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# Existence results of some $p(u)$-Laplacian systems

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**Abstract**

In this paper, we consider the existence of weak solutions for some $p(u)$-Laplacian problems with Dirichlet boundary conditions. Here the exponent of nonlinearity $p$ depends on the solution $u$ itself. Existence results for the associated boundary-value local problem are given by using a singular perturbation technique combined with the theory of Sobolev spaces with exponent variables.

Keywords: $p(u)$-Laplacian, elliptic problems, Sobolev spaces
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## 1 Introduction

Let $\Omega$ be a bounded domain of $\mathbb{R}^d, d \geq 2$, The purpose of this work is to investigate the existence of weak solutions to the boundary value problems given by

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(u)-2}\nabla u) = f + g(u)|\nabla u|^{p(u)-1} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f$ is a given data, $p$ is the nonlinear exponent function

$$p : \mathbb{R} \to [1, \ +\infty) \tag{1.1}$$

and

$$g : \mathbb{R} \to \mathbb{R} \tag{1.2}$$

is a bounded and continuous function that belongs to $L^1(\mathbb{R})$, and satisfying the following sign condition

$$-g(u)|\nabla u|^{p(.)-1}.u \geq 0. \tag{1.3}$$

In recent years, the study of this kind of problems arouses much interest with the development of elastic mechanics, electro-rheological fluids, image restoration, etc. We refer the readers to [13, 4, 5, 18, 1, 12, 10, 14, 15, 11, 16, 17, 19]. Existence results for different elliptic systems originating from the thermistor problem and from the modelling of thermorheological fluids, already obtained by Zhikov in [20, 21, 22, 23] and by Antontsev and Rodrigues in [2]. The

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majority of proofs in these works are based on the Schauder fixed-point theorem. Many diffusion and reaction-diffusion equations with distinct nonlocal terms have been studied by different authors like the pioneer works by Chipot et al. [6, 7, 8, 9]. In [6], Chipot et al. considered the $p(u)$-Laplacian problem of the following prototype

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(u)-2}\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Usually the motivation to study non-local problems is the fact that in reality physical measurements of certain quantities are not made in a punctual way but through a local averages. As pointed out in [3], the main difficulty in the analysis of these $p(u)$-problems relies in the fact that their weak formulation cannot be written as equalities in terms of duality in fixed Banach spaces. Indeed, Their sequences of solutions correspond to different exponents and therefore belong to possible distinct Sobolev spaces.

## 2 Preliminaries

The exponent function $p$ depend on the solution $u$ and therefore it depend on the space variable $x$, hence we can write the power $p$ as a variable exponent $q(x)$ in the following sense,

$$q(x) = p(u(x)) \ .$$

This allows us to look for the weak solutions to the problem (1) in a Sobolev space with variable exponents.
Let $\Omega$ be a bounded domain of $\mathbb{R}^d, d \geq 2$, we denote by $\mathcal{Q}(\Omega)$ the set of all Lebesgue-measurable functions $q : \Omega \to [1, \infty)$ and we define

$$q_- := \operatorname*{ess\,inf}_{x\in\Omega} q(x) \ , \ q_+ := \operatorname*{ess\,sup}_{x\in\Omega} q(x). \tag{2.1}$$

For any $q \in \mathcal{Q}(\Omega)$ , we introduce the variable exponent Lebesgue space by:

$$L^{q(\cdot)}(\Omega) = \{ \ u : \Omega \to \mathbb{R} \ / \ \rho_{q(\cdot)}(u) \ := \int_\Omega |u(x)|^{q(x)} dx < \infty\}. \tag{2.2}$$

Equipped with the Luxembourg norm

$$\|u\|_{q(\cdot)} \ := \inf \left\{\lambda > 0 \ : \ \rho_{q(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1\right\}, \tag{2.3}$$

$L^{q(\cdot)}(\Omega)$ becomes a Banach space. If

$$1 \leq q_- \leq q_+ < \infty, \tag{2.4}$$

$L^{q(\cdot)}(\Omega)$ is separable and, in particular, $C_0^\infty(\Omega)$ and $L^\infty(\Omega) \cap L^{q(\cdot)}(\Omega)$ are dense in $L^{q(\cdot)}(\Omega)$. Moreover, If we restrict (2.4) to

$$1 < q_- \leq q_+ < \infty, \tag{2.5}$$

then $L^{q(\cdot)}(\Omega)$ is reflexive. The dual space of $L^{q(\cdot)}(\Omega)$ is $L^{q'(\cdot)}(\Omega)$ , where $q'(x)$ is the generalised Hölder conjugate of $q(x)$ ,

$$\frac{1}{q(x)} + \frac{1}{q'(x)} = 1.$$

From (2.1) and (2.5), we can see that

$$1 < (q_+)' \leq \operatorname*{ess\,inf}_{x\in\Omega} q'(x) \leq \operatorname*{ess\,sup}_{x\in\Omega} q'(x) \leq (q_-)' < \infty.$$

The next proposition shows that there is a gap between the modular and the norm in $L^{q(\cdot)}(\Omega)$.

**Proposition 2.1.** If (2.5) holds, for $u \in L^{q(x)}(\Omega)$, then the following assertions hold

$$\min\left\{\|u\|_{q(\cdot)}^{q_-},\ \|u\|_{q(\cdot)}^{q_+}\right\} \le \rho_{q(\cdot)}(u) \le \max\left\{\|u\|_{q(\cdot)}^{q_-},\ \|u\|_{q(\cdot)}^{q_+}\right\},$$

$$\min\left\{\rho_{q(\cdot)}(u)^{\frac{1}{q_-}}\ ,\ \rho_{q(\cdot)}(u)^{\frac{1}{q_+}}\right\} \le \|u\|_{q(\cdot)} \le \max\left\{\rho_{q(\cdot)}(u)^{\frac{1}{q_-}}\ ,\ \rho_{q(\cdot)}(u)^{\frac{1}{q_+}}\right\}. \tag{2.6}$$

$$\|u\|_{q(\cdot)}^{q_-} - 1 \le \rho_{q(\cdot)}(u) \le \|u\|_{q(\cdot)}^{q_+} + 1. \tag{2.7}$$

**Proposition 2.2.** (Generalised Young's inequality)
For some positive constant $C(\delta)$ and any $\delta > 0$ we have:

$$|uv| \le \delta \frac{|u|^{q(x)}}{q(x)} + C(\delta)\frac{|v|^{q'(x)}}{q(x)}.$$

**Proposition 2.3.** (Generalised Hölder inequality)
i) For any functions $u \in L^{q(\cdot)}(\Omega)$ and $v \in L^{q'(\cdot)}(\Omega)$, we have:

$$\int_\Omega uvdx \le \left(\frac{1}{q_-} + \frac{1}{q'_-}\right)\|u\|_{q(\cdot)}\|v\|_{q'(\cdot)} \le 2\|u\|_{q(\cdot)}\|v\|_{q'(\cdot)}. \tag{2.8}$$

ii) For all $q$ satisfying to (2.4), we have the following continuous embedding,

$$L^{q(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega) \text{ whenever } q(x) \ge r(x) \text{ for a.e. } x \in \Omega. \tag{2.9}$$

We define also the generalised Sobolev space by

$$W^{1,q(\cdot)}(\Omega) := \{u \in L^{q(\cdot)}(\Omega) : \nabla u \in L^{q(\cdot)}(\Omega)\},$$

which is a Banach space for the norm

$$\|u\|_{1,q(\cdot)} := \|u\|_{q(\cdot)} + \|\nabla u\|_{q(\cdot)}. \tag{2.10}$$

The space $W^{1,q(\cdot)}(\Omega)$ is separable if (2.4) holds, and is reflexive when (2.5) is satisfied. We have as in (2.9)

$$W^{1,q(\cdot)}(\Omega) \hookrightarrow W^{1,r(\cdot)}(\Omega) \text{ whenever } q(x) \ge r(x) \text{ for a.e. } x \in \Omega. \tag{2.11}$$

Now, we introduce the following function space

$$W_0^{1,q(\cdot)}(\Omega) := \{u \in \mathrm{W}_0^{1,1}(\Omega) : \nabla u \in L^{q(\cdot)}(\Omega)\},$$

which we endow with the norm

$$\|u\|_{W_0^{1,q(\cdot)}(\Omega)} := \|u\|_1 + \|\nabla u\|_{q(\cdot)}. \tag{2.12}$$

If $q \in C(\overline{\Omega})$ , then the norm in $W_0^{1,q(\cdot)}(\Omega)$ is equivalent to $\|\nabla u\|_{q(\cdot)}$.
If $\Omega$ is a bounded domain with $\partial\Omega$ Lipschitz-continuous and $q$ is log-Hölder continuous, then $C_0^\infty(\Omega)$ is dense in $W_0^{1,q(.)}(\Omega)$. Recall that a function $q$ is log-Hölder continuous, if

$$\exists C > 0 :\ |q(x) - q(y)| \le \frac{C}{\ln\left(\frac{1}{|x-y|}\right)} \quad \forall x, y \in \Omega,\quad |x - y| < \frac{1}{2}. \tag{2.13}$$

If $q$ is a measurable function in $\Omega$ satisfying to $1 \le q_- \le q_+ < d$ and the Log-Hölder continuity property (2.13), then

$$\|u\|_{q^*(\cdot)} \le C\|\nabla u\|_{q(\cdot)} \quad \forall u \in W_0^{1,q(\cdot)}(\Omega)\ ,$$

for some positive constant $C$ depending on $q_+$, $d$ and on the constant of (2.13), where

$$q^*(x) := \begin{cases} \frac{dq(x)}{d-q(x)} & \text{if } q(x) < d \\ \infty & \text{if } q(x) \ge d. \end{cases}$$

On the other hand, if $q$ satisfies (2.13) and $q_- > d$, then

$$\|u\|_\infty \le C\|\nabla u\|_{q(\cdot)} \quad \forall u \in W_0^{1,q(\cdot)}(\Omega)\ ,$$

and where $C$ is another positive constant depending on $q_-$, $d$ and on the constant of (2.13).

**Lemma 2.4.** [6] Assume that

$$1 < \alpha \le q_n(x) \le \beta < \infty \quad \forall n \in \mathbb{N},$$

$$\textit{for a.e. } x \in \Omega, \textit{ for some constants } \alpha \textit{ and } \beta, \tag{2.14}$$

$$q_n \to q \textit{ a.e. in } \Omega, \textit{ as } n \to \infty, \tag{2.15}$$

$$\nabla u_n \to \nabla u \textit{ in } L^1(\Omega)^d, \textit{ as } n \to \infty, \tag{2.16}$$

$$\||\nabla u_n|^{q_n(x)}\|_1 \le C, \textit{ for some positive constant } C \textit{ not depending on } n. \tag{2.17}$$

*Then* $Du \in L^{q(\cdot)}(\Omega)^d$ *and*

$$\liminf_{n\to\infty} \int_\Omega |\nabla u_n|^{q_n(x)} dx \ge \int_\Omega |\nabla u|^{q(x)} dx. \tag{2.18}$$

**Lemma 2.5.** [7, 13] *For all* $\xi$ , $\eta \in \mathbb{R}^d$, *the following assertions hold true*:

$$2 \le p < \infty \Rightarrow \frac{1}{2^{p-1}} |\xi - \eta|^p \le \left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right) \cdot (\xi - \eta); \tag{2.19}$$

$$1 < p < 2 \Rightarrow (p-1)|\xi - \eta|^2 \le \left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right) \cdot (\xi - \eta) \left(|\xi|^p + |\eta|^p\right)^{\frac{2-p}{p}} \tag{2.20}$$

## 3 The principal result

In this section, we prove the existence of weak solutions for the local problem (1), so we define the set where we are going to look for these solutions as

$$W_0^{1,p(u)}(\Omega) := \{u \in W_0^{1,1}(\Omega) : \int_\Omega |\nabla u|^{p(u)} dx < \infty\}.$$

If $1 < p(u) < \infty$ for all $u \in \mathbb{R}$, this set is a Banach space for the norm $\|u\|_{W_0^{1,p(\cdot)}(\Omega)}$ defined at (2.12) which is equivalent to $\|\nabla u\|_{p(u)}$ in the case of $p(u) \in C(\overline{\Omega})$ . If for some constant $\alpha, 1 < \alpha \le p, p$ continuous, then, in view of (2.11), $W_0^{1,p(u)}(\Omega)$ is a closed subspace of $W_0^{1,\alpha}(\Omega)$, as a consequence it is separable and reflexive.

**Definition 3.1.** Let the function $p$ given in (1) be continuous and assume that

$$1 < \alpha \le p(u) \le \beta < \infty \quad \forall u \in \mathbb{R}, \tag{3.1}$$

for some constants $\alpha$ and $\beta$. Assume also that

$$f \in W^{-1,\alpha'}(\Omega). \tag{3.2}$$

A function $u \in W_0^{1,p(u)}(\Omega)$ is said to be a weak solution to the problem (1), if

$$\int_\Omega |\nabla u|^{p(u)-2} \nabla u \cdot \nabla v dx = \int_\Omega g(u) |\nabla u|^{p(u)-1} v dx + \langle f, v\rangle \quad \forall v \in W_0^{1,p(u)}(\Omega),$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $(W_0^{1,p(u)}(\Omega))'$ and $W_0^{1,p(u)}(\Omega)$.

Note that $q = p(u) \in \mathcal{Q}(\Omega)$ and the essential infimum $q_-$ and the essential supremum $q_+$ satisfy to (2.4) for all $u \in W_0^{1,p(u)}(\Omega)$ .

**Theorem 3.2.** *Let $\Omega \subset \mathbb{R}^d, d \geq 2$, be a bounded domain with $\partial\Omega$ Lipschitz-continuous. Assume that*

$$p : \mathbb{R} \to \mathbb{R} \ \text{ is a Lipschitz} - \text{continuous function} \tag{3.3}$$

*and that condition* (3.2) *holds. If*

$$d < \alpha \leq p(u) \leq \beta < \infty \quad \forall u \in \mathbb{R}, \tag{3.4}$$

*then there exists, at least, one weak solution to the problem* (1) *in the sense of Definition 3.1.*

**Proof** The proof of Theorem 3.2 is divided into several steps.
**Step 1: Approximation** For each $\varepsilon > 0$, we consider the auxiliary problem

$$\begin{cases} -div(|\nabla u|^{p(u)-2}\nabla u) - g(u)|\nabla u|^{p(u)-1} \\ +\varepsilon\left(-div(|\nabla u|^{\beta-2}\nabla u) - g(u)|\nabla u|^{\beta-1}\right) = f \ \text{ in } \Omega, \\ u = 0 \quad \text{ on } \partial\Omega. \end{cases} \tag{3.5}$$

For an exponent function $p$ satisfying (3.3) and (3.4), a function $u$ is said to be a weak solution to the regularized problem (3.5), if

$$\begin{cases} u \in W_0^{1,\beta}(\Omega), \\ \displaystyle\int_\Omega |\nabla u|^{p(u)-2}\nabla u \cdot \nabla v dx - \int_\Omega g(u)|\nabla u|^{p(u)-1} v dx \\ +\varepsilon\left(\displaystyle\int_\Omega |\nabla u|^{\beta-2}\nabla u \cdot \nabla v dx - \int_\Omega g(u)|\nabla u|^{\beta-1} v dx\right) = \langle f, v\rangle \quad \forall v \in W_0^{1,\beta}(\Omega), \end{cases}$$

where here $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W^{-1,\alpha'}(\Omega)$ and $W_0^{1,\alpha}(\Omega)$.

**Claim 3.3.** For each $\varepsilon > 0$ there exists a weak solution $u_\varepsilon$ to the problem (3.5).

**Proof of Claim** Let $w \in L^2(\Omega)$, from (3.4), we have

$$d < \alpha \leq p(w) \leq \beta < \infty \quad \text{for a.e.} \quad x \in \Omega. \tag{3.6}$$

therefore,

$$f \in W^{-1,\alpha'}(\Omega) \subset W^{-1,\beta'}(\Omega) \ ,$$

and, by the usual theory of monotone operators, there exists a unique $u = u_w$ solution to the problem

$$\begin{cases} u \in W_0^{1,\beta}(\Omega), \\ \displaystyle\int_\Omega |\nabla u|^{p(w)-2}\nabla u \cdot \nabla v dx - \int_\Omega g(u)|\nabla u|^{p(w)-1} v dx \\ +\varepsilon\left(\displaystyle\int_\Omega |\nabla u|^{\beta-2}\nabla u \cdot \nabla v dx - \int_\Omega g(u)|\nabla u|^{\beta-1} v dx\right) = \langle f, v\rangle \quad \forall v \in W_0^{1,\beta}(\Omega), \end{cases} \tag{3.7}$$

We take $v = u$ as a test function in (3.7) and using the Hölder inequality, we derive

$$\begin{aligned} \int_\Omega |\nabla u|^{p(w)} dx - \int_\Omega g(u)|\nabla u|^{p(w)-1} u dx + \varepsilon\left(\int_\Omega |\nabla u|^\beta dx - \int_\Omega g(u)|\nabla u|^{\beta-1} u dx\right) \\ = \int_\Omega f \nabla u dx. \end{aligned}$$

Then,

$$\varepsilon\int_\Omega |\nabla u|^\beta dx \leq \int_\Omega f\nabla u dx + \varepsilon\int_\Omega g(u)|\nabla u|^{\beta-1} u dx.$$

Thanks to Young's inequality and the fact that $g \in L^1(\mathbb{R})$, we have

$$\varepsilon\int_\Omega |\nabla u|^\beta dx \leq C_1\|\nabla u\|_\beta + C_2\int_\Omega |\nabla u|^\beta dx + C_3\int_\Omega |u|^\beta dx.$$

Thus

$$\varepsilon\|\nabla u\|_\beta^\beta \le C_1\|\nabla u\|_\beta + C_2\|\nabla u\|_\beta^\beta + C_4\|\nabla u\|_\beta^\beta.$$

We get

$$\|\nabla u\|_\beta \le C, \tag{3.8}$$

where $C = C(\alpha, \beta, \Omega, \varepsilon, f)$ independent of $w$. Since $\beta > d \ge 2$ one has $W_0^{1,\beta}(\Omega) \hookrightarrow L^2(\Omega)$, compactly and

$$\|u\|_2 = \|u_w\|_2 \le C,$$

for some positive constant $C = C(\alpha, \beta,\ \Omega,\ \varepsilon,\ f\ ,\ d)$ independent of $w$. Let us consider the mapping

$$B \ni w \to u_w \in B, \tag{3.9}$$

where $B := \{v \in L^2(\Omega)\ :\ \|v\|_2 \le C\}$. From Schauder's fixed point theorem, it is clear that this mapping will have a fixed point provided it is continuous. To prove this, we suppose that $w_n$ is a sequence in $L^2(\Omega)$ such that

$$w_n \to w \quad \text{in} \quad L^2(\Omega), \quad \text{as} \quad n \to \infty. \tag{3.10}$$

For every $n \in \mathbb{N}$, let $u_n$ be the solution to the problem (3.7) associated to $w = w_n$. By (3.8), we have

$$\|\nabla u_n\|_\beta \le C.$$

Hence, for some subsequence still noted $u_n$ and some $u$ we have

$$u_n \rightharpoonup u \quad \text{in} \quad W_0^{1,\beta}(\Omega), \quad \text{as} \quad n \to \infty, \tag{3.11}$$

$$u_n \to u \quad \text{in} \quad L^2(\Omega), \quad \text{as} \quad n \to \infty. \tag{3.12}$$

By (3.7), one has

$$\begin{aligned} &\int_\Omega \left(|\nabla u_n|^{p(w_n)-2}\nabla u_n + \varepsilon|\nabla u_n|^{\beta-2}\nabla u_n\right)\cdot\nabla v dx \\ &- \int_\Omega \left(g(u_n)|\nabla u_n|^{p(w_n)-1} + \varepsilon g(u_n)|\nabla u_n|^{\beta-1}\right) v dx = \langle f, v\rangle \quad \forall v \in W_0^{1,\beta}(\Omega). \end{aligned} \tag{3.13}$$

Using the monotonicity, we have also

$$\begin{aligned} &\int_\Omega \left(|\nabla u_n|^{p(w_n)-2}\nabla u_n + \varepsilon|\nabla u_n|^{\beta-2}\nabla u_n\right)\cdot\nabla(u_n - v)dx \\ &- \int_\Omega \left(g(u_n)|\nabla u_n|^{p(w_n)-1} + \varepsilon g(u_n)|\nabla u_n|^{\beta-1}\right)(u_n - v)dx \\ &- \int_\Omega \left(|\nabla v|^{p(w_n)-2}\nabla v + \varepsilon|\nabla v|^{\beta-2}\nabla v\right)\cdot\nabla(u_n - v)dx \\ &+ \int_\Omega \left(g(v)|\nabla v|^{p(w_n)-1} + \varepsilon g(v)|\nabla v|^{\beta-1}\right)(u_n - v)dx \ge 0 \quad \forall v \in W_0^{1,\beta}(\Omega). \end{aligned} \tag{3.14}$$

Taking $v = u_n - v$ in (3.13) and using (3), we obtain

$$\begin{aligned} &\langle f, u_n - v\rangle - \int_\Omega \left(|\nabla v|^{p(w_n)-2}\nabla v - \varepsilon|\nabla v|^{\beta-2}\nabla v\right)\cdot\nabla(u_n - v)dx \\ &+ \int_\Omega \left(g(v)|\nabla v|^{p(w_n)-1} + \varepsilon g(v)|\nabla v|^{\beta-1}\right)(u_n - v)dx \ge 0 \quad \forall v \in W_0^{1,\beta}(\Omega). \end{aligned} \tag{3.15}$$

In view of (3.10) one can assume that for some subsequence

$$w_n \to w \quad \text{a.e. in} \quad \Omega, \quad \text{as} \quad n \to \infty.$$

By using (3.3), we can apply Lebesgue's theorem, therefore

$$|\nabla v|^{p(w_n)-2}\nabla v \to |\nabla v|^{p(w)-2}\nabla v \quad \text{strongly in} \quad L^{\beta'}(\Omega)^d, \quad \text{as} \quad n \to \infty \tag{3.16}$$

and

$$g(v)|\nabla v|^{p(w_n)-1} \to g(v)|\nabla v|^{p(w)-1} \quad \text{strongly in} \quad L^{\beta'}(\Omega)^d, \quad \text{as} \quad n \to \infty. \tag{3.17}$$

Using (3.11), (3.16) and (3.17) we can pass to the limit in (3.15), we obtain

$$\begin{aligned} &\langle f, u-v\rangle - \int_\Omega \left(|\nabla v|^{p(w)-2}\nabla v + \varepsilon|\nabla v|^{\beta-2}\nabla v\right)\cdot\nabla(u-v)dx \\ &+\int_\Omega \left(g(v)|\nabla v|^{p(w)-1} + \varepsilon g(v)|\nabla v|^{\beta-1}\right)(u-v)dx \geq 0 \quad \forall v\in W_0^{1,\beta}(\Omega). \end{aligned} \tag{3.18}$$

Taking $v = u \pm \delta z$, with $z \in W_0^{1,\beta}(\Omega)$ and $\delta > 0$, we get

$$\begin{aligned} \pm\Big[\langle f, z\rangle &- \int_\Omega \left(|\nabla(u\pm\delta z)|^{p(w)-2}\nabla(u\pm\delta z) + \varepsilon|\nabla(u\pm\delta z)|^{\beta-2}\nabla(u\pm\delta z)\right)\cdot\nabla z dx \\ &+\int_\Omega \left(g(u\pm\delta z)|\nabla u\pm\delta z|^{p(w)-1}(u\pm\delta z) + \varepsilon g(u\pm\delta z)|\nabla u\pm\delta z|^{\beta-1}(u\pm\delta z)\right) z dx\Big] \geq 0. \end{aligned} \tag{3.19}$$

By letting $\delta \to 0$ in (3.19), we obtain

$$\begin{aligned} &\int_\Omega \left(|\nabla u|^{p(w)-2}\nabla u + \varepsilon|\nabla u|^{\beta-2}\nabla u\right)\cdot\nabla z dx \\ &-\int_\Omega \left(g(u)|\nabla u|^{p(w)-1} + \varepsilon g(u)|\nabla v|^{\beta-1}\right) z dx = \langle f, z\rangle \quad \forall z\in W_0^{1,\beta}(\Omega). \end{aligned}$$

Thus $u = u_w$. In view of (3.12) and since the limit is uniquely determined, we conclude that

$$u_{w_n} \to u_w \quad \text{strongly in} \quad L^2(\Omega), \quad \text{as} \quad n\to\infty,$$

we deduce that the mapping (3.9) is continuous and thus concludes the proof of the claim.

It means that, for each $\varepsilon > 0$ there exists $u_\varepsilon \in W_0^{1,\beta}(\Omega)$ such that

$$\begin{aligned} &\int_\Omega |\nabla u_\varepsilon|^{p(u_\varepsilon)-2}\nabla u_\varepsilon\nabla v dx - \int_\Omega g(u_\varepsilon)|\nabla u_\varepsilon|^{p(u_\varepsilon)-1} v dx + \varepsilon\int_\Omega |\nabla u_\varepsilon|^{\beta-2}\nabla u_\varepsilon\cdot\nabla v dx \\ &-\varepsilon\int_\Omega g(u_\varepsilon)|\nabla u_\varepsilon|^{\beta-1} v dx = \langle f, v\rangle \quad \forall v\in W_0^{1,\beta}(\Omega). \end{aligned} \tag{3.20}$$

Recall that

$$d < \alpha \leq p(u_\varepsilon) \leq \beta < \infty \quad \forall\varepsilon > 0, \quad \text{for a.e.} \quad x\in\Omega.$$

**Step 2: Passage to the limit as $\varepsilon \to 0$**
We take $v = u_\varepsilon$ in (3), we obtain

$$\begin{aligned} &\int_\Omega \left(|\nabla u_\varepsilon|^{p(u_\varepsilon)} - g(u_\varepsilon)|\nabla u_\varepsilon|^{p(u_\varepsilon)-1}u_\varepsilon\right)dx \\ &+\varepsilon\int_\Omega \left(|\nabla u_\varepsilon|^{\beta} - g(u_\varepsilon)|\nabla u_\varepsilon|^{\beta-1}u_\varepsilon\right)dx = \langle f, u_\varepsilon\rangle. \end{aligned}$$

By using the Hölder inequality, we get

$$\begin{aligned} \int_\Omega |\nabla u_\varepsilon|^\alpha dx \leq C\||\nabla u_\varepsilon|^\alpha\|_{\frac{p(u_\varepsilon)}{\alpha}} &\leq C\left(\int_\Omega |\nabla u_\varepsilon|^{p(u_\varepsilon)}dx + 1\right)^{\overline{\frac{1}{\left(\frac{p(u_\varepsilon)}{\alpha}\right)_-}}} \\ &\leq C\left(\int_\Omega |\nabla u_\varepsilon|^{p(u_\varepsilon)}dx + 1\right), \end{aligned} \tag{3.21}$$

where $C = C(\alpha, \beta, \Omega)$. Therefore

$$\langle f, u_\varepsilon\rangle \leq C_1||\nabla u_\varepsilon||_\alpha \leq C_1\Big(\int_\Omega |\nabla u_\varepsilon|^{p(u_\varepsilon)}dx + 1\Big)^{\frac{1}{\alpha}}. \tag{3.22}$$

From (3.22) and by using Young's inequality, we obtain

$$\int_\Omega |\nabla u_\varepsilon|^{p(u_\varepsilon)}dx + \varepsilon\|\nabla u_\varepsilon\|_\beta^\beta \leq C. \tag{3.23}$$

Using (3.21) and (3.22), we can also deduce that

$$||\nabla u_\varepsilon||_\alpha \leq C, \tag{3.24}$$

where $C$ is a positive constant which does not depend on $\varepsilon$.
Therefore, by the compact embedding $W_0^{1,\alpha}(\Omega) \hookrightarrow L^2(\Omega)$, for some subsequence still denoted by $u_n$ and some $u$ we have

$$u_{\varepsilon_n} \rightharpoonup u \quad \text{in} \quad W_0^{1,\alpha}(\Omega), \quad \text{as} \quad n \to \infty \tag{3.25}$$

$$\nabla u_{\varepsilon_n} \rightharpoonup \nabla u \quad \text{in} \quad L^\alpha(\Omega)^d, \quad \text{as} \quad n \to \infty \tag{3.26}$$

$$\begin{aligned} u_{\varepsilon_n} &\to u \quad \text{in} \quad L^2(\Omega), \quad \text{as} \quad n \to \infty \\ u_{\varepsilon_n} &\to u \quad \text{a.e. in} \quad \Omega, \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.27}$$

Note that, from (3.4), $u$ is Hölder-continuous and, by (3.3), so does $p(u)$ .
From (3.27), one has also

$$p(u_{\varepsilon_n}) \to p(u) \quad \text{a.e. in} \quad \Omega, \quad \text{as} \quad n \to \infty. \tag{3.28}$$

We recall that

$$d < \alpha \leq p(u_{\varepsilon_n}) \leq \beta < \infty \quad \forall n \in \mathbb{N}, \quad for \quad a.e. \quad x \in \Omega. \tag{3.29}$$

We replace $u_{\varepsilon_n}$ by $u_\varepsilon$ in (3.23), using (3.26), (3.28) and (3.29) so, we can apply the Lemma 2.4, we obtain

$$u \in W_0^{1,p(u)}(\Omega). \tag{3.30}$$

By the monotonicity, we have

$$\begin{aligned} &\int_\Omega \Big(|\nabla u_{\varepsilon_n}|^{p(u_{\varepsilon_n})-2}\nabla u_{\varepsilon_n} + \varepsilon|\nabla u_{\varepsilon_n}|^{\beta-2}\nabla u_{\varepsilon_n}\Big) \cdot \nabla(u_{\varepsilon_n} - v)dx \\ &\quad - \int_\Omega \Big(g(u_{\varepsilon_n})|\nabla u_{\varepsilon_n}|^{p(u_{\varepsilon_n})-1} + \varepsilon g(u_{\varepsilon_n})|\nabla u_{\varepsilon_n}|^{\beta-1}\Big)(u_{\varepsilon_n} - v)dx \\ &\quad\quad - \int_\Omega \Big(|\nabla v|^{p(u_{\varepsilon_n})-2}\nabla v + \varepsilon|\nabla v|^{\beta-2}\nabla v\Big) \cdot \nabla(u_{\varepsilon_n} - v)dx \\ &\quad\quad\quad + \int_\Omega \Big(g(v)|\nabla v|^{p(u_{\varepsilon_n})-1} + \varepsilon g(v)|\nabla v|^{\beta-1}\Big)(u_{\varepsilon_n} - v)dx \geq 0 \\ &\forall v \in W_0^{1,\beta}(\Omega). \end{aligned} \tag{3.31}$$

By replacing $u_{\varepsilon_n}$ with $u_\varepsilon$ and taking $u_{\varepsilon_n} - v$ in the place of $v$ in (3), we can rewrite (3) as

$$\begin{aligned} &\langle f, u_{\varepsilon_n} - v\rangle - \Big(\int_\Omega \Big(|\nabla v|^{p(u_{\varepsilon_n})-2}\nabla v + \varepsilon|\nabla v|^{\beta-2}\nabla v\Big) \cdot \nabla(u_{\varepsilon_n} - v)dx \\ &\quad - \int_\Omega \Big(g(v)|\nabla v|^{p(u_{\varepsilon_n})-1} + \varepsilon g(v)|\nabla v|^{\beta-1}\Big)(u_{\varepsilon_n} - v)dx\Big) \geq 0, \end{aligned} \tag{3.32}$$

say for all $v \in C_0^\infty(\Omega)$ .
By using (3.28),and the Lebesgue theorem, we have for such a $v$

$$|\nabla v|^{p(u_{\varepsilon_n})-2}\nabla v \to |\nabla v|^{p(u)-2}\nabla v \quad \text{in} \quad L^{\alpha'}(\Omega)^d, \quad \text{as} \quad n \to \infty \tag{3.33}$$

and

$$g(v)|\nabla v|^{p(u_{\varepsilon_n})-1} \to g(v)|\nabla v|^{p(u)-1} \quad \text{in} \quad L^{\alpha'}(\Omega)^d, \quad \text{as} \quad n \to \infty. \tag{3.34}$$

From (3.24), (3.25), (3.33) and (3.34), we can pass to the limit in (3) as $n \to \infty$ therefore

$$\begin{aligned} &\langle f, u - v\rangle - \Big(\int_\Omega |\nabla v|^{p(u)-2}\nabla v \cdot \nabla(u - v)dx \\ &\quad - \int_\Omega g(v)|\nabla v|^{p(u)-1}(u - v)dx\Big) \geq 0 \quad \forall v \in C_0^\infty(\Omega). \end{aligned} \tag{3.35}$$

From the assumptions (3.3) and (3.4), $p(u)$ is Hölder-continuous which implies that $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(u)}(\Omega)$. Thus, (3.34) holds true also for all $v \in W_0^{1,p(u)}(\Omega)$.
So we can take $v = u \pm \delta z$, where $z \in W_0^{1,p(u)}(\Omega)$ and $\delta > 0$, in (3.34) we get

$$\pm\left(\langle f, z\rangle - \Big(\int_\Omega |\nabla u|^{p(u)-2}\nabla u\cdot\nabla z dx - \int_\Omega g(v)|\nabla v|^{p(u)-1} z dx\Big)\right) \geq 0. \tag{3.36}$$

Which implies,

$$\int_\Omega |\nabla u|^{p(u)-2}\nabla u\cdot\nabla z dx - \int_\Omega g(v)|\nabla v|^{p(u)-1} z dx = \langle f, z\rangle \quad \forall z \in W_0^{1,p(u)}(\Omega). \tag{3.37}$$

That's completes the proof of Theorem 3.2.

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