([71]) A function $T : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous triangular norm (briefly, a t–norm) if T satisfies the following conditions:

(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a,1) = a$ for all $a \in [0,1]$;
(d) $T(a,b) \leq T(c,d)$ whenever $a \leq c$ and $b \leq d$ for all $a,b,c,d \in [0,1]$.

Typical examples of continuous t–norms are $T_P(a,b) = ab$, $T_M(a,b) = \min(a,b)$ and $T_L(a,b) = \max(a+b-1,0)$ (the Łukasiewicz t-norm).

Recall (see [29], [30]) that if T is a t–norm and $\{x_n\}$ is a given sequence of numbers in $[0,1]$, $T_{i=1}^{n} x_i$ is defined recurrently by

$$T_{i=1}^{n} x_i = \begin{cases} x_1, & \text{if } n = 1, \\ T(T_{i=1}^{n-1} x_i, x_n), & \text{if } n \geq 2. \end{cases}$$

$T_{i=n}^{\infty} x_i$ is defined as $T_{i=1}^{\infty} x_{n+i}$.

It is known ([30]) that for the Łukasiewicz t-norm the following implication holds:

$$\lim_{n\to\infty} (T_L)_{i=1}^{\infty} x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^{\infty} (1 - x_n) < \infty.$$

Definition 3.2. ([72]) A *Random Normed space* (briefly, RN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous t–norm, and μ is a function from X into D^+ such that, the following conditions hold:

(RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
(RN2) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X$, $\alpha \neq 0$;
(RN3) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

Definition 3.3. Let (X, μ, T) be a RN-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $\epsilon > 0$ and $\lambda > 0$, there exists positive integer N such that $\mu_{x_n - x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.

(2) A sequence $\{x_n\}$ in X is called *Cauchy* if, for every $\epsilon > 0$ and $\lambda > 0$, there exists positive integer N such that $\mu_{x_n - x_m}(\epsilon) > 1 - \lambda$ whenever $n \geq m \geq N$.

(3) A RN-space (X, μ, T) is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X. A complete RN-space is said to be random Banach space.

Theorem 3.4. ([71]) *If (X, μ, T) is a RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

The theory of random normed spaces (RN-spaces) is important as a generalization of deterministic result of linear normed spaces and also in the study of random operator equations. The RN-spaces may also provide us the appropriate tools to study the geometry of nuclear physics and have important application in quantum particle physics. The generalized Hyers-Ulam stability of different functional equations in random normed spaces, RN-spaces and fuzzy normed spaces has been recently studied in, Alsina [3], Mirmostafaee, Mirzavaziri and Moslehian [50, 51, 52], Miheţ and Radu [44]-[47], Miheţ, Saadati and Vaezpour [48, 49], Baktash et. al [8] and Saadati et. al. [70].

From now on, we use the following abbreviation for a given function f:

$$Df(x_1,..,x_n) := \sum_{k=2}^{n}(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n})f(\sum_{i=1,i\neq i_1,\dots,i_{n-k+1}}^{n} a_i x_i - \sum_{r=1}^{n-k+1} a_{i_r}x_{i_r}) + f(\sum_{i=1}^{n} a_i x_i) - 2^{n-1}a_1 f(x_1).$$

Theorem 3.5. *Let X be a real linear space, (Y,Λ,T) be a complete RN-space and $\xi : X^n \to D^+$ ($n\in\mathbb{N}$, $n\geq 2$ and $\xi(x_1,...,x_n)$ is denoted by $\xi_{x_1,...,x_n}$) be a function such that*

$$\lim_{m\to\infty}\xi_{a_1^m x_1,\dots,a_1^m x_n}(|a_1|^m t) = 1 \tag{3.1}$$

for all $x_1,...,x_n\in X$, $t>0$ and

$$\lim_{m\to\infty} T_{\ell=1}^{\infty}(\xi_{a_1^{m+\ell-1}x,0,\dots,0}(2^{n-\ell-1}|a_1|^{m+\ell-1}t)) = 1 \tag{3.2}$$

for all $x\in X$ and all $t>0$. Suppose that $f : X\to Y$ is a function satisfying

$$\Lambda_{Df(x_1,\dots,x_n)}(t) \geq \xi_{x_1,\dots,x_n}(t) \tag{3.3}$$

for all $x_1,...,x_n\in X$ and $t>0$. Then there exists a unique additive function $A : X\to Y$ such that

$$\Lambda_{f(x)-A(x)}(t) \geq T_{\ell=1}^{\infty}(\xi_{a_1^{\ell-1}x,0,\dots,0}(2^{n-\ell-1}|a_1|^{\ell}t)) \tag{3.4}$$

for all $x\in X$ and $t>0$.

Proof. Putting $x_1 = x$ and $x_i = 0$ $(i=2,...,n)$ in (3.3), we obtain that

$$\Lambda_{\left(\sum_{k=2}^{n}\left(\sum_{i_1=2}^{k}\sum_{i_2=i_1+1}^{k+1}\cdots\sum_{i_{n-k+1}=i_{n-k}+1}^{n}\right)f(a_1x)+f(a_1x)-2^{n-1}a_1f(x)\right)}(t) \geq \xi_{x,0,\dots,0}(t)$$

for all $x\in X$ and $t>0$, that is,

$$\Lambda_{(\binom{n-1}{n-1}+\binom{n-1}{n-2}+\dots+\binom{n-1}{1}+1)f(a_1x)-2^{n-1}a_1f(x)}(t) \geq \xi_{x,0,\dots,0}(t)$$

for all $x\in X$ and $t>0$. It follows from last inequality that

$$\Lambda_{(1+\sum_{\ell=1}^{n-1}\binom{n-1}{\ell})f(a_1x)-2^{n-1}a_1f(x)}(t) \geq \xi_{x,0,\dots,0}(t)$$

for all $x\in X$ and $t>0$, hence by using the relation $1+\sum_{\ell=1}^{n-1}\binom{n-1}{\ell} = 2^{n-1}$, gives

$$\Lambda_{2^{n-1}f(a_1x)-2^{n-1}a_1f(x)}(t) \geq \xi_{x,0,\dots,0}(t)$$

for all $x\in X$ and $t>0$. So

$$\Lambda_{\frac{f(a_1x)}{a_1}-f(x)}(t) \geq \xi_{x,0,\dots,0}(2^{n-1}|a_1|t) \geq \xi_{x,0,\dots,0}(2^{n-2}|a_1|t)$$

for all $x \in X$ and $t > 0$, which implies that

$$\Lambda_{\frac{f(a_1^{\ell+1}x)}{a_1^{\ell+1}} - \frac{f(a_1^{\ell}x)}{a_1^{\ell}}}(t) \geq \xi_{a_1^{\ell}x,0,\ldots,0}(2^{n-1}|a_1|^{\ell+1}t) \tag{3.5}$$

for all $x \in X$, $t > 0$ and $\ell \in \mathbb{N}$. It follows from (3.5) and (RN_3) that

$$\begin{aligned}\Lambda_{\frac{f(a_1^2x)}{a_1^2} - f(x)}(t) &\geq T(\Lambda_{\frac{f(a_1^2x)}{a_1^2} - \frac{f(a_1x)}{a_1}}(\frac{t}{2}), \Lambda_{\frac{f(a_1x)}{a_1} - f(x)}(\frac{t}{2}))\\ &\geq T(\xi_{a_1x,0,\ldots,0}(2^{n-2}|a_1|^2t), \xi_{x,0,\ldots,0}(2^{n-2}|a_1|t))\\ &\geq T(\xi_{a_1x,0,\ldots,0}(2^{n-3}|a_1|^2t), \xi_{x,0,\ldots,0}(2^{n-2}|a_1|t))\end{aligned}$$

for all $x \in X$ and $t > 0$. Thus

$$\Lambda_{\frac{f(a_1^mx)}{a_1^m} - f(x)}(t) \geq T_{\ell=1}^{m}(\xi_{a_1^{\ell-1}x,0,\ldots,0}(2^{n-\ell-1}|a_1|^{\ell}t)) \tag{3.6}$$

for all $x \in X$ and $t > 0$. In order to prove the convergence of the sequence $\{\frac{f(a_1^mx)}{a_1^m}\}$, we replace x with $a_1^{m'}x$ in (3.6) to find that

$$\Lambda_{\frac{f(a_1^{m+m'}x)}{a_1^{m+m'}} - \frac{f(a_1^{m'}x)}{a_1^{m'}}}(t) \geq T_{\ell=1}^{m}(\xi_{a_1^{m'+\ell-1}x,0,\ldots,0}(2^{n-\ell-1}|a_1|^{m'+\ell}t))$$

for all $x \in x$ and all $t > 0$. Since the right hand side of the inequality tends to 1 as m' and m tend to infinity, the sequence $\{\frac{f(a_1^mx)}{a_1^m}\}$ is a Cauchy sequence. Therefore, one can define the function $A : X \to Y$ by

$$A(x) := \lim_{m\to\infty} \frac{1}{a_1^m} f(a_1^m x)$$

for all $x \in X$. Now, if we replace $x_1, \ldots, x_n$ with $a_1^m x_1, \ldots, a_1^m x_n$ in (3.3), respectively, it follows that

$$\Lambda_{\frac{Df(a_1^mx_1,\ldots,a_1^mx_n)}{a_1^m}}(t) \geq \xi_{a_1^mx_1,\ldots,a_1^mx_n}(|a_1|^m t) \tag{3.7}$$

for all $x_1, \ldots, x_n \in x$ and all $t > 0$. By letting $m \to \infty$ in (3.7), gives $DA(x_1, \ldots, x_n) = 0$ thus A satisfies (1.3). Hence by Theorem 2.1, the function $A : X \to Y$ is additive.

To prove (3.4) take the limit as $m \to \infty$ in (3.6).

Finally, to prove the uniqueness of the additive function A subject to (3.4), let us assume that there exists a additive function A' which satisfies (3.4). Since $A(a_1^m x) = a_1^m A(x)$ and $A'(a_1^m x) = a_1^m A'(x)$ for all $x \in X$ and $m \in \mathbb{N}$, from (3.4) it follows that

$$\begin{aligned}\Lambda_{A(x)-A'(x)}(t) &= \Lambda_{A(a_1^mx)-A'(a_1^mx)}(|a_1|^m t)\\ &\geq T(\Lambda_{A(a_1^mx)-f(a_1^mx)}(|a_1|^{m-1}t), \Lambda_{f(a_1^mx)-A'(a_1^mx)}(|a_1|^{m-1}t))\\ &\geq T(T_{\ell=1}^{\infty}(\xi_{a_1^{m+\ell-1}x,0,\ldots,0}(2^{n-\ell-1}|a_1|^{m+\ell-1}t))\\ &\qquad , T_{\ell=1}^{\infty}(\xi_{a_1^{m+\ell-1}x,0,\ldots,0}(2^{n-\ell-1}|a_1|^{m+\ell-1}t)))\end{aligned} \tag{3.8}$$

for all $x \in X$ and all $t > 0$. By letting $m \to \infty$ in (3.8), we find that $A = A'$. □

4. Approximately additive functions in non-Archimedean spaces

In 1897, Hensel [31] has introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications [42, 75, 67, 76].

A non-Archimedean field is a field K equipped with a function (valuation) $| \,.\, |$ from K into $[0, \infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$, and $|r + s| \leq \max\{|r|, |s|\}$ for all $r, s \in K$. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in N$. An example of a non-Archimedean valuation is the function $| \,.\, |$ taking everything but 0 into 1 and $|0| = 0$. This valuation is called trivial.

Definition 4.1. Let X be a vector space over a scalar field K with a non–Archimedean non-trivial valuation $| \,.\, |$. A function $\| \,.\, \| : X \to \mathbb{R}$ is a non–Archimedean norm (valuation) if it satisfies the following conditions:

(NA1) $\|x\| = 0$ if and only if $x = 0$;

(NA2) $\|rx\| = |r|\|x\|$ for all $r \in K$ and $x \in X$;

(NA3) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in X$ (the strong triangle inequality).

Then $(X, \| \,.\, \|)$ is called a non–Archimedean space.

Remark 4.2. Thanks to the inequality

$$\|x_m - x_l\| \leq \max\{\|x_{j+1} - x_j\| : l \leq j \leq m - 1\} \quad (m > l)$$

a sequence $\{x_m\}$ is Cauchy if and only if $\{x_{m+1} - x_m\}$ converges to zero in a non–Archimedean space. By a complete non–Archimedean space we mean one in which every Cauchy sequence is convergent.

The most important examples of non-Archimedean spaces are p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: "for $x, y > 0$, there exists $n \in \mathbb{N}$ such that $x < ny$."

Example 4.3. Let p be a prime number. For any nonzero rational number $x = \frac{a}{b}p^{n_x}$ such that a and b are integers not divisible by p, define the p-adic absolute value $|x|_p := p^{-n_x}$. Then $| \,.\, |$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $| \,.\, |$ is denoted by $\mathbb{Q}_p$ which is called the p-adic number field.

Note that if $p > 3$, then $|2^n| = 1$ in for each integer n.

Arriola and Beyer [6] investigated stability of approximate additive functions $f : \mathbb{Q}_p \to \mathbb{R}$. They showed that if $f : \mathbb{Q}_p \to \mathbb{R}$ is a continuous function for which there exists a fixed ϵ :

$$|f(x + y) - f(x) - f(y)| \leq \epsilon$$

for all $x, y \in Q_p$, then there exists a unique additive function $T : \mathbb{Q}_p \to \mathbb{R}$ such that

$$|f(x) - T(x)| \leq \epsilon$$

for all $x \in \mathbb{Q}_p$. Additionally in 2007, Moslehian and Rassias [54] proved the generalized Hyers–Ulam stability of the Cauchy functional equation and the quadratic functional equation in non–Archimedean normed spaces.

Theorem 4.4. *Let G is an additive group, X is a complete non–Archimedean space and $\psi : G^n \to [0, \infty)$ be a function such that*

$$\lim_{m\to\infty} \frac{1}{|a_1|^m} \psi(a_1^m x_1, ..., a_1^m x_n) = 0 \tag{4.1}$$

for all $x_1, ..., x_n \in G$, and

$$\tilde{\psi}(x) := \lim_{m\to\infty} \max\{\frac{1}{|a_1|^\ell}\psi(a_1^\ell x, 0, ..., 0) : \ 0 \le \ell < m\} \tag{4.2}$$

for each $x \in G$, exists. Suppose that $f : G \to X$ is a function satisfying

$$\|Df(x_1, ..., x_n)\| \le \psi(x_1, ..., x_n) \tag{4.3}$$

for all $x_1, ..., x_n \in G$. Then there exists a additive function $A : G \to X$ such that

$$\|f(x) - A(x)\| \le \frac{1}{|2^{n-1}a_1|}\tilde{\psi}(x) \tag{4.4}$$

for all $x \in G$. Moreover, if

$$\lim_{\jmath\to\infty}\lim_{m\to\infty} \max\{\frac{1}{|a_1|^\ell}\psi(a_1^\ell x, 0, ..., 0) : \ \jmath \le \ell < m + \jmath\} = 0$$

then A is the unique additive function satisfying (4.4).

Proof. Putting $x_1 = x$ and $x_i = 0$ $(i = 2, ..., n)$ in (4.3), we get

$$\|f(x) - \frac{1}{a_1}f(a_1x)\| \le \frac{1}{|2^{n-1}a_1|}\psi(x, 0, ..., 0) \tag{4.5}$$

for all $x \in G$. Replacing x by $a_1^{m-1}x$ in (4.5), we have

$$\|\frac{1}{a_1^{m-1}}f(a_1^{m-1}x) - \frac{1}{a_1^m}f(a_1^m x)\| \le \frac{1}{|2^{n-1}a_1^m|}\psi(a_1^{m-1}x, 0, ..., 0) \tag{4.6}$$

for all $x \in G$. It follows from (4.6) and (4.1) that the sequence $\{\frac{1}{a_1^m}f(a_1^m x)\}$ is Cauchy. Since X is complete, we conclude that $\{\frac{1}{a_1^m}f(a_1^m x)\}$ is convergent. So one can define the function $A : X \to Y$ by

$$A(x) := \lim_{m\to\infty} a_1^m f(\frac{x}{a_1^m})$$

for all $x \in G$. It follows from (4.5) and (4.6) by using induction that

$$\|f(x) - \frac{1}{a_1^m}f(a_1^m x)\| \le \frac{1}{|2^{n-1}a_1|}\max\{\frac{1}{|a_1|^\jmath}\psi(a_1^\jmath x, 0, ..., 0) : \ 0 \le \jmath < m\} \tag{4.7}$$

for all $m \in \mathbb{N}$ and all $x \in G$. By taking m to approach infinity in (4.7) and using (4.2), we obtain (4.4). By (4.1) and (4.3), we get

$$\begin{aligned}\|DA(x_1, ..., x_n)\| &= \lim_{m\to\infty}\frac{1}{|a_1|^m}\|Df(a_1^m x_1, ..., a_1^m x_n)\| \\ &\le \lim_{m\to\infty}\frac{1}{|a_1|^m}\psi(a_1^m x_1, ..., a_1^m x_n) = 0\end{aligned}$$

for all $x_1, ..., x_n \in G$. Therefore the function $A : G \to X$ satisfies (1.3). By Theorem 2.1, the function $A : X \to Y$ is additive.

If A' is another additive function satisfying (4.4), then

$$\begin{aligned}\|A(x)-A'(x)\| &= \lim_{\jmath\to\infty}\frac{1}{|a_1|^{\jmath}}\|A(a_1^{\jmath}x)-A'(a_1^{\jmath}x)\| \\ &\le \lim_{\jmath\to\infty}\frac{1}{|a_1|^{\jmath}}\max\{\ \|A(a_1^{\jmath}x)-f(a_1^{\jmath}x)\|,\|f(a_1^{\jmath}x)-A'(a_1^{\jmath}x)\|\ \} \\ &\le \frac{1}{|2^{n-1}a_1|}\lim_{\jmath\to\infty}\lim_{m\to\infty}\max\{\frac{1}{|a_1|^{\ell}}\psi(a_1^{\ell}x,0,...,0):\ \jmath\le\ell<m+\jmath\}=0\end{aligned}$$

for all $x\in G$, so $A=A'$. This completes the proof of the uniqueness of A. □

Corollary 4.5. *Let $\eta:[0,\infty)\to[0,\infty)$ be a function satisfying*
(i) $\eta(|a_1|t)\le\eta(|a_1|)\eta(t)$ for all $t\ge 0$;
(ii) $\eta(|a_1|)<|a_1|$.
Suppose that $\varepsilon>0$ and G be a normed space and let $f:G\to X$ satisfying

$$\|Df(x_1,...,x_n)\|\le\varepsilon\sum_{i=1}^{n}\eta(\|x_i\|)$$

for all $x_1,...,x_n\in G$. Then there exists a unique additive function $A:G\to X$ such that

$$\|f(x)-A(x)\|\le\frac{\varepsilon}{|2^{n-1}a_1|}\ \eta(\|x\|)$$

for all $x\in G$.

Proof. Defining $\psi:G\times G\to[0,\infty)$ by $\psi(x_1,...,x_n):=\varepsilon\sum_{i=1}^{n}\eta(\|x_i\|)$, we have

$$\lim_{m\to\infty}\frac{1}{|a_1|^m}\psi(a_1^m x_1,...,a_1^m x_n)\le\lim_{m\to\infty}(\frac{\eta(|a_1|)}{|a_1|})^m\psi(x_1,...,x_n)=0$$

for all $x_1,...,x_n\in G$. We have

$$\tilde{\psi}(x):=\lim_{m\to\infty}\max\{\frac{1}{|a_1|^{\ell}}\psi(a_1^{\ell}x,0,...,0):\ 0\le\ell<m\}=\psi(x,0,...,0)$$

and

$$\lim_{\jmath\to\infty}\lim_{m\to\infty}\max\{\frac{1}{|a_1|^{\ell}}\psi(a_1^{\ell}x,0,...,0):\ \jmath\le\ell<m+\jmath\}=\lim_{\jmath\to\infty}\frac{1}{|a_1|^{\jmath}}\psi(a_1^{\jmath}x,0,...,0)=0$$

for all $x\in G$. □

Remark 4.6. The classical example of the function η is the function $\eta(t)=t^p$ for all $t\in[0,\infty)$, where $p>1$ with the further assumption that $|a_1|<1$.

Remark 4.7. We can formulate similar statements to Theorem 4.4 in which we can define the sequence $A(x):=\lim_{m\to\infty}a_1^m f(\frac{x}{a_1^m})$ under suitable conditions on the function ψ then obtain similar result to Corollary 4.5 for $p<1$.

5. Approximately additive functions in p–Banach spaces

We consider some basic concepts concerning p–normed spaces.

Definition 5.1. (See [9, 68]). Let X be a real linear space. A function $\| \,.\, \| : X \to \mathbb{R}$ is a quasi-norm (valuation) if it satisfies the following conditions:

(QN1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$;

(QN2) $\|\lambda.\, x\| = |\lambda|.\|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$;

(QN3) There is a constant $M \geq 1$: $\|x + y\| \leq M(\|x\| + \|y\|)$ for all $x, y \in X$. Then $(X, \| \,.\, \|)$ is called a quasi-normed space.

The smallest possible M is called the modulus of concavity of $\| \,.\, \|$. A quasi-Banach space is a complete quasi-normed space.

A quasi-norm $\| \,.\, \|$ is called a p–norm $(0 < p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a p–Banach space.

By the Aoki-Rolewicz Theorem [68], each quasi-norm is equivalent to some p–norm (see also [9]). Since it is much easier to work with p–norms, henceforth we restrict our attention mainly to p–norms. Moreover in [73], J. Tabor has investigated a version of Hyers-Rassias-Gajda Theorem (see [20, 56]) in quasi-Banach spaces.

Our main result in this section is the following:

Theorem 5.2. *Let $\ell \in \{-1, 1\}$ be fixed, X be a p–normed space, Y be a p–Banach space and $\varphi : X^n \to [0, \infty)$ be a function such that*

$$\lim_{m\to\infty} \frac{1}{|a_1|^{m\ell}} \varphi(a_1^{m\ell} x_1, ..., a_1^{m\ell} x_n) = 0 \tag{5.1}$$

for all $x_1, ..., x_n \in X$, and

$$\widetilde{\varphi}(x) := \sum_{j=\frac{1-\ell}{2}}^{\infty} \frac{1}{|a_1|^{\ell j p}} \varphi^p(a_1^{\ell j} x, 0, ..., 0) < \infty \tag{5.2}$$

for all $x \in X$ (denoted $(\varphi(x_1, ..., x_n))^p$ by $\varphi^p(x_1, ..., x_n)$). Suppose that $f : X \to Y$ is a function that satisfies

$$\|Df(x_1, ..., x_n)\| \leq \varphi(x_1, ..., x_n) \tag{5.3}$$

for all $x_1, ..., x_n \in X$. Furthermore, assume that $f(0) = 0$ in (5.3) for the case $\ell = 1$. Then there exists a unique additive function $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{2^{n-1}|a_1|^{\frac{1+\ell}{2}}} [\widetilde{\varphi}(\frac{x}{a_1^{\frac{1-\ell}{2}}})]^{\frac{1}{p}} \tag{5.4}$$

for all $x \in X$.

Proof. For $\ell = 1$, putting $x_1 = x$ and $x_i = 0$ $(i = 2, ..., n)$ in (5.3), we obtain

$$\|2^{n-1} f(a_1 x) - 2^{n-1} a_1 f(x)\| \leq \varphi(x, 0, ..., 0) \tag{5.5}$$

for all $x \in X$. So

$$\|f(x) - \frac{1}{a_1} f(a_1 x)\| \leq \frac{1}{2^{n-1}|a_1|} \varphi(x, 0, ..., 0) \tag{5.6}$$

for all $x \in X$. Replacing x by a_1x in (5.6) and dividing by a_1 and summing the resulting inequality with (5.6), we get

$$\|f(x) - \frac{1}{a_1^2}f(a_1^2x)\| \le \frac{1}{2^{n-1}|a_1|}(\varphi(x, 0, ..., 0) + \frac{\varphi(a_1x, 0, ..., 0)}{|a_1|}) \tag{5.7}$$

for all $x \in X$. Hence

$$\|\frac{1}{a_1^l}f(a_1^lx) - \frac{1}{a_1^m}f(a_1^mx)\|^p \le \frac{1}{2^{(n-1)p}\ |a_1|^p}\sum_{j=l}^{m-1}\frac{1}{|a_1|^{jp}}\varphi^p(a_1^jx, 0, ..., 0) \tag{5.8}$$

for all nonnegative integers m and l with $m > l$ and for all $x \in X$. It follows from (5.1) and (5.8) that the sequence $\{\frac{1}{a_1^m}f(a_1^mx)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{a_1^m}f(a_1^mx)\}$ converges. Therefore, one can define the function $A : X \to Y$ by

$$A(x) := \lim_{m\to\infty}\frac{1}{a_1^m}f(a_1^mx)$$

for all $x \in X$. By (5.2) for $\ell = 1$ and (5.3),

$$\begin{aligned}\|DA(x_1, ..., x_n)\| &= \lim_{m\to\infty}\frac{1}{|a_1|^m}\|Df(a_1^mx_1, ..., a_1^mx_n)\| \\ &\le \lim_{m\to\infty}\frac{1}{|a_1|^m}\varphi(a_1^mx_1, ..., a_1^mx_n) = 0\end{aligned}$$

for all $x_1, ..., x_n \in X$. So $DA(x_1, ..., x_n) = 0$. By Theorem 2.1, the function $A : X \to Y$ is additive. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (5.8), we get the inequality (5.4) for $\ell = 1$.

Now, let $A' : X \to Y$ be another additive function satisfying (1.3) and (5.4). So

$$\begin{aligned}\|A(x) - A'(x)\|^p &= \frac{1}{|a_1|^{mp}}\|A(a_1^mx) - A'(a_1^mx)\|^p \\ &\le \frac{1}{|a_1|^{mp}}(\|A(a_1^mx) - f(a_1^mx)\|^p + \|A'(a^mx) - f(a^mx)\|^p) \\ &\le \frac{2}{|a_1|^{mp}2^{(n-1)p}|a_1|^p}\widetilde{\varphi}(a_1^mx)\end{aligned}$$

which tends to zero as $m \to \infty$ for all $x \in X$. So we can conclude that $A(x) = A'(x)$ for all $x \in X$. This proves the uniqueness of A.

Also, for $\ell = -1$, it follows from (5.5) that

$$\|f(x) - a_1f(\frac{x}{a_1})\| \le \frac{1}{2^{n-1}}\varphi(\frac{x}{a_1}, 0, ..., 0)$$

for all $x \in X$. Hence

$$\|a_1^lf(\frac{x}{a_1^l}) - a_1^mf(\frac{x}{a_1^m})\|^p \le \frac{1}{2^{(n-1)p}}\sum_{j=l}^{m-1}|a_1|^{jp}\varphi^p(\frac{x}{a_1^{j+1}}, 0, ..., 0) \tag{5.9}$$

for all nonnegative integers m and l with $m > l$ and for all $x \in X$. It follows from (5.9) that the sequence $\{a^mf(\frac{x}{a^m})\}$ is a Cauchy sequence for all $x \in X$. Since Y

is complete, the sequence $\{a^m f(\frac{x}{a^m})\}$ converges. So one can define the function $A : X \to Y$ by

$$A(x) := \lim_{m\to\infty} a_1^m f(\frac{x}{a_1^m})$$

for all $x \in X$. By (5.2) for $\ell = -1$ and (5.3),

$$\|DA(x_1, ..., x_n)\| = \lim_{m\to\infty} |a_1|^m \|Df(\frac{x_1}{a_1^m}, ..., \frac{x_n}{a_1^m})\| \leq \lim_{m\to\infty} |a_1|^m \varphi(\frac{x_1}{a_1^m}, ..., \frac{x_n}{a_1^m}) = 0$$

for all $x_1, ..., x_n \in X$. So $DA(x_1, ..., x_n) = 0$. By Theorem 2.1, the function $A : X \to Y$ is additive. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (5.9), we get the inequality (5.4) for $\ell = -1$. The rest of the proof is similar to the proof of previous section. □

Corollary 5.3. *Let ε, λ_i $(1 \leq i \leq n)$ be non-negative real numbers such that $\lambda_i < 1$ or $\lambda_i > 1$ $(1 \leq i \leq n)$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies*

$$\|Df(x_1, ..., x_n)\| \leq \varepsilon \sum_{i=1}^{n} \|x_i\|^{\lambda_i} \tag{5.10}$$

for all $x_1, ..., x_n \in X$. Then there exists a unique additive function $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{\varepsilon}{2^{n-1}||a_1|^p - |a_1|^{\lambda_1 p}|^{\frac{1}{p}}} \|x\|^{\lambda_1}$$

for all $x \in X$.

Proof. In Theorem 5.2, put $\varphi(x_1, ..., x_n) := \varepsilon \sum_{i=1}^{n} \|x_i\|^{\lambda_i}$ for all $x_1, ..., x_n \in X$. □

6. Approximately additive functions by using alternative fixed point

Baker [7] was the first author who applied the fixed point method in the study of Hyers–Ulam stability (see also [2]). A systematic study of fixed point theorems in nonlinear analysis is due to Isac and Rassias; cf. [35, 36]. Recently, Cădariu and Radu [11] applied the fixed point method to the investigation of the Cauchy additive functional equation [12, 55]. Using such a clever idea, they could present a short, simple proof for the Hyers-Ulam stability of Cauchy and Jensen functional equations (see also [18, 40, 53]).

We now introduce one of fundamental results of fixed point theory. For the proof, refer to [43, 69]. For an extensive theory of fixed point theorems and other nonlinear methods, the reader is referred to the book of Hyers, Isac and Rassias [33].

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if and only if d satisfies:

(GM_1) $d(x, y) = 0$ if and only if $x = y$;
(GM_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(GM_3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity.

Let (X, d) be a generalized metric space. An operator $T : X \to X$ satisfies a Lipschitz condition with Lipschitz constant L if there exists a constant $L \geq 0$ such that

$$d(Tx, Ty) \leq Ld(x, y)$$

for all $x, y \in X$. If the Lipschitz constant L is less than 1, then the operator T is called a strictly contractive operator.

We recall the following theorem by Margolis and Diaz.

Theorem 6.1. *Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive function $T : \Omega \to \Omega$ with Lipschitz constant L. Then for each given $x \in \Omega$, either*

$$d(T^m x, T^{m+1}x) = \infty \quad \text{for all } m \geq 0,$$

or other exists a natural number m_0 such that

- ★ $d(T^m x, T^{m+1}x) < \infty$ *for all* $m \geq m_0$;
- ★ *the sequence $\{T^m x\}$ is convergent to a fixed point y^* of T;*
- ★ *y^* is the unique fixed point of T in*

$$\Lambda = \{y \in \Omega : d(T^{m_0}x, y) < \infty\};$$

- ★ $d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ *for all* $y \in \Lambda$.

By using the idea of Cădariu and Radu, we will prove the stability of the general n-dimensional additive functional equation (1.3).

Theorem 6.2. *Let X be a real vector space and Y be a real Banach space. Suppose that $\ell \in \{-1, 1\}$ be fixed and $f : X \to Y$ a function for which there exists a function $\varphi : X^n \to [0, \infty)$ that satisfying (5.1) and (5.3) for all $x_1, ..., x_n \in X$. If there exists $0 < L = L(\ell) < 1$ such that the function $x \mapsto \psi(x) = \varphi(\frac{x}{a_1}, 0, ..., 0)$ has the property*

$$\psi(x) \leq L \,.\, |a_1|^\ell \,.\, \psi(\frac{x}{a_1^\ell}) \tag{6.1}$$

for all $x \in X$. Then there exists a unique additive function $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{L^{\frac{\ell+1}{2}}}{2^{n-1}(1-L)} \; \psi(x) \tag{6.2}$$

for all $x \in X$.

Proof. Let Ω be the set of all functions $g : X \to Y$ and introduce a generalized metric on Ω as follows:

$$d(g, h) = d_\psi(g, h) = \inf\{K \in (0, \infty) : \; \|g(x) - h(x)\| \leq K\psi(x), \; x \in X\}$$

It is easy to show that (Ω, d) is a generalized complete metric space [11].

Now we define a function $T : \Omega \to \Omega$ by $T\ g(x) = \frac{1}{a_1^\ell}\ g(a_1^\ell x)$ for all $x \in X$. Note that for all $g, h \in \Omega$,

$$\begin{aligned}
d(g, h) < K \;\; &\Rightarrow \;\; \|g(x) - h(x)\| \leq K\psi(x), && \textit{for all } x \in X,\\
&\Rightarrow \;\; \|\frac{1}{a_1^\ell} g(a_1^\ell x) - \frac{1}{a_1^\ell} h(a_1^\ell x)\| \leq \frac{1}{|a_1|^\ell} \; K \; \psi(a_1^\ell x), && \textit{for all } x \in X,\\
&\Rightarrow \;\; \|\frac{1}{a_1^\ell} g(a_1^\ell x) - \frac{1}{a_1^\ell} h(a_1^\ell x)\| \leq L \; K \; \psi(x), && \textit{for all } x \in X,\\
&\Rightarrow \;\; d(T\ g, T\ h) \leq L\ K.
\end{aligned}$$

Hence we see that $d(T\ g, T\ h) \leq L\ d(g, h)$ for all $g, h \in \Omega$, that is, T is a strictly self-function of Ω with the Lipschitz constant L.

Putting $x_1 = x$ and $x_i = 0$ $(i = 2, ..., n)$ in (5.3), we have (5.5) for all $x \in X$, thus, by using (6.1) with the case $\ell = 1$, we obtain that

$$\|f(x) - \frac{1}{a_1} f(a_1 x)\| \leq \frac{1}{2^{n-1}} \frac{1}{|a_1|}\ \varphi(x, 0, ..., 0) = \frac{1}{2^{n-1}} \frac{1}{|a_1|}\ \psi(a_1 x) \leq \frac{L}{2^{n-1}}\ \psi(x)$$

for all $x \in X$, that is, $d(f, Tf) \leq \frac{L}{2^{n-1}} < \infty$.

Also, if we substitute $x = \frac{x}{a_1}$ in (5.5) and use (6.1) with the case $\ell = -1$, then we see that

$$\|f(x) - a_1 f(\frac{x}{a_1})\| \leq \frac{1}{2^{n-1}}\ \psi(x)$$

for all $x \in X$, that is, $d(f, Tf) \leq \frac{1}{2^{n-1}} < \infty$.

Now, from the fixed point alternative in both cases, it follows that there exists a fixed point A of T in Ω such that

$$A(x) = \lim_{m\to\infty} \frac{1}{a_1^{m\ell}} f(a_1^{m\ell} x)$$

for all $x \in X$, since $\lim_{m\to\infty} d(T^m f, A) = 0$.

Also, if we replace $x_1, ..., x_n$ with $a_1^{m\ell} x_1, ..., a_1^{m\ell} x_n$ in (5.3), respectively, and divide by $a_1^{m\ell}$, then it follows from (5.1)that

$$\begin{aligned}\|DA(x_1, ..., x_n)\| &= \lim_{m\to\infty} \frac{1}{|a_1|^m} \|Df(a_1^m x_1, ..., a_1^m x_n)\| \\ &\leq \lim_{m\to\infty} \frac{1}{|a_1|^m} \varphi(a_1^m x_1, ..., a_1^m x_n) = 0\end{aligned}$$

for all $x_1, ..., x_n \in X$, so $DA(x_1, ..., x_n) = 0$. Thus the function A is additive.

According to the fixed point alterative, since A is the unique fixed point of T in the set $\Lambda = \{g \in \Omega : d(f, g) < \infty\}$, A is the unique function such that

$$\|f(x) - A(x)\| \leq K\ \psi(x)$$

for all $x \in X$ and $K > 0$. Again using the fixed point alterative, gives

$$d(f, A) \leq \frac{1}{1-L} d(f, Tf) \leq \frac{L^{\frac{\ell+1}{2}}}{2^{n-1}(1-L)}$$

so we conclude that

$$\|f(x) - A(x)\| \leq \frac{L^{\frac{\ell+1}{2}}}{2^{n-1}(1-L)}\ \psi(x)$$

for all $x \in X$. This completes the proof. □

Corollary 6.3. *Let X be a normrd space and Y be a Banach space. Let ε, λ_i $(1 \leq i \leq n)$ be non-negative real numbers such that $\lambda_i < 1$ or $\lambda_i > 1$ $(1 \leq i \leq n)$. Suppose that $f : X \to Y$ is a function satisfying (5.10) for all $x_1, ..., x_n \in X$. Then there exists a unique additive function $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{\varepsilon}{2^{n-1}||a_1| - |a_1|^{\lambda_1}|} \|x\|^{\lambda_1} \tag{6.3}$$

for all $x \in X$.

Proof. In Theorem 6.2, put $\varphi(x_1, ..., x_n) := \varepsilon \sum_{i=1}^{n} \|x_i\|^{\lambda_i}$ for all $x_1, ..., x_n \in X$. Then the relation (5.1) is true for $\lambda_i < 1$ or $\lambda_i > 1$ and also the inequality (6.1) holds with $L = |a_1|^{(\lambda_1 - 1)\ell}$. Thus from (6.2), yields (6.3). □

Corollary 6.4. *Assume that $\theta \geq 0$ is fixed. Let $f : X \to Y$ be a function such that*

$$\|Df(x_1, ..., x_n)\| \leq \theta$$

for all $x_1, ..., x_n \in X$, then there exists a unique addtive function $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{\theta}{n2^{n-1}(|a_1| - 1)}$$

holds for all $x \in X$.

Proof. Letting $\lambda_1 = 0$, $\varepsilon = \frac{\theta}{n}$ and applying corollary 6.3. □

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