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On Noether's conservation laws of the Sine-Gordon equation using moving frames

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Abstract

The wide significance of the problem of finding conservation laws in a great number of applications in mechanics and physics is beyond any doubt. The aim of this paper is to obtain conservation laws of the Sine-Gordon equation via the concept of moving frames and the variational principle. For this purpose, we first present a Lagrangian whose Euler-Lagrange equation is the Sine-Gordon equation, and then by Noether's First Theorem and Mansfield's method, we obtain the space of conservation laws in terms of invariants and the adjoint representation of a moving frame, for that Lagrangian, which is invariant under HRT group action.

Keywords: Sine-Gordon equation, moving frame, conservation laws, normalized differential invariants, syzygy
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1 Introduction

The nonlinear Sine-Gordon equation is a nonlinear partial difefrential equation arises in various physical applications such as Josephson junction transmission lines, charge density waves, relativistic field theory, motion of dislocations in crystals, and so on. In the study of partial differential equations, conservation laws have many considerable uses. They are momentous for investigating integrability and establishing existence and uniqueness of solutions. In addition, they play an essential role in descriptions of physically conserved quantities such as mass, energy and momentum, as well as charge and other constants of motion. In this paper, we will deal with the Sine-Gordon equation given in the form

$$u_{tt} - c^2 u_{xx} + m^2 \text{sin} u = 0, \tag{1.1}$$

where c and m is a known constant, $u = u(x,t)$ represents the wave displacement at position x and time t. In the case of mechanical transmission line, $u = u(x,t)$ describes an angle of rotation of the pendulums. There are various methods for finding conservation laws of the Sine-Gordon equation. In 1918, Emmy Noether in pivotal paper [7], proved the substantial result that for systems arising from a variational principle, every conservation law of the system comes from a Lie group action that leave the Lagrangian invariant (Theorem 4.29 of [8]). Recently in [2, 4, 5], Mansfield

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and Gonçalves considered diverse Lagrangians, which are invariant under a Lie group action, where independent variables are invariant. In recent works [3], Mansfield and Gonçalves considered invariant Lagrangians under a Lie group action, where independent variables are no longer invariant. They presented the mathematical structure behind both the Euler-Lagrange equations and the set of conservation laws, and they proved that Noether's conservation laws can be displayed as the product of adjoint representation of a right moving frame, which is equivariant, and a matrix where the columns are vectors of invariants.

Our goal in this paper is to find conservation laws of the equation (1.1) by applying Noether's Theorem and moving frames. In this paper, after giving a Lagrangian whose Euler-Lagrange equation is the Sine-Gordon equation, first we take a suitable moving frame for Hyperbolic Rotation- Translation (HRT) group action that leave the Lagrangian invariant and obtain syzygies between normalized differential invariants, then according to [3], we calculate Noether's conservation laws of equation (1.1), in terms of the Adjoint representation of the moving frame, vectors of invariants and a matrix which represents the group action on the 1-forms.

In section 2, we will briefly give some background on moving frames, differential invariants of a group action, invariant differential operators, and invariant forms. Throughout section 2 we will use the group action of a hyperbolic group on the space $(x, t, u(x, t))$, which the hyperbolic group is symmetry group of the Euler-Lagrange equation and variational problem.

In section 3, we concentrate on invariant calculus of variations and find the adjoint representation associated to the hyperbolic group action. Then, we end this section with the calculation of Noether's conservation laws associated to equation (1.1), in terms of vectors of invariants, the adjoint representation of the moving frame and a matrix which represents the group action on the 1-forms.

2 Moving frames, differential invariants of a group action and invariant forms

The Sine-Gordon equation (1.1) is the Euler-Lagrange equation for the variational problem

$$\Phi[u] = \iint \frac{1}{2}(-u_t^2 + c^2u_x^2 - 2m^2\text{cos}u)dxdt, \tag{2.1}$$

in other words, the equation (1.1) is the Euler-Lagrange equation of the Lagrangian

$$L = \frac{1}{2}(-u_t^2 + c^2u_x^2 - 2m^2\text{cos}u).$$

So, variational symmetry Hyperbolic Rotation-Translation (HRT) group G of the functional $\Phi[u]$ with infinitesimal generators

$$-c\partial_x + \partial_t, \qquad c\partial_x + \partial_t, \qquad c^2t\partial_x + x\partial_t, \tag{2.2}$$

is a symmetry group of the Sine-Gordon equation (1.1) (see Theorem 4.14 of [8]). Note that in [6] we have explained the structure of Hyperbolic Rotation-Translation group and formation steps of this group.

We remind that a group action of G on M is a map

$$G \times M \to M, \qquad (g, z) \;\to\; \tilde{z} = g\cdot z,$$

which satisfies either $g\cdot(h\cdot z) = (gh)\cdot z$, called a *left action*, or $g\cdot(h\cdot z) = (hg)\cdot z$, called a *right action*.

The action of the Lie group G associated to vector fields (2.2) on a 2-dimensional manifold M with coordinates (x, t), is given as follows

$$\begin{aligned}
\tilde{x} &= -c\alpha + c\beta + \frac{1}{2}(e^{c\theta}x + e^{-c\theta}x + ce^{c\theta}t - ce^{-c\theta}t) = -c\alpha + c\beta + x\cosh(c\theta) + ct\sinh(c\theta),\\
\tilde{t} &= \frac{1}{2c}(2c\alpha + 2c\beta + e^{c\theta}x - e^{-c\theta}x + ce^{c\theta}t + ce^{-c\theta}t) = \alpha + \beta + \frac{x}{c}.\sinh(c\theta) + t\cosh(c\theta),
\end{aligned}$$

where α, β and θ are constants that parametrize the group action.

Definition 2.1. We say, two smooth surfaces $\mathcal{K}$ and $\mathcal{O}$ contained in $\mathbb{R}^n$, such that, $\dim(\mathcal{K}) = \alpha$, $\dim(\mathcal{O}) = \beta$, $0 \leqslant \alpha, \beta \leqslant n$, $\alpha + \beta \geqslant n$, intersect *transversally* if for every $x \in \mathcal{K} \cap \mathcal{O}$, the tangent spaces $T_x\mathcal{K}$ and $T_x\mathcal{O}$, as subspaces of $T_x\mathbb{R}^n$, satisfy

$$T_x\mathcal{K} + T_x\mathcal{O} = T_x\mathbb{R}^n.$$

Suppose G is a Lie group which acts freely and regularly on some domain Ω in smooth manifold M, then as given in page 115 of [5], for every $x \in \Omega$, there is a neighbourhood $\mathcal{U}$ of x such that the following hold.
- The group orbits all have the same dimension of the group and foliate $\mathcal{U}$.
- There is a surface $\mathcal{K} \subset \mathcal{U}$ that crosses these orbits transversally at a single point. This surface is called the *cross section*.
- If $\mathcal{O}(z)$ represents the orbit through z, then the element $g \in G$ taking $z \in \mathcal{U}$ to $\{k\} = \mathcal{O}(z) \cap \mathcal{K}$ is unique.

Now by above conditions, we define a *right moving frame* as the map $\rho : \mathcal{U} \to G$ that sends an element $z \in \mathcal{U}$ to the unique group element $g = \rho(z)$ such that

$$\rho(z) \cdot z = k, \qquad \{k\} = \mathcal{O}(z) \cap \mathcal{K}.$$

According to [5] in page 117, for obtaining the right moving frame, first we define the cross section $\mathcal{K}$ as the locus of the set of equations $\psi_j(z) = 0$, for $j = 1, \cdots, r = dim(G)$, then, to obtain the group element that takes z to k, we solve the so called *normalization equations*

$$\psi_j(\tilde{z}) = \psi_j(g \cdot z) = 0, \qquad j = 1, \cdots, r,$$

for the r group parameters that describe the Lie group near its identity element, which yields the frame ρ in parametric form.

We now consider the group action G associated to vector fields (2.2) on the space $(x, t, u(x,t))$, where u is invariant.

Example 2.2. Consider the group action G on the space $(x, t, u(x,t))$ as follows

$$\begin{pmatrix} \tilde{x} \\ \tilde{t} \end{pmatrix} = \begin{pmatrix} \cosh(c\theta) & c\sinh(c\theta) \\ \frac{1}{c}\sinh(c\theta) & \cosh(c\theta) \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} + \begin{pmatrix} -c\alpha + c\beta \\ \alpha + \beta \end{pmatrix}, \qquad \tilde{u} = u\,, \tag{2.3}$$

where α, β and θ are constants that parametrize the group action. The prolonged action on u_x and u_t is given explicitly by

$$g \cdot u_x = \tilde{u}_x = \tilde{D}_x \tilde{u}, \qquad g \cdot u_t = \tilde{u}_t = \tilde{D}_t \tilde{u}.$$

The transformed total differentiation operators $\tilde{D}_i$ are defined by

$$\tilde{D}_i = \frac{d}{d\tilde{x}_i} = \sum_{k=1}^{p} \left((d\tilde{x}/dx)^{-T} \right)_{ik} D_k,$$

where $d\tilde{x}/dx$ is the Jacobian matrix. So,

$$\tilde{u}_x = \cosh(c\theta)u_x - \frac{1}{c}\sinh(c\theta)u_t, \qquad \tilde{u}_t = -c\sinh(c\theta)u_x + \cosh(c\theta)u_t.$$

If we take M to be the space with coordinates $(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \cdots)$, then the action is locally free near the identity of hyperbolic group G and regular. So, if we take the normalization equations to be $\tilde{x} = 0$, $\tilde{t} = 0$ and $\tilde{u}_x = 0$, we obtain

$$\alpha = -\frac{1}{2c} . \frac{(ct - x)(cu_x - u_t)}{\sqrt{-c^2 u_x^2 + u_t^2}}, \qquad \beta = \frac{1}{2c} . \frac{cxu_x + c^2 tu_x + xu_t + ctu_t}{\sqrt{-c^2 u_x^2 + u_t^2}},$$

$$\theta = \frac{1}{c} . \ln \left(\frac{\sqrt{-c^2 u_x^2 + u_t^2}}{cu_x - u_t} \right), \tag{2.4}$$

as the frame in parametric form.

As given in page 128 of [5], if $\mathbf{z} = (z_1, \cdots, z_n) \in M$, and the normalization equations are $\tilde{z}_i = c_i$ for $i = 1, \cdots, r = dim(G)$, then the components of

$$\rho(\mathbf{z}) \cdot \mathbf{z} = (c_1, \cdots, c_r, I(z_{r+1}), \cdots, I(z_n)),$$

where

$$I(z_k) = g \cdot z_k|_{g=\rho(z)}, \qquad k = r+1, \cdots, n,$$

are all invariants.

Definition 2.3. For any prolonged action in the jet space $M = J^n(X \times U)$, the invariantized jet coordinates known as the *normalized differential invariants* are denoted as

$$J^i = I(x_i) = \tilde{x}_i|_{g=\rho(z)}, \qquad I^\alpha_k = I(u^\alpha_k) = \tilde{u}^\alpha_k|_{g=\rho(z)},$$

which is the original Fels and Olver notation [1].

According to Theorem 10.3 in page 38 of [1] (Replacement Theorem), any invariant is a function of the $I(z_k)$. Particularly, the set $\{J^i, I^\alpha_k\}$ is a complete set of differential invariants for a prolonged action.

Example 2.2 (cont.). The normalized differential invariants up to order two are as follows

$$\begin{aligned} g\cdot z &= (\tilde{x}, \tilde{t}, \tilde{u}, \tilde{u}_x, \tilde{u}_t, \tilde{u}_{xx}, \tilde{u}_{xt}, \tilde{u}_{tt})\,\big|_{g=\rho(z)} \\ &= (J^x, J^t, I^u, I^u_1, I^u_2, I^u_{11}, I^u_{12}, I^u_{22}) \\ &= (0, 0, u, 0, -\sqrt{-c^2u_x^2 + u_t^2}, -\frac{u_{xx}u_t^2 - 2u_{xt}u_xu_t + u_{tt}u_x^2}{c^2u_x^2 - u_t^2}, \\ &\quad -\frac{-c^2u_{xx}u_xu_t + c^2u_{xt}u_x^2 + u_{xt}u_t^2 - u_{tt}u_xu_t}{c^2u_x^2 - u_t^2}, -\frac{c^4u_{xx}u_x^2 - 2c^2u_{xt}u_xu_t + u_{tt}u_t^2}{c^2u_x^2 - u_t^2}). \end{aligned}$$

If we define the *Invariant differential operators* as

$$\mathcal{D}_i = \tilde{D}_i\,\big|_{g=\rho(z)}\ , \qquad \text{where} \qquad \tilde{D}_i = \frac{d}{d\tilde{x}_i} = \sum_{k=1}^{p} \left(\left({}^{d\tilde{x}}\!/_{dx} \right)^{-T} \right)_{ik} D_k,$$

according to Example 4.5.1 in [5], we know in general

$$\mathcal{D}_i I^\alpha_k \neq I^\alpha_{ki}.$$

Definition 2.4. The *Invariant differentiation* of the jet coordinates, J^i and I^α_k, denoted as

$$\mathcal{D}_j J^i = \delta_{ij} + N_{ij}\ , \qquad \mathcal{D}_j I^\alpha_K = I^\alpha_{Kj} + M^\alpha_{Kj}\ ,$$

where N_{ij} and M^α_{Kj} are the *Correction terms*, and δ_{ij} is the Kronecker delta. For more information on correction terms see §4.5 in [5].

Now let I^α_J and I^α_L be two generating differential invariants, and let $JK = LM$ such that $I^\alpha_{JK} = I^\alpha_{LM}$. Thus, as given in [5], we will have the so called *Syzygies* or *Differential identities*

$$\mathcal{D}_K I^\alpha_J - \mathcal{D}_M I^\alpha_L = M^\alpha_{JK} - M^\alpha_{LM}.$$

For obtaining the Correction terms, we define the *Infinitesimals of the prolonged group action* with respect to the group parameters a_j, evaluated at the identity element e, as

$$\xi^i_j = \frac{\partial \tilde{x}_i}{\partial a_j}\,|_{g=e}\ , \qquad \phi^\alpha_{K,j} = \frac{\partial \tilde{u}^\alpha_K}{\partial a_j}\,|_{g=e}\ .$$

Now, let the normalization equations be $\{\psi_\lambda(z) = 0,\ \lambda = 1, \cdots, r\}$ and suppose the n variables actually occurring in the $\psi_\lambda(z)$ are $\zeta_1, \cdots, \zeta_n$ such that m of these are independent variables and $n-m$ of them are dependent variables and their derivatives.

Let ϕ denote the $r \times n$ matrix

$$\phi_{ij} = \left(\frac{\partial(g\cdot\zeta_j)}{\partial g_i}\,|_{g=e} \right)(I),$$

and define $\mathbf{T}$ to be the invariant $p \times n$ total derivative matrix as follows,

$$\mathbf{T}_{ij} = I\left(\frac{D}{Dx_i}\zeta_j \right),$$

and $\mathbf{J}$ to be the $n \times r$ matrix

$$\mathbf{J}_{ij} = \frac{\partial \psi_j(I)}{\partial I(\zeta_i)},$$

that is, transpose of the Jacobian matrix of the normalization equations $\psi_1, \cdots, \psi_r$, with invariantised arguments.

So, the correction terms can be obtained as follows, that has been proved in [5].

Theorem 2.5. *The formulae for the Correction terms are*

$$N_{ij} = \sum_{l=1}^{r} \mathbf{K}_{jl} \xi_l^i(I), \qquad M_{Kj}^{\alpha} = \sum_{l=1}^{r} \mathbf{K}_{jl} \phi_{K,l}^{\alpha}(I),$$

where l is the index for the group parameters, $r = \dim(G)$, and the $p \times r$ correction matrix $\mathbf{K}$, is given by

$$\mathbf{K} = -\mathbf{T}\mathbf{J}(\phi\mathbf{J})^{-1}.$$

Now, we calculate the syzygies of the transformation (2.3) and the invariant differentiation of the jet coordinates in Example 2.2.

Example 2.2 (cont.). If we set $u = u(x,t,\tau)$ and $\tilde{\tau} = \tau$ and take the normalization equations as before, we obtain

$$\begin{aligned}
\tilde{u}_\tau \big|_{g=\rho(z)} &= I_3^u = u_\tau,\\
\tilde{u}_t \big|_{g=\rho(z)} &= I_2^u = -\sqrt{-c^2u_x^2 + u_t^2},\\
\tilde{u}_{xx} \big|_{g=\rho(z)} &= I_{11}^u = -\frac{u_{xx}u_t^2 - 2u_{xt}u_xu_t + u_{tt}u_x^2}{c^2u_x^2 - {u_t}^2},\\
\tilde{u}_{xt} \big|_{g=\rho(z)} &= I_{12}^u = -\frac{-c^2u_{xx}u_xu_t + c^2u_{xt}u_x^2 + u_{xt}u_t^2 - u_{tt}u_xu_t}{c^2u_x^2 - u_t^2},\\
\tilde{u}_{tt} \big|_{g=\rho(z)} &= I_{22}^u = -\frac{c^4u_{xx}u_x^2 - 2c^2u_{xt}u_xu_t + u_{tt}u_t^2}{c^2u_x^2 - u_t^2}.
\end{aligned}$$

According to Theorem 2.5, we obtain the invariant differentiation of the jet coordinates as follows,

$$\begin{aligned}
&\mathcal{D}_x I_2^u = I_{12}^u, && \mathcal{D}_t I_2^u = I_{22}^u, && \mathcal{D}_\tau I_2^u = I_{23}^u,\\
&\mathcal{D}_x I_{11}^u = I_{111}^u - \frac{2I_{11}^uI_{12}^u}{I_2^u}, && \mathcal{D}_t I_{11}^u = I_{112}^u - \frac{2(I_{12}^u)^2}{I_2^u}, && \mathcal{D}_\tau I_{11}^u = I_{113}^u - \frac{2I_{12}^uI_{13}^u}{I_2^u},\\
&\mathcal{D}_x I_{22}^u = I_{122}^u - \frac{2c^2I_{11}^uI_{12}^u}{I_2^u}, && \mathcal{D}_t I_{22}^u = I_{222}^u - \frac{2c^2(I_{12}^u)^2}{I_2^u}, && \mathcal{D}_\tau I_{22}^u = I_{223}^u - \frac{2c^2I_{12}^uI_{13}^u}{I_2^u},\\
&\mathcal{D}_x I_{12}^u = I_{112}^u - \frac{I_{11}^u}{I_2^u}(c^2I_{11}^u + I_{22}^u), && \mathcal{D}_t I_{12}^u = I_{122}^u - \frac{I_{12}^u}{I_2^u}(c^2I_{11}^u + I_{22}^u), && \mathcal{D}_\tau I_{12}^u = I_{123}^u - \frac{I_{13}^u}{I_2^u}(c^2I_{11}^u + I_{22}^u).
\end{aligned}$$

We know that there are two ways to reach I_{113}^u and since both ways must be equal, we get the following syzygy between I_3^u and I_{11}^u:

$$\mathcal{D}_3 I_{11}^u = \left((\mathcal{D}_1)^2 - \frac{2I_{12}^u\mathcal{D}_1}{I_2^u} + \frac{I_{11}^u\mathcal{D}_2}{I_2^u} \right) I_3^u,$$

similarly, there are two possibilities to obtain I_{223}^u, so we get a syzygy between I_3^u and I_{22}^u and the syzygy is:

$$\mathcal{D}_3 I_{22}^u = \left((\mathcal{D}_2)^2 - \frac{c^2I_{12}^u\mathcal{D}_1}{I_2^u} \right) I_3^u.$$

Finally, there are two syzygies between I_3^u and I_{12}^u, which are as follows:

$$\mathcal{D}_3 I_{12}^u = \left(\mathcal{D}_1\mathcal{D}_2 - \frac{I_{22}^u\mathcal{D}_1}{I_2^u} \right) I_3^u, \tag{2.5}$$

$$\mathcal{D}_3 I_{12}^u = \left(\mathcal{D}_2\mathcal{D}_1 + \frac{I_{12}^u \mathcal{D}_2}{I_2^u} - \frac{c^2 I_{11}^u \mathcal{D}_1}{I_2^u} - \frac{I_{22}^u \mathcal{D}_1}{I_2^u} \right) I_3^u. \tag{2.6}$$

To prove the relations (2.5) and (2.6), from the normalization equations in Example 2.2 we get the following table of infinitesimals:

$$\begin{array}{c} \\ \alpha \\ \beta \\ \theta \end{array} \begin{array}{cccccccccc} x & t & \tau & u & u_x & u_t & u_{xx} & u_{xt} & u_{tt} & \dots \end{array} \\ \begin{pmatrix} -c & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ c & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ c^2 t & x & 0 & 0 & -u_t & -c^2 u_x & -2u_{xt} & -c^2 u_{xx} - u_{tt} & -2c^2 u_{xt} & \dots \end{pmatrix},$$

also, according to the formulas in description after Definition 2.4 and Theorem 2.5, we obtain

$$\phi = \left(\begin{array}{c|ccc} & x & t & u_x \\ \alpha & -c & 1 & 0 \\ \beta & c & 1 & 0 \\ \theta & 0 & 0 & -I_2^u \end{array} \right),$$

$$\mathbf{J} = \left(\begin{array}{c|ccc} & \psi_1(I) & \psi_2(I) & \psi_3(I) \\ J^1 & 1 & 0 & 0 \\ J^2 & 0 & 1 & 0 \\ I_1^u & 0 & 0 & 1 \end{array} \right),$$

$$\mathbf{T} = \left(\begin{array}{c|ccc} & x & t & u_x \\ x & 1 & 0 & I_{11}^u \\ t & 0 & 1 & I_{12}^u \\ \tau & 0 & 0 & I_{13}^u \end{array} \right),$$

$$\mathbf{K} = \left(\begin{array}{cccc} & a & b & \theta \\ x & \frac{1}{2c} & -\frac{1}{2} & \frac{I_{11}^u}{I_2^u} \\ t & -\frac{1}{2c} & -\frac{1}{2} & \frac{I_{12}^u}{I_2^u} \\ \tau & 0 & 0 & \frac{I_{13}^u}{I_2^u} \end{array} \right).$$

Now, since we know that there are several ways in which to reach I_{123}^u, thus we have the invariant differentiation formulae,

$$\mathcal{D}_3 I_{12}^u = I_{123}^u + M_{123}^u = I_{123}^u + \begin{pmatrix} 0 & 0 & \frac{I_{13}^u}{I_2^u} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -c^2 I_{11}^u - I_{22}^u \end{pmatrix} = I_{123}^u - \frac{(c^2 I_{11}^u + I_{22}^u) I_{13}^u}{I_2^u},$$

$$\mathcal{D}_2\mathcal{D}_1 I_3^u = \mathcal{D}_2 I_{13}^u = I_{123}^u + M_{132}^u = I_{123}^u + \begin{pmatrix} -\frac{1}{2c} & -\frac{1}{2} & \frac{I_{12}^u}{I_2^u} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -I_{23}^u \end{pmatrix} = I_{123}^u - \frac{I_{12}^u I_{23}^u}{I_2^u},$$

$$\mathcal{D}_1\mathcal{D}_2 I_3^u = \mathcal{D}_1 I_{23}^u = I_{123}^u + M_{231}^u = I_{123}^u + \begin{pmatrix} \frac{1}{2c} & -\frac{1}{2} & \frac{I_{11}^u}{I_2^u} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -c^2 I_{13}^u \end{pmatrix} = I_{123}^u - \frac{c^2 I_{11}^u I_{13}^u}{I_2^u},$$

hence, by placing the second two relations in the first relation, we obtain the following two syzygies between I_3^u and I_{12}^u:

$$\mathcal{D}_3 I_{12}^u = \left(\mathcal{D}_1\mathcal{D}_2 - \frac{I_{22}^u \mathcal{D}_1}{I_2^u} \right) I_3^u,$$

$$\mathcal{D}_3 I^u_{12} = \left(\mathcal{D}_2\mathcal{D}_1 + \frac{I^u_{12}\mathcal{D}_2}{I^u_2} - \frac{c^2 I^u_{11}\mathcal{D}_1}{I^u_2} - \frac{I^u_{22}\mathcal{D}_1}{I^u_2}\right) I^u_3.$$

We see that from equations (2.5) and (2.6), the invariant operators $\mathcal{D}_x$ and $\mathcal{D}_t$ do not commute. So, the invariant total differentiation operators do not commute. In fact, we have the theorem below.

Theorem 2.6. [1] *Denote the invariantized derivatives of the infinitesimals* ξ^k_l, *for* $k, i = 1, \cdots, p$ *and* $l = 1, \cdots, r$, *by*

$$\Xi^k_{li} = \tilde{D}_i \xi^k_l(\tilde{z})\,\big|_{g=\rho(z)}\ ,$$

then the commutators are given by

$$[\mathcal{D}_i, \mathcal{D}_j] = \sum_{k=1}^{p} A^k_{ij}\mathcal{D}_k, \qquad A^k_{ij} = \sum_{l=1}^{r} K_{jl}\Xi^k_{li} - K_{il}\Xi^k_{lj}.$$

Definition 2.7. The *Invariant one-forms* are denoted as

$$I(dx_i) = d\tilde{x}_i\,\big|_{g=\rho(z)} = \Big(\sum_{j=1}^{p} D_j(\tilde{x}_i)dx_j\Big)\,\big|_{g=\rho(z)}\ .$$

Theorem 2.8. [3] *Consider the set of invariant total differentiation operators,* $\{\mathcal{D}_i\}$, *and the set of invariant one-forms,* $\{I(dx_j)\}$. *So if*

$$\mathcal{D}_i(I(dx_j)) = \sum_{k=1}^{p} B^k_{ij} I(dx_k),$$

then $B^j_{ki} = A^i_{jk}$.

By above theorem, we can verify that an invariant total differentiation operator $\mathcal{D}_i$ sends invariant differential forms to invariant differential forms. In fact, if $\mathcal{D}_i$ is the invariant differentiation operator and ω is a form, then $\mathcal{D}_i(\omega)$ denote as a *Lie derivative*. For more details see [3].

Finally, in the end of this section, from the Theorem 2.8 we obtain the Lie derivatives of $I(dx_j)$ with respect to $\mathcal{D}_i$ for the hyperbolic group action on (x, t, τ) as in Example 2.2, that will be required in the next section.

Conclusion 2.9. Let $g \in G$ act on $(x, t, \tau, u(x, t, \tau))$ that has been given in Example 2.2. Then the Lie derivatives of $I(dx_j)$ with respect to $\mathcal{D}_i$ are as shown in Table 1.

Table 1: Lie derivatives of the $I(dx_j)$ with respect to the $\mathcal{D}_i$.

Lie derivative	$I(dx)$	$I(dt)$	$I(d\tau)$
$\mathcal{D}_x$	$c^2.\frac{I^u_{11}}{I^u_2}I(dt)$	$-\frac{I^u_{12}}{I^u_2}I(dt) - \frac{I^u_{13}}{I^u_2}I(d\tau)$	0
$\mathcal{D}_t$	$c^2\big(-\frac{I^u_{11}}{I^u_2}I(dx) - \frac{I^u_{13}}{I^u_2}I(d\tau)\big)$	$\frac{I^u_{12}}{I^u_2}I(dx)$	0
$\mathcal{D}_\tau$	$c^2.\frac{I^u_{13}}{I^u_2}I(dt)$	$\frac{I^u_{13}}{I^u_2}I(dx)$	0

3 Invariant calculus of variations and structure of Noether's conservation laws

In this section, we will calculate the Noether's Conservation Laws of the Sine-Gordon Equation by the concept of Invariant Calculus of Variations as formulated by Gonçalves and Mansfield [2, 3, 4] and Mansfield [5]. Suppose the Lagrangian $\bar{L}[\mathbf{u}]$ of the variational problem $\bar{\Phi}[\mathbf{u}] = \int \bar{L}[\mathbf{u}]\,d\mathbf{x}$ to be a smooth function of $\mathbf{x} = (x_1, \cdots, x_p)$, $\mathbf{u} = (u^1, \cdots, u^q)$ and finitely many derivatives of u^α, where $\bar{\Phi}[\mathbf{u}]$ is invariant under some group action with finite set of generators $\{\kappa_1, \cdots, \kappa_N\}$. So as given in [3], we can rewrite $\bar{\Phi}[\mathbf{u}]$ as $\Phi[\kappa] = \int L[\kappa]\,I(d\mathbf{x})$, in which $I(d\mathbf{x}) =$

$I(dx_1)\cdots I(dx_p)$ denotes the invariant volume form and $d\mathbf{x} = dx_1 \cdots dx_p$ is the standard volume form. Now, we suppose the functional $\bar{\Phi}[\mathbf{u}]$ be extremized by $\mathbf{x} \to (\mathbf{x}, \mathbf{u}(\mathbf{x}))$, then for a small perturbation of $\mathbf{u}$

$$0 = \frac{d}{d\varepsilon}\,|_{\varepsilon=0}\, \bar{\Phi}[\mathbf{u}+\varepsilon\mathbf{v}] = \int \sum_{\alpha=1}^{q} \Big[E^{\alpha}(\bar{L})v^{\alpha} + \sum_{i=1}^{p} \frac{d}{dx_i}\Big(\frac{\partial \bar{L}}{\partial u_i^{\alpha}} v^{\alpha} + \cdots\Big)\Big] d\mathbf{x},$$

where

$$E^{\alpha} = \sum_{K} (-1)^{K} \frac{D^{|K|}}{Dx_1^{k_1} Dx_2^{k_2} \cdots Dx_p^{k_p}} \frac{\partial}{\partial u_K^{\alpha}},$$

is the Euler operator with respect to the dependent variable u^{α}. And symbolically,

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \bar{\Phi}[\mathbf{u}+\varepsilon\mathbf{v}] = \frac{d}{d\tau}\,\big|_{u_\tau = v} \bar{\Phi}[\mathbf{u}]\ .$$

According to [3], we have

$$\begin{aligned} 0 &= \mathcal{D}_t \int L[\kappa]\, I(d\mathbf{x}) = \mathcal{D}_{p+1} \int L[\kappa]\, I(d\mathbf{x}) \\ &= \int \Big(\sum_{\alpha} E^{\alpha}(L) I_{\tau}^{\alpha} I(dx) + \sum_{i=1}^{p} \mathcal{D}_i \big[\sum_{j=1}^{p+1} F_{ij} I(dx_1) \cdots \widehat{I(dx_j)} \cdots I(dx_{p+1}) \big] \Big), \end{aligned}$$

where $E^{\alpha}(L)$ are the invariantized Euler-Lagrange equations, F_{ij} depends on $I_{K,p+1}^{\alpha}$ and I_J^{α} with K and J multi-indices of differentiation with respect to x_i, for $i = 1, \cdots, p$, and

$$I(dx_1) \cdots \widehat{I(dx_j)} \cdots I(dx_{p+1}) = I(dx_1) \cdots I(dx_{j-1}) I(dx_{j+1}) \cdots I(dx_{p+1}).$$

Theorem 3.1. [3] *The process of calculating the invariantized Euler-Lagrange equations produces boundary terms*

$$\int \sum_{i=1}^{p} \mathcal{D}_i \Big(\sum_{j=1}^{p+1} F_{ij} I(dx_1) \cdots \widehat{I(dx_j)} \cdots I(dx_{p+1}) \Big),$$

that can be written as

$$\int \sum_{i=1}^{p} d\Big((-1)^{i-1} \Big[\sum_{K,\alpha} I_{K,\tau}^{\alpha} C_{K,i}^{\alpha} \Big] I(dx_1) \cdots \widehat{I(dx_j)} \cdots I(dx_{p+1}) \Big),$$

where K is a multi-index of differentiation with respect to x_i, for $i = 1, \cdots, p$, and $C_{K,i}^{\alpha}$ are functions of I_J^{α}, with J a multi-index of differentiation with respect to x_i.

Now, we consider the variational problem (2.1), that its Euler-Lagrange equation is the nonlinear Sine-Gordon equation (1.1).

Example 3.2. Consider the variational problem

$$\Phi[u] = \iint \frac{1}{2}\big(-u_t^2 + c^2 u_x^2 - 2m^2 \cos u \big) dx dt, \tag{3.1}$$

which is invariant under the action (2.3). To find the invariantized Euler-Lagrange equation, introduce a dummy invariant independent variable τ to effect the variation, and set $u = u(x,t,\tau)$, therefore $\tilde{u}_{\tau}\,\big|_{g=\rho(z)} = I_3^u = u_{\tau}$. Rewriting the above variational problem in terms of the invariants of the group action yields

$$\iint \frac{1}{2}\Big(-(I_2^u)^2 + c^2 (I_1^u)^2 - 2m^2 \cos(I^u) \Big) I(dx) I(dt). \tag{3.2}$$

To obtain the invariantized Euler-Lagrange equation and boundary terms, after differentiating (3.2) under the integral sign we obtain

$$\begin{aligned}
&\mathcal{D}_\tau \iint \frac{1}{2}\Big(-(I_2^u)^2 + c^2(I_1^u)^2 - 2m^2\cos(I^u)\Big) I(dx)I(dt) \\
&= \iint \Big[\big(-I_2^u.\mathcal{D}_\tau(I_2^u) + c^2 I_1^u.\mathcal{D}_\tau(I_1^u) + m^2\sin(I^u).\mathcal{D}_\tau(I^u)\big) I(dx)I(dt) \\
&\quad + \frac{1}{2}\left(-(I_2^u)^2 + c^2(I_1^u)^2 - 2m^2\cos(I^u)\right)\mathcal{D}_\tau\left(I(dx)I(dt)\right)\Big].
\end{aligned}$$

Using Table 1 we see that $\mathcal{D}_\tau\left(I(dx)I(dt)\right) = 0$. Then performing integration by parts, and substituting I_1^u equal to zero in the second integral yields

$$\iint \left(I_{22}^u - c^2 I_{11}^u + m^2\sin(I^u)\right) I(dx)I(dt) + \iint \mathcal{D}_t\Big(-I_2^u I_3^u I(dx)I(dt)\Big).$$

Thus, we obtain the invariantized Euler-Lagrange equation

$$E^u(L) = I_{22}^u - c^2 I_{11}^u + m^2\sin(I^u) = u_{tt} - c^2 u_{xx} + m^2\sin u.$$

Therefore, according to Theorem 3.1 the boundary terms can be written as

$$\iint d(I_2^u I_3^u I(dx)). \tag{3.3}$$

Note that to find the boundary terms (3.3), after differentiating the multiple integral (3.2) under the integral sign, we see that since the syzygy $\mathcal{D}_3 I_{12}^u$ has not appear in the result of the differentiation, thus none of the equations (2.5) or (2.6) have included. But if the syzygy $\mathcal{D}_3 I_{12}^u$ appeared in the multiple integral after differentiation, then if we include any of the equations (2.5) or (2.6), we get equivalent boundary terms. Because, we claim that the equations (2.5) and (2.6) are equal:

Proof . We prove that the equations

$$\mathcal{D}_3 I_{12}^u = \left(\mathcal{D}_1\mathcal{D}_2 - \frac{I_{22}^u\mathcal{D}_1}{I_2^u}\right) I_3^u$$

and

$$\mathcal{D}_3 I_{12}^u = \left(\mathcal{D}_2\mathcal{D}_1 + \frac{I_{12}^u\mathcal{D}_2}{I_2^u} - \frac{c^2 I_{11}^u\mathcal{D}_1}{I_2^u} - \frac{I_{22}^u\mathcal{D}_1}{I_2^u}\right) I_3^u$$

are equal. To prove it suffices to show that

$$\left(\mathcal{D}_1\mathcal{D}_2 - \frac{I_{22}^u\mathcal{D}_1}{I_2^u}\right) I_3^u = \left(\mathcal{D}_2\mathcal{D}_1 + \frac{I_{12}^u\mathcal{D}_2}{I_2^u} - \frac{c^2 I_{11}^u\mathcal{D}_1}{I_2^u} - \frac{I_{22}^u\mathcal{D}_1}{I_2^u}\right) I_3^u$$

or

$$\mathcal{D}_1\mathcal{D}_2 = \mathcal{D}_2\mathcal{D}_1 + \frac{I_{12}^u\mathcal{D}_2}{I_2^u} - \frac{c^2 I_{11}^u\mathcal{D}_1}{I_2^u}.$$

According to Theorem 2.6, we have

$$[\mathcal{D}_i, \mathcal{D}_j] = \sum_{k=1}^{p} A_{ij}^k \mathcal{D}_k, \qquad A_{ij}^k = \sum_{l=1}^{r} K_{jl}\Xi_{li}^k - K_{il}\Xi_{lj}^k, \qquad \Xi_{li}^k = \tilde{D}_i \xi_l^k(\tilde{z})\,\big|_{g=\rho(z)}\,.$$

On the other hand, from the formulas in description after Definition 2.4, we obtain

$$\begin{aligned}
\xi_1^1 &= -c, & \xi_1^2 &= 1, & \xi_1^3 &= 0, \\
\xi_2^1 &= c, & \xi_2^2 &= 1, & \xi_2^3 &= 0, \\
\xi_3^1 &= c^2 t, & \xi_3^2 &= x, & \xi_3^3 &= 0.
\end{aligned}$$

Hence,

$$\Xi^2_{31} = \tilde{D}_x \xi^2_3(\tilde{z}) = 1,$$
$$\Xi^1_{32} = \tilde{D}_y \xi^1_3(\tilde{z}) = c^2,$$

and $\Xi^k_{li} = 0$, if $\Xi^k_{li} \neq \Xi^2_{31}$, $\Xi^k_{li} \neq \Xi^1_{32}$,

$$A^1_{12} = -c^2 K_{13} = B^1_{21}, \qquad A^2_{13} = K_{33} = B^1_{32},$$
$$A^1_{21} = c^2 K_{13} = B^2_{11}, \qquad A^2_{31} = -K_{33} = B^3_{12},$$
$$A^1_{32} = -c^2 K_{33} = B^3_{21}, \qquad A^2_{12} = K_{23} = B^1_{22},$$
$$A^1_{23} = c^2 K_{33} = B^2_{31}, \qquad A^2_{21} = -K_{23} = B^2_{12}.$$

So, we have

$$[\mathcal{D}_1, \mathcal{D}_2] = A^1_{12}\mathcal{D}_1 + A^2_{12}\mathcal{D}_2 = -c^2 K_{13}\mathcal{D}_1 + K_{23}\mathcal{D}_2 = -\frac{c^2 I^u_{11}\mathcal{D}_1}{I^u_2} + \frac{I^u_{12}\mathcal{D}_2}{I^u_2},$$

on the other hand,

$$[\mathcal{D}_1, \mathcal{D}_2] = \mathcal{D}_1\mathcal{D}_2 - \mathcal{D}_2\mathcal{D}_1,$$

thus, we conclude that

$$\mathcal{D}_1\mathcal{D}_2 - \mathcal{D}_2\mathcal{D}_1 = -\frac{c^2 I^u_{11}\mathcal{D}_1}{I^u_2} + \frac{I^u_{12}\mathcal{D}_2}{I^u_2},$$

or

$$\mathcal{D}_1\mathcal{D}_2 = \mathcal{D}_2\mathcal{D}_1 + \frac{I^u_{12}\mathcal{D}_2}{I^u_2} - \frac{c^2 I^u_{11}\mathcal{D}_1}{I^u_2}.$$

□

Now, we present an important theorem that has been proved in [3].

Theorem 3.3. *Let $\int L(k_1, k_2, \cdots)\, I(d\mathbf{x})$ be invariant under $G \times M \to M$, where $M = J^n(X, U)$, with generating invariants κ_j, for $j = 1, \cdots .N$. Introduce a dummy invariant variable t to effect the variation and then integration by parts yields*

$$\mathcal{D}_t \int L(k_1, k_2, \cdots)\, I(d\mathbf{x})$$
$$= \int \Big[\sum_\alpha E^\alpha(L) I^\alpha_t I(d\mathbf{x}) + \sum_{k=1}^{p} d\Big((-1)^{k-1}\Big(\sum_{J,\alpha} I^\alpha_{Jt} C^\alpha_{J,k}\Big) I(dx_1)\cdots \widehat{I(dx_k)} \cdots I(dx_{p+1})\Big)\Big],$$

where this defines the vectors $\mathbf{C}^\alpha_k = (C^\alpha_{J,k})$. Recall that $E^\alpha(L)$ are the invariantized Euler- Lagrange equations and $I^\alpha_{Jt} = I(u^\alpha_{Jt})$, where J is a multi-index of differentiation with respect to the variables x_i, for $i = 1, \cdots, p$. Let $(a_1, \cdots, a_r)$ be the coordinates of G near the identity e, and $\mathbf{v}_i$, for $i = 1, \cdots, r$, the associated infinitesimal vector fields. Furthermore, let $Ad(g)$ be the Adjoint representation of G with respect to these vector fields. For each dependent variable, define the matrices of characteristics to be,

$$\mathcal{Q}^\alpha(\tilde{z}) = (\widetilde{D_K(Q^\alpha_i)}), \qquad \alpha = 1, \cdots, q,$$

where K is a multi-index of differentiation with respect to the x_k, and

$$Q^\alpha_i = \phi^\alpha_i - \sum_{k=1}^{p} \xi^k_i u^\alpha_k = \frac{\partial \widetilde{u^\alpha}}{\partial a_i}|_{g=e} - \sum_{k=1}^{p} \frac{\partial \widetilde{x_k}}{\partial a_i}|_{g=e}\, u^\alpha_k,$$

are the components of the q-tuple $\mathbf{Q}_i$ known as the characteristic of the vector field $\mathbf{v}_i$. Let $\mathcal{Q}^\alpha(J, I)$, for $\alpha = 1, \cdots, q$, be the invariantization of the above matrices. Then, the r conservation laws obtained via Noether's Theorem can be written in the form,

$$d(Ad(\rho)^{-1}(\upsilon_1, \cdots, \upsilon_p) M_{\mathcal{J}} d^{p-1}\hat{\mathbf{x}}) = 0\,,$$

where

$$\upsilon_k = \sum_{\alpha} (-1)^{k-1} (\mathcal{Q}^{\alpha}(J,I)\mathbf{C}_k^{\alpha} + L(\Xi(J,I))_k),$$

are the vectors of invariants, with $(\Xi(J,I))_k$ *the* k^{th} *column of* $\Xi(J,I)$*,* $M_{\mathcal{J}}$ *is the matrix of first minors of the Jacobian matrix evaluated at the frame,* $\mathcal{J} = {d\tilde{x}}/{dx}\,|_{g=\rho(z)}$ *, and*

$$d^{p-1}\hat{x} = \begin{pmatrix} \widehat{dx_1} dx_2 \cdots dx_p \\ dx_1 \widehat{dx_2} dx_3 \cdots dx_p \\ \vdots \\ dx_1 \cdots dx_{p-1} \widehat{dx_p} \end{pmatrix} = \begin{pmatrix} dx_2 dx_3 \cdots dx_p \\ dx_1 dx_3 \cdots dx_p \\ \vdots \\ dx_1 dx_2 \cdots dx_{p-1} \end{pmatrix}.$$

Lemma 3.4. *The inverse of the Adjoint representation of the hyperbolic group G with respect to its generating vector fields evaluated at the frame (2.4) is*

$$Ad(\rho(z))^{-1} = \begin{bmatrix} \dfrac{cu_x - u_t}{\sqrt{-c^2u_x^2 + u_t^2}} & 0 & 0 \\ 0 & \dfrac{\sqrt{-c^2u_x^2 + u_t^2}}{cu_x - u_t} & 0 \\ -\dfrac{1}{2}\dfrac{(ct-x)(cu_x - u_t)}{\sqrt{-c^2u_x^2 + u_t^2}} & \dfrac{1}{2}\dfrac{(x+ct)\sqrt{-c^2u_x^2 + u_t^2}}{cu_x - u_t} & 1 \end{bmatrix}. \tag{3.4}$$

Proof . Consider the action (2.3) and let it act on the infinitesimal vector fields generating the hyperbolic group G ,

$$\mathbf{v}_1 = -c\partial_x + \partial_t, \qquad \mathbf{v}_2 = c\partial_x + \partial_t, \qquad \mathbf{v}_3 = c^2 t\partial_x + x\partial_t,$$

as follow

$$\begin{aligned}
& g.\big(\alpha'(-c\partial_x + \partial_t) + \beta'(c\partial_x + \partial_t) + \gamma'(c^2 t\,\partial_x + x\partial_t)\big) \\
= \;& \alpha'(-c\partial_{\tilde{x}} + \partial_{\tilde{t}}) + \beta'(c\partial_{\tilde{x}} + \partial_{\tilde{t}}) + \gamma'(c^2\tilde{t}\,\partial_{\tilde{x}} + \tilde{x}\partial_{\tilde{t}}) \\
= \;& \alpha'(-c\cosh(c\theta)\partial_x + \sinh(c\theta)\partial_t - c\sinh(c\theta)\,\partial_x + \cosh(c\theta)\,\partial_t) \\
& +\beta'(c\cosh(c\theta)\partial_x - \sinh(c\theta)\partial_t - c\sinh(c\theta)\,\partial_x + \cosh(c\theta)\,\partial_t) \\
& +\gamma'\big[c^2(\alpha + \beta + \frac{x}{c}\sinh(c\theta) + t\cosh(c\theta))(\cosh(c\theta)\partial_x - \frac{1}{c}\sinh(c\theta)\partial_t) \\
& +(-c\alpha + c\beta + x\cosh(c\theta) + ct\sinh(c\theta))(-c\sinh(c\theta)\partial_x + \cosh(c\theta)\partial_t)\big] \\
= \;& \alpha'[(\cosh(c\theta) + \sinh(c\theta))(-c\partial_x + \partial_t)] + \beta'[(\cosh(c\theta) - \sinh(c\theta))(c\partial_x + \partial_t)] \\
& +\gamma'\big[-c\alpha\big(\cosh(c\theta) + \sinh(c\theta)\big)(-c\partial x + \partial t) \\
& +c\beta\big(\cosh(c\theta) - \sinh(c\theta)\big)(c\partial x + \partial t) + (c^2 t\partial x + x\partial t)\big] \\
= \;& \begin{pmatrix} \alpha' & \beta' & \gamma' \end{pmatrix} \begin{pmatrix} \cosh(c\theta) + \sinh(c\theta) & 0 & 0 \\ 0 & \cosh(c\theta) - \sinh(c\theta) & 0 \\ -c\alpha\,(\cosh(c\theta) + \sinh(c\theta)) & c\beta\,(\cosh(c\theta) - \sinh(c\theta)) & 1 \end{pmatrix} \\
& \times \begin{pmatrix} -c\partial_x + \partial_t \\ c\partial_x + \partial_t \\ c^2 t\partial_x + x\partial_t \end{pmatrix},
\end{aligned}$$

where the above 3×3 matrix, $Ad(g)$, is the Adjoint representation of G with respect to its generating infinitesimal vector fields. So $Ad(g)^{-1}$ is

$$Ad(g)^{-1} = \begin{bmatrix} \cosh(c\theta) - \sinh(c\theta) & 0 & 0 \\ 0 & \cosh(c\theta) + \sinh(c\theta) & 0 \\ c\alpha & -c\beta & 1 \end{bmatrix}.$$

Now evaluating $Ad(g)^{-1}$ at the frame (2.4), we obtain

$$Ad(\rho(z))^{-1} = \begin{bmatrix} \dfrac{cu_x - u_t}{\sqrt{-c^2u_x^2+u_t^2}} & 0 & 0 \\ 0 & \dfrac{\sqrt{-c^2u_x^2+u_t^2}}{cu_x - u_t} & 0 \\ -\dfrac{1}{2}\dfrac{(ct-x)(cu_x-u_t)}{\sqrt{-c^2u_x^2+u_t^2}} & \dfrac{1}{2}\dfrac{(x+ct)\sqrt{-c^2u_x^2+u_t^2}}{cu_x-u_t} & 1 \end{bmatrix}.$$

□

Now, we calculate the Noether's conservation laws of the nonlinear Sine-Gordon equation (1.1), namely, the Euler-Lagrange equations for the variational problem (3.1).

Theorem 3.5. *The three Noether's conservation laws of Euler-Lagrange equations for the variational problem*

$$\iint \frac{1}{2}\big(- u_t^2 + c^2u_x^2 - 2m^2\cos u\big)dxdt,$$

are

$$d\left(\begin{bmatrix} \dfrac{cu_x - u_t}{\sqrt{-c^2u_x^2+u_t^2}} & 0 & 0 \\ 0 & \dfrac{\sqrt{-c^2u_x^2+u_t^2}}{cu_x - u_t} & 0 \\ -\dfrac{1}{2}\dfrac{(ct-x)(cu_x-u_t)}{\sqrt{-c^2u_x^2+u_t^2}} & \dfrac{1}{2}\dfrac{(x+ct)\sqrt{-c^2u_x^2+u_t^2}}{cu_x-u_t} & 1 \end{bmatrix} \right.$$
$$\times \begin{bmatrix} \dfrac{1}{2}c(I_2^u)^2 + cm^2\cos(I^u) & -\dfrac{1}{2}(I_2^u)^2 + m^2\cos(I^u) \\ -\dfrac{1}{2}c(I_2^u)^2 - cm^2\cos(I^u) & -\dfrac{1}{2}(I_2^u)^2 + m^2\cos(I^u) \\ 0 & 0 \end{bmatrix}.$$
$$\left. \times \begin{bmatrix} \dfrac{-u_t}{\sqrt{-c^2u_x^2+u_t^2}} & \dfrac{-u_x}{\sqrt{-c^2u_x^2+u_t^2}} \\ \dfrac{-c^2u_x}{\sqrt{-c^2u_x^2+u_t^2}} & \dfrac{-u_t}{\sqrt{-c^2u_x^2+u_t^2}} \end{bmatrix} \begin{bmatrix} dt \\ dx \end{bmatrix} \right) = 0.$$

Proof . According to Theorem 3.3, the elements of C_i^u correspond to the coefficients of the $I_{J\tau}^{\alpha}$ in (3.3), as follows:

$$C_1^u = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad C_2^u = \begin{bmatrix} -I_2^u \\ 0 \\ 0 \end{bmatrix},$$

and the $(\Xi(J,I))_i$, for $i = 1, 2$, are

$$(\Xi(J,I))_1 = \begin{matrix} \\ \alpha \\ \beta \\ \theta \end{matrix} \begin{matrix} \xi^x \\ \begin{pmatrix} -c \\ c \\ 0 \end{pmatrix} \end{matrix}, \qquad (\Xi(J,I))_2 = \begin{matrix} \\ \alpha \\ \beta \\ \theta \end{matrix} \begin{matrix} \xi^t \\ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \end{matrix}.$$

Since $I_1^u = 0$, the invariantized matrix of characteristics is,

$$\mathcal{Q}^u(J,I) = \begin{matrix} \\ \alpha \\ \beta \\ \theta \end{matrix} \begin{matrix} \begin{matrix} Q^u & D_x(Q^u) & D_t(Q^u) \end{matrix} \\ \begin{pmatrix} -I_2^u & cI_{11}^u - I_{12}^u & cI_{12}^u - I_{22}^u \\ -I_2^u & -cI_{11}^u - I_{12}^u & -cI_{12}^u - I_{22}^u \\ 0 & -I_2^u & 0 \end{pmatrix} \end{matrix},$$

thus, the vectors of invariants are

$$\upsilon_1 = \begin{bmatrix} \frac{1}{2}c(I_2^u)^2 + cm^2\cos(I^u) \\ -\frac{1}{2}c(I_2^u)^2 - cm^2\cos(I^u) \\ 0 \end{bmatrix}, \qquad \upsilon_2 = \begin{bmatrix} -\frac{1}{2}(I_2^u)^2 + m^2\cos(I^u) \\ -\frac{1}{2}(I_2^u)^2 + m^2\cos(I^u) \\ 0 \end{bmatrix},$$

and according to Lemma 3.4, the inverse of the Adjoint representation $Ad(\rho)^{-1}$ is as (3.4). Finally, the Jacobian matrix $\mathcal{J}$ is

$$\begin{aligned} \mathcal{J} &= \begin{bmatrix} \frac{\partial\tilde{x}}{\partial x}\big|_{g=\rho(z)} & \frac{\partial\tilde{x}}{\partial t}\big|_{g=\rho(z)} \\ \frac{\partial\tilde{t}}{\partial x}\big|_{g=\rho(z)} & \frac{\partial\tilde{t}}{\partial t}\big|_{g=\rho(z)} \end{bmatrix} = \begin{bmatrix} \cosh(c\theta) & c.\sinh(c\theta) \\ \frac{1}{c}.\sinh(c\theta) & \cosh(c\theta) \end{bmatrix} \\ &= \begin{bmatrix} \frac{-u_t}{\sqrt{-c^2u_x^2+u_t^2}} & \frac{-c^2u_x}{\sqrt{-c^2u_x^2+u_t^2}} \\ \frac{-u_x}{\sqrt{-c^2u_x^2+u_t^2}} & \frac{-u_t}{\sqrt{-c^2u_x^2+u_t^2}} \end{bmatrix} \end{aligned}$$

and its matrix of first minors, $M_{\mathcal{J}}$, is

$$M_{\mathcal{J}} = \begin{bmatrix} \frac{-u_t}{\sqrt{-c^2u_x^2+u_t^2}} & \frac{-u_x}{\sqrt{-c^2u_x^2+u_t^2}} \\ \frac{-c^2u_x}{\sqrt{-c^2u_x^2+u_t^2}} & \frac{-u_t}{\sqrt{-c^2u_x^2+u_t^2}} \end{bmatrix}.$$

Thus, the conservation laws are

$$d\Big(Ad(\rho)^{-1}.\big[\upsilon_1\ \upsilon_2\big].M_{\mathcal{J}}.d^1\hat{x}\Big) = 0, \qquad \text{where,} \qquad d^1\hat{x} = \begin{bmatrix} dt \\ dx \end{bmatrix}.$$

□

4 Concluding Remarks

We see that the three Noether's conservation laws of the Sine-Gordon equation (1.1) are in terms of vectors of invariants, the adjoint representation of the moving frame and a matrix which represents the group action on the 1-forms. Also, we notice that since we have already proved that equations (2.5) and (2.6) are equivalent, for calculation of boundary terms if we substitute $\mathcal{D}_\tau I_{12}^u$ by equation (2.6) instead of equation (2.5), or we use a combination of the two; in any case the conservation laws are equivalent.

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References

[1] M. Fels and P.J. Olver, *Moving coframes: II. Regularization and theoretical foundations*, Acta Appl. Math. **55** (1999), no. 2, 127–208.

[2] T.M.N. Gonçalves and E.L. Mansfield, *Moving frames and conservation laws for Euclidean invariant Lagrangians*, Stud. Appl. Math. **130** (2013), no. 2, 134–166.

[3] T. M. N. Gonçalves and E. L. Mansfield, *Moving Frames and Noether's Conservation Laws the General Case*, Vol. 4. Forum of Mathematics, Sigma. Cambridge University Press, 2016.

[4] T.M.N. Gonçalves and E.L. Mansfield, *On moving frames and Noether's conservation laws*, Stud. Appl. Math. **128** (2012), no. 1, 1–29.

[5] E.L. Mansfield, *A practical guide to the invariant calculus*, Vol. 26. Cambridge University Press, 2010.

[6] Y. Masoudi and M. Nadjafikhah, *Moving frames and conservation laws of a Lagrangian invariant under the Hyperbolic Rotation-Translation group*, Hokkaido Math. J. **47** (2018), no. 3, 557–579.

[7] E. Noether, *Invariant variation problems*, Transport Theory Statistic. Phys. **1** (1971), no. 3, 186–207.

[8] P.J. Olver, *Applications of Lie groups to differential equations*, Vol. 107. Springer Science and Business Media, 2000.