Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 2597–2602
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.24630.2784

Bernstein-type inequalities for a zero-preserving operator on the space of polynomials

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(Communicated by Ali Jabbari)

Abstract

In this paper, we study zero-preserving character of a linear operator on the space of complex-polynomials which also preserve Bernstein-type inequalities for polynomials.

Keywords: Gauss Lucas theorem, Inequalities in the Complex Domain, polynomials
2020 MSC: 30C10, 26D10, 41A17

1 Bernstein's Inequality

Let $\mathcal{P}_n$ denote the space of polynomials of degree at most n over the field of complex numbers. If $P \in \mathcal{P}_n$, then according to Bernstein's inequality [3],

$$\max_{|z|=1}|P'(z)| \leq n \max_{|z|=1}|P(z)|. \tag{1.1}$$

The result is sharp and equality in (1.1) holds if $P(z) = az^n$, $a \neq 0$. In other words, the Bernstein's inequality gives us the exact constant C_n in the inequality

$$\max_{|z|=1}|T[P](z)| \leq C_n \max_{|z|=1}|P(z)| \tag{1.2}$$

for the operator $T \equiv \dfrac{d}{dz}$. In this case $C_n = n$.

This inequality of Bernstein has an analogue [2] for trigonometric polynomials which states that if $t(\theta) = \sum_{k=-n}^{n} a_k e^{ik\theta}$ is a trigonometric polynomial of degree n with $|t(\theta))| \leq 1$ for $0 \leq \theta < 2\pi$ then

$$|t'(\theta)| \leq n \qquad \text{for} \quad 0 \leq \theta < 2\pi. \tag{1.3}$$

Note that if $P(z)$ is a polynomial of degree n, then $t(\theta) = P(e^{i\theta})$ is a trigonometric polynomial with $\left|\frac{1}{M}t(\theta)\right| \leq 1$ for $\theta \in \mathbb{R}$, where $M = \max_{|z|=1}|P(z)|$. By applying (1.3) to $\frac{1}{M}t(\theta)$, one can get inequality (1.1).

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Received: September 2021 *Accepted:* June 2022

Bernstein's inequality for trigonometric polynomials has played a fundamental role in harmonic analysis, approximation theory [9] and in the study of random trigonometric series [7]. It also has found its usage in the theory of Banach spaces [13, p. 20-21].

Many mathematician's have studied this problem of characterization of C_n for different operators defined on $\mathcal{P}_n$ (for more details see [14, P. 538]). Jain [6], studied the operator $T_\alpha[P](z) := zP'(z) - \alpha P(z)$ and proved that if $P \in \mathcal{P}_n$ and $\alpha \in \mathbb{C}$ with $|\alpha| \leq n/2$, then

$$\max_{|z|=1} |zP'(z) - \alpha P(z)| \leq |n - \alpha| \max_{|z|=1} |P(z)|. \tag{1.4}$$

That is, for this operator $C_n = |n - \alpha|$. One can easily observe that Bernstein's inequality is a special of Jain's result and follows by taking $\alpha = 0$.

Let $P \in \mathcal{P}_n$ with $|P(z)| \leq |Mz^n|$ for $|z| \leq 1$, then the inequality (1.1) can be reformulated as:

$$|P'(z)| \leq \left| \frac{d}{dz}(Mz^n) \right| \qquad \text{for} \quad |z| = 1. \tag{1.5}$$

As an extension of Berntein's inequality, Malik and Vong [10] proved that if an nth degree polynomial $F(z)$ has all zeros in $|z| \leq 1$ and $P \in \mathcal{P}_n$ with $|P(z)| \leq |F(z)|$ for $|z| = 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \leq n/2$

$$|zP'(z) - \alpha P(z)| \leq |zF'(z) - \alpha F(z)| \qquad \text{for} \quad |z| \geq 1. \tag{1.6}$$

The inequality (1.4) can be obtained from (1.6) by taking $F(z) = Mz^n$, where $M = \max_{|z|=1} |P(z)|$.

2 Zero-Preserving Linear Operator on $\mathcal{P}_n$

A linear operator $T : \mathcal{P}_n \to \mathcal{P}_n$ is said to *preserve zeros* if, for every $P \in \mathcal{P}_n$ having all its zeros in $|z| \leq 1$, the polynomial $T[P](z)$ also has all its zeros in $|z| \leq 1$. Rahman and Schmeisser [14, p. 538] called such class of operators as B_n-operators.

By Gauss-Lucas theorem [14, p. 71], the ordinary derivative is a B_n operator. The zero-preserving property of the ordinary derivative and its linearity play lead roles in the proof of inequality (1.6) given in [14]. In fact, this inequality holds for every operator on $\mathcal{P}_n$ satisfying these two properties. In this direction, Rahman and Schmeisser [14, p. 538] proved the following:

Theorem 2.1. Let $F(z)$ be a polynomial of degree n having all its zeros in $|z| \leq 1$ and $P \in \mathcal{P}_n$ such that $|P(z)| \leq |F(z)|$ for $|z| = 1$, then for any B_n-operator T, we have

$$|T[P](z)| \leq |T[F](z)| \qquad \text{for} \quad |z| \geq 1. \tag{2.1}$$

The inequality is sharp and equality holds if and only if $P(z) = e^{i\theta} F(z)$, $\theta \in \mathbb{R}$.

In this paper, we first study the zero-preserving character of the operator

$$T_{m,\alpha}[P](z) = zP^{(m)}(z) - \alpha P^{(m-1)}(z), \quad \text{where}\, m \in \mathbb{N} \ \text{ with } \ m \leq n, \tag{2.2}$$

defined on the space of polynomials $\mathcal{P}_n$ and $\alpha \in \mathbb{C}$. This operator involve the mth and $(m-1)$th derivatives of $P(z)$. Moreover, $P^{(0)}(z) = P(z)$. In this direction, we first prove the following theorem.

Theorem 2.2. 1 Let $P(z)$ be a polynomial of degree n and has all zeros in $|z| \leq r$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) \leq \dfrac{n-m+1}{2}$, then all the zeros of $T_{m,\alpha}[P](z)$, given by (2.2), are also in $|z| \leq r$.

For the proof of this theorem, we need the following generalized version of Walsh's Coincidence theorem, due to A. Aziz [1], for the case when the circular region is a circle.

Lemma 2.3. Let $G(z_1, z_2, ..., z_n)$ be a symmetric n-linear form of total degree m, $m \leq n$, in $z_1, z_2, ..., z_n$ and let $\mathcal{C} : |z| \leq r$ be a circle containing the n points $w_1, w_2, ..., w_n$. Then in $\mathcal{C}$ there exists atleast one point β such that

$$G(\beta, \beta, \ldots, \beta) = G(w_1, w_2, \ldots, w_n).$$

Proof .[Proof of Theorem 2.2] Let w be any zero of the polynomial $T_{m,\alpha}[P](z)$, then

$$wP^{(m)}(w) - \alpha P^{(m-1)}(w) = 0. \tag{2.3}$$

This expression is linear and symmetric in the zeros of $P(z)$. By lemma 2.3, w will also satisfy the equation obtained by replacing $P(z)$ in (2.3) by $(z-\beta)^n$, where β is a suitable complex number with $|\beta| \le r$. This implies

$$\begin{aligned} &n(n-1)\ldots(n-m+1)(w-\beta)^{n-m}w \\ &\qquad - \alpha\, n(n-1)\ldots.(n-m+2)(w-\beta)^{n-m+1} = 0 \end{aligned}$$

or

$$n(n-1)\ldots(n-m+2)(w-\beta)^{n-m}\{(n-m+1)w - \alpha(w-\beta)\} = 0 \tag{2.4}$$

Since $\Re(\alpha) \le \frac{n-m+1}{2}$, then $\Re\left(\frac{\alpha}{n-m+1}\right) \le \frac{1}{2}$. This implies that

$$\left|\frac{\alpha}{n-m+1}\right| \le \left|\frac{\alpha}{n-m+1} - 1\right|$$

or

$$|\alpha| \le |\alpha - (n-m+1)| \tag{2.5}$$

The equation (2.4) implies that

$$(w-\beta) = 0 \text{ or } (n-m+1)w - \alpha(w-\beta) = 0.$$

Equivalently,

$$w = \beta \text{ or } w = \tfrac{\alpha\beta}{\alpha-(n-m+1)}.$$

This further implies by using (2.5) that,

$$|w| = |\beta| \text{ or } |w| = \frac{|\alpha||\beta|}{|\alpha-(n-m+1)|} \le \frac{|\alpha||\beta|}{|\alpha|}.$$

Thus,

$$\Rightarrow |w| \le |\beta| \le r$$

Hence, it follows that all the zeros of $T_{m,\alpha}[P](z)$ also lie in $|z| \le r$. This completes the proof. □

The linearity of $T_{m,\alpha}[P](z)$ is not difficult to verify. Hence, the Theorem 2.2 can also be formulated as:

Theorem 2.4. The operator $T_{m,\alpha}[P](z)$ given by (2.2) is a B_n-operator on $\mathcal{P}_n$, if $\Re(\alpha) \le \frac{n-m+1}{2}$.

Since $T_{m,\alpha}[P](z)$ is a B_n-operator on the space of polynomials with a constraint on α, then by applying theorem 2.1 to $T_{m,\alpha}[P](z) = zP^{(m)}(z) - \alpha P^{(m-1)}(z)$, we obtain the following extension of the inequality (1.6).

Corollary 2.5. Let $F(z)$ be a polynomial of degree n and has all its zeros in $|z| \le 1$. Further, let $P \in \mathcal{P}_n$ with

$$|P(z)| \le |F(z)| \qquad \text{for} \quad |z| = 1.$$

Then for any $\alpha \in \mathbb{C}$ with $\Re(\alpha) \le \dfrac{n-m+1}{2}$ and $|z| \ge 1$,

$$|zP^{(m)}(z) - \alpha P^{(m-1)}(z)| \le |zF^{(m)}(z) - \alpha F^{(m-1)}(z)|, \quad m \le n. \tag{2.6}$$

The bound is sharp and inequality (2.1) becomes equality if $P(z) = e^{i\theta}F(z)$, $\theta \in \mathbb{R}$.

The next Corollary follows by taking $F(z) = \max_{|z|=1}|P(z)|z^n$ in Corollary 2.5.

Corollary 2.6. Let $P \in \mathcal{P}_n$, then for any $\alpha \in \mathbb{C}$ with $\Re(\alpha) \leq \dfrac{n-m+1}{2}$,

$$\max_{|z|=1}|zP^{(m)}(z) - \alpha P^{(m-1)}(z)| \leq \frac{n!}{(n-m+1)!}|\alpha - (n-m+1)|\max_{|z|=1}|P(z)|. \tag{2.7}$$

The result is sharp and equality in (2.7) holds if $P(z) = az^n$.

Thus, $C_n = \frac{n!}{(n-m+1)!}|\alpha - (n-m+1)|$ is the exact constant for the operator $T_{m,\alpha}$.

Remark 2.7. For $m = 1$, the inequalities (2.6) and (2.7) reduces to (1.6) and (1.4) respectively. Moreover, the the inequalities (2.6) and (2.7) hold for $|\alpha| \leq n/2$, while as, the Corollaries 2.5 and 2.6 also show that the range of α for which these inequalities hold extends from $|\alpha| \leq n/2$ to $\Re(\alpha) \leq n/2$.

3 Polynomials with Constraints

Let $\mathcal{P}_n^0$ denotes the set of polynomials of degree at n and having no zero in $|z| < 1$. It was proved by P.D. Lax [8] that if $P \in \mathcal{P}_n^0$ then

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{2}\max_{|z|=1}|P(z)|.$$

This inequality strengthens the Bernstien's inequality for polynomials not vanishing in $|z| < 1$. It was earlier conjectured by P. Erdös.

The exact constant C_n in Corollary 2.6 can also be strengthened for $P \in \mathcal{P}_n^0$ by using the following result of Rahman and Schmeisser [14, p. 539].

Lemma 3.1. Let $P \in \mathcal{P}_n^0$ and $\varphi_n(z) = z^n$, then for any B_n-operator T,

$$|T[P](z)| \leq \frac{(|T[1](z)| + |T[\varphi_n](z)|)}{2}\max_{|z|=1}|P(z)| \qquad \text{for} \quad |z| \geq 1.$$

If we take $T = T_{m,\alpha}$ in Lemma 3.1, then

$$T_{m,\alpha}[\varphi_n](z) = \frac{n!}{(n-m+1)!}((n-m+1) - \alpha)z^{n-m+1}$$

and

$$T_{m,\alpha}[1](z) = -\delta_{m1}\alpha, .$$

where δ_{m1} denotes Kronecker delta.

Thus, we have the following Erdös-Lax type inequality for $T_{m,\alpha}$.

Theorem 3.2. Let $P \in \mathcal{P}_n^0$, then for every $\alpha \in \mathbb{C}$ with $\Re(\alpha) \leq \dfrac{n-m+1}{2}$,

$$\max_{|z|=1}|zP^{(m)}(z) - \alpha P^{(m-1)}(z)| \leq C_n^m \max_{|z|=1}|P(z)|,$$

where

$$C_n^m = \delta_{m1}|\alpha| + \frac{n!}{(n-m+1)!}\frac{|\alpha - (n-m+1)|}{2}. \tag{3.1}$$

The inequality is sharp and $P(z) = az^n + b$ is an extremal polynomial, $|a| = |b| \neq 0$.

A polynomial $f(z)$ of degree n is said to be self-inversive if $f(z) = \sigma q(z)$, where $q(z) = z^n\overline{f(1/\overline{z})}$ and $|\sigma| = 1$. The Theorem 3.2 also holds if $P(z)$ is a self-inversive polynomials. The following lemma due to Rahman and Schmeisser [14, p. 539] is needed for the proof.

Lemma 3.3. Let $P \in \mathcal{P}_n$, $Q(z) = z^n\overline{P(1/\overline{z})}$ and $\varphi_n(z) = z^n$, then for any B_n-operator T

$$|T[P](z)| + |T[Q](z)| \leq (|T[1](z)| + |T[\varphi_n](z)|) \max_{|z|=1}|P(z)| \quad |z| \geq 1$$

Theorem 3.4. Let $P(z)$ be a self-inversive polynomial of degree n, then for every $\alpha \in \mathbb{C}$ with $\Re(\alpha) \leq \dfrac{n-m+1}{2}$,

$$\max_{|z|=1}|zP^{(m)}(z) - \alpha P^{(m-1)}(z)| \leq C_n^m \max_{|z|=1}|P(z)|, \tag{3.2}$$

where C_n^m is given by (3.1). The inequality (3.2) is sharp and $P(z) = az^n + \bar{a}$ is an extremal polynomial, $a \in \mathbb{C} \setminus \{0\}$.

Proof . Let $Q(z) = z^n\overline{P(1/\overline{z})}$ then $P(z) = \sigma Q(z)$ for some unit modulus complex number σ. This implies that

$$\left|zP^{(m)}(z) - \alpha P^{(m-1)}(z)\right| = \left|zQ^{(m)}(z) - \alpha Q^{(m-1)}(z)\right| \qquad \forall z \in \mathbb{C}. \tag{3.3}$$

Moreover, from Lemma 3.3 with $T = T_{m,\alpha}$, we have

$$\begin{aligned} &\left|zP^{(m)}(z) - \alpha P^{(m-1)}(z)\right| + \left|zQ^{(m)}(z) - \alpha Q^{(m-1)}(z)\right| \\ &\leq \left(\delta_{m1}|\alpha| + \frac{n!}{(n-m+1)!}|\alpha - (n-m+1)|\right) \max_{|z|=1}|P(z)|. \end{aligned} \tag{3.4}$$

The inequality (3.2) follows by combining inequalities (3.3) and (3.4). This completes the proof. □

4 Concluding Remarks and Open problems

1. If one refers to proof of Theorem 2.1, we can conclude that the linearity and zero-preserving property of B_n-operator plays a fundamental role in the proof. There are operators on $\mathcal{P}_n$ which have zero-preserving property like Nagy's generalized derivative (see [4, 16]) but are not linear. A natural question one can ask here is that, whether theorem 2.1, holds for operators which are not linear but does preserve location of zeros, or does there exist non-linear zero-preserving operators on $\mathcal{P}_n$ which satisfy the conclusion of Theorem 2.1.

2. For $m = 1$, the operator $T_{m,\alpha}$ takes the form of Simirnov operator (see [5, 15]). In [15, Chapter V], Simirnov proved that his operator preserves inequalities between polynomials. According to his result, for $m = 1$, the Corollary 2.5 holds for all α's with $\alpha/n \in \bar{\Omega}_{|z|}$ where $\Omega_{|z|}$ denotes the image of the disc $\{t : |t| \leq |z|\}$ under the mapping $\phi(t) = \frac{t}{1+t}$. In this regard, the extension of $T_{m,\alpha}$ in Simirnov's settings is a plausible question to ask.

5 Acknowledgment

The authors are thankful to the referee for comments and suggestions.

References

[1] A. Aziz, *On the location of the zeros of certain composite polynomials*, Pacific J. Math. **118** (1985), 17–26.

[2] S.N. Bernstein, *Sur. ĺordre de la meilleure appromation des functions continues par des Polynomes de degré donné*, Mem. Acad. Roy. Belgique, **12** (1912), no. 4, 1–103.

[3] S. Bernstein, *Sur la limitation des dérivées des polynomes*, C. R. Math. Acad. Sci. Paris **190** (1930), 338–340.

[4] P.L. Cheung, T.W. Ng, J. Tsai and S.C.P. Yam, *Higher-order, polar and Sz.-Nagy's generalized derivatives of random polynomials with independent and identically distributed zeros on the unit circle*, Comput. Meth. Funct. Theory **15** (2015), no. 1, 159–186.

[5] E.G. Ganenkova and V.V. Starkov, *The Möbius transformation and Smirnov's inequality for polynomials*, Math. Notes 105 (2019), no. 2, 58–68.

[6] V.K. Jain, *Generalization of certain well known inequalities for polynomials*, Glas. Mat. **32** (1997), 45–51.

[7] J.P. Kahane, *Some Random Series of Functions*, Cambridge University Press, Cambridge, 1985.

[8] P.D. Lax, *Proof of a conjecture of P. Erdös on the derivative of a polynomial*, Bull. Amer. Math. Soc. **50** (1944), 509–513.

[9] G.G. Lorentz, *Approximation of Functions*, Chelsea Publishing Company, NewYork, 1986.

[10] M.A. Malik and M.C. Vong, *Inequalities concerning the derivative of polynomials*, Rend. Circ. Mat. Palermo **34** (1985), 422–426.

[11] M. Marden, *Geometry of polynomials*, American Mathematical Society, Providence, 1985.

[12] G.V. Milovanović, D.S. Mitrinović and Th. M. Rassias, *Topics in Polynomials: Extremal Properties, Inequalities, Zeros,* World Scientific Publishing Company, Singapore, 1994.

[13] G. Pisier, *The Volume of Convex Bodies and Banach Space Geometry*, Volume 94, Cambridge University Press, Cambridge, 1989.

[14] Q.I. Rahman and G. Schmeisser, *Analytic theory of Polynomials*, Clarendon Press Oxford, 2002.

[15] V.I. Smirnov and N.A. Lebedev, *Constructive Theory of Functions of a Complex Variable*, Nauka, Moscow, Leningrad, 1964.

[16] S. Gulzar, N.A. Rather and F.A. Bhat, *The location of critical points of polynomials*, Asian-Eur. J. Math. **12** (2019), no. 1, 1950087.