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On a solvable system of difference equations via some number sequences

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Abstract

In this paper, we show that the following three-dimensional rational system of difference equations

$$x_n = \frac{z_{n-1}z_{n-3}}{bx_{n-2}+az_{n-3}},\ y_n = \frac{x_{n-1}x_{n-3}}{dy_{n-2}+cx_{n-3}},\ z_n = \frac{y_{n-1}y_{n-3}}{fz_{n-2}+ey_{n-3}},\ n \in \mathbb{N}_0,$$

where the parameters a, b, c, d, e, f and the initial values x_{-i}, y_{-i}, z_{-i}, $i \in \{1, 2, 3\}$, are real numbers, can be solved in explicit form. In addition, the solutions of aforementioned systems according to the special cases of the parameters are given in closed form. Later, the forbidden set of the initial values for aforementioned system is described. Finally, an application and numerical examples to support our results are given.

Keywords: Interlacing indices, solution, forbidden set, Fibonacci number, Pell number
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1 Introduction

There are different types of difference equations in theory of difference equations. One of them is

$$x_{n+1} = a_n x_n + b_n,\ n \in \mathbb{N}_0. \tag{1.1}$$

The equation (1.1) is non-homogeneous linear difference equation of the first-order with variable coefficients. Equation (1.1) and its special cases were solved by Levy and Lessman in [23]. The general solution of equation (1.1) can be written in the following form

$$x_n = x_0 \prod_{k=0}^{n-1} a_k + \sum_{i=0}^{n-1} b_i \prod_{k=i+1}^{n-1} a_k,\ n \in \mathbb{N}.$$

If every $n \in \mathbb{N}_0$, $a_n = 0$, equation (1.1) is expressed as

$$x_{n+1} = b_n,\ n \in \mathbb{N}_0. \tag{1.2}$$

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Then, the general solution of equation (1.2) is $x_n = b_{n-1}$ for $n \in \mathbb{N}$.
If every $n \in \mathbb{N}_0$, $b_n = 0$, then equation (1.1) turns into the following homogeneous linear difference equation of the first-order with variable coefficients

$$x_{n+1} = a_n x_n,\ n \in \mathbb{N}_0. \tag{1.3}$$

The general solution of equation (1.3) is

$$x_n = x_0 \prod_{k=0}^{n-1} a_k,\ n \in \mathbb{N}.$$

For the case when the sequences $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are constant, that is, $a_n = a$, $b_n = b$, $n \in \mathbb{N}_0$, equation (1.1) becomes

$$x_{n+1} = ax_n + b,\ n \in \mathbb{N}_0, \tag{1.4}$$

which defines non-homogeneous linear difference equation of the first-order with constant coefficients. The general solution of equation (1.4) is

$$x_n = a^n x_0 + b\frac{1-a^n}{1-a},\ n \in \mathbb{N}, \tag{1.5}$$

if $a \neq 1$ and while

$$x_n = x_0 + bn,\ n \in \mathbb{N}, \tag{1.6}$$

if $a = 1$.
In equation (1.4), if $b = 0$, we obtain the following equation

$$x_{n+1} = ax_n,\ n \in \mathbb{N}_0. \tag{1.7}$$

The solution of equation (1.7) has the following form

$$x_n = a^n x_0,\ n \in \mathbb{N}. \tag{1.8}$$

Another well-known important difference equation is

$$x_{n+1} = ax_n + bx_{n-1},\ n \in \mathbb{N}_0. \tag{1.9}$$

De Moivre solved the homogeneous linear difference equation (1.9) in [6]. The general solution of the sequence $(x_n)_{n\geq -1}$, is given by

$$x_n = \frac{(\lambda_2 x_{-1} - x_0)\,\lambda_1^{n+1} + (x_0 - \lambda_1 x_{-1})\,\lambda_2^{n+1}}{\lambda_2 - \lambda_1},\ n \geq -1, \tag{1.10}$$

when $b \neq 0$ and $a^2 + 4b \neq 0$,

$$x_n = (x_0\,(n+1) - x_{-1}\lambda_1 n)\,\lambda_1^n,\ n \geq -1, \tag{1.11}$$

when $b \neq 0$ and $a^2 + 4b = 0$, where λ_1 and λ_2 are the roots of the polynomial $P(\lambda) = \lambda^2 - a\lambda - b = 0$. Also, the roots of characteristic equation are $\lambda_{1,2} = \frac{a\pm\sqrt{a^2+4b}}{2}$.
Recently, non-linear difference equations have been studied by mathematicians. A simple example for non-linear difference equation which can be solved in explicit form is the following difference equation

$$x_n = \frac{bx_{n-1} + a}{dx_{n-1} + c},\ n \in \mathbb{N}_0, \tag{1.12}$$

where the initial value x_{-1} is real number. Equation (1.12) is called Riccati difference equation.
If $\begin{vmatrix} b & a \\ d & c \end{vmatrix} = 0$, then equation (1.12) is trivial such that $x_n = \frac{a}{c}$ for $n \in \mathbb{N}_0$. If $d = 0$, equation (1.12) turns into the linear equation

$$x_n = \frac{b}{c}x_{n-1} + \frac{a}{c},\ n \in \mathbb{N}_0. \tag{1.13}$$

From (1.5)-(1.6), the general solution of equation (1.13) is

$$x_n = \left(\frac{b}{c}\right)^{n+1} x_{-1} + \frac{a}{c}\frac{1-\left(\frac{b}{c}\right)^{n+1}}{1-\frac{b}{c}},$$

if $\frac{b}{c} \neq 1$ and while

$$x_n = x_{-1} + \frac{a}{c}(n+1),$$

if $\frac{b}{c} = 1$.
If $d \neq 0 \neq (b+c)$ and $\begin{vmatrix} b & a \\ d & c \end{vmatrix} \neq 0$, by means of the change of variables

$$x_n = \frac{b+c}{d} y_n - \frac{c}{d},\ n \geq -1. \tag{1.14}$$

Using (1.14) in equation (1.12) we have

$$y_n = \frac{-R + y_{n-1}}{y_{n-1}},\ n \in \mathbb{N}_0, \tag{1.15}$$

where the parameter $R = \frac{bc-ad}{(b+c)^2}$, and it is called Riccati number. By using the change of variable

$$y_n = \frac{z_{n+1}}{z_n},\ n \geq -1,$$

then equation (1.15) transforms into the following second order linear difference equations,

$$z_{n+1} = z_n - R z_{n-1},\ n \in \mathbb{N}_0. \tag{1.16}$$

From (1.10)-(1.11), the general solution of equation (1.16) is

$$z_n = \begin{cases} \frac{(\lambda_1 y_{-1} - R)\lambda_1^n - (\lambda_2 y_{-1} - R)\lambda_2^n}{\lambda_1 - \lambda_2}, & \text{if } R \neq \frac{1}{4}, \\ \left(\frac{2y_{-1} + (2y_{-1}-1)n}{2}\right)\left(\frac{1}{2}\right)^n, & \text{if } R = \frac{1}{4}, \end{cases} \quad n \in \mathbb{N}_0,$$

where $\lambda_1 = \frac{1+\sqrt{1-4R}}{2}$, $\lambda_2 = \frac{1-\sqrt{1-4R}}{2}$, $\lambda_1\lambda_2 = R = \frac{bc-ad}{(b+c)^2}$, the initial values $z_{-1} = 1$ and $z_0 = y_{-1}$.
Then, the solution of equation (1.15) is given by

$$y_n = \begin{cases} \frac{(\lambda_1 y_{-1} - R)\lambda_1^{n+1} - (\lambda_2 y_{-1} - R)\lambda_2^{n+1}}{(\lambda_1 y_{-1} - R)\lambda_1^n - (\lambda_2 y_{-1} - R)\lambda_2^n}, & \text{if } R \neq \frac{1}{4}, \\ \frac{2y_{-1} + (2y_{-1}-1)(n+1)}{4y_{-1} + (4y_{-1}-2)n}, & \text{if } R = \frac{1}{4}, \end{cases} \quad n \in \mathbb{N}_0.$$

Moreover, the solution of equation (1.12) is given by

$$x_n = \begin{cases} \frac{b+c}{d}\frac{\left(\lambda_1 \frac{dx_{-1}+c}{b+c} - R\right)\lambda_1^{n+1} - \left(\lambda_2 \frac{dx_{-1}+c}{b+c} - R\right)\lambda_2^{n+1}}{\left(\lambda_1 \frac{dx_{-1}+c}{b+c} - R\right)\lambda_1^{n} - \left(\lambda_2 \frac{dx_{-1}+c}{b+c} - R\right)\lambda_2^{n}} - \frac{c}{d}, & \text{if } R \neq \frac{1}{4}, \\ \frac{b+c}{d}\frac{2\frac{dx_{-1}+c}{b+c} + \left(2\frac{dx_{-1}+c}{b+c} - 1\right)(n+1)}{4\frac{dx_{-1}+c}{b+c} + \left(4\frac{dx_{-1}+c}{b+c} - 2\right)n} - \frac{c}{d}, & \text{if } R = \frac{1}{4}, \end{cases} \quad n \in \mathbb{N}_0. \tag{1.17}$$

For more details, see [22].
Difference equations or systems which transform into linear or Riccati difference equations by applying appropriate transformations have engaged attention of many mathematicians (see, e.g. [2, 3, 5, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41]).
For example, following difference equations

$$x_{n+1} = \frac{x_n x_{n-2}}{\pm x_{n-1} \mp x_{n-2}},\ n \in \mathbb{N}_0, \tag{1.18}$$

reduced to the Riccati difference equation under convenient transformations in [1]. Later, in [7, 8], equations in (1.18) were generalized to the following equations

$$x_{n+1} = \frac{a x_{n-l} x_{n-k}}{b x_{n-p} + c x_{n-q}} \quad \text{and} \quad x_{n+1} = \frac{a x_{n-l} x_{n-k}}{b x_{n-p} - c x_{n-q}},\ n \in \mathbb{N}_0,$$

where $r := \max\{l, k, p, q\}$ is non-negative integer, a, b, c are positive constants.

Then, the equations in (1.18) were expanded to the following systems of difference equations

$$x_{n+1} = \frac{y_n y_{n-2}}{x_{n-1} + y_{n-2}},\ y_{n+1} = \frac{x_n x_{n-2}}{\pm y_{n-1} \pm x_{n-2}},\ n \in \mathbb{N}_0. \tag{1.19}$$

in [4]. The solutions of systems in given (1.19) were found by using induction.

But, two-dimensional systems of difference equations in (1.19) were extended to the following two-dimensional system of difference equations with constant coefficients

$$x_{n+1} = \frac{y_n y_{n-2}}{b x_{n-1} + a y_{n-2}},\ y_{n+1} = \frac{x_n x_{n-2}}{d y_{n-1} + c x_{n-2}},\ n \in \mathbb{N}_0, \tag{1.20}$$

and system (1.20) was solved using convenient transformations in [24].

A natural question is if any of the corresponding three-dimensional relatives to equation (1.18) and system (1.19) is also solvable. Here we give a positive answer to the question. Namely, we consider the following system

$$x_n = \frac{z_{n-1} z_{n-3}}{b x_{n-2} + a z_{n-3}},\ y_n = \frac{x_{n-1} x_{n-3}}{d y_{n-2} + c x_{n-3}},\ z_n = \frac{y_{n-1} y_{n-3}}{f z_{n-2} + e y_{n-3}},\ n \in \mathbb{N}_0, \tag{1.21}$$

where the parameters a, b, c, d, e, f and the initial values $x_{-i}, y_{-i}, z_{-i},\ i \in \{1, 2, 3\}$, are real numbers. We solve system (1.21) in explicit form. Then, the solutions of system (1.21) according to the special cases of the parameters are given in closed form. Later, the forbidden set of the initial values for system (1.21) is described. Finally, an application and numerical examples to support our results are given.
The following definition will help us to find solutions.

Definition 1.1. [25] The general equation

$$x_{n+1} = h\left(x_{n+1-(k+1)}, x_{n+1-2(k+1)}, \ldots, x_{n+1-l(k+1)}\right),\quad n \in \mathbb{N}_0,$$

where $l \in \mathbb{N}$ and $k \in \mathbb{N}_0$ is a difference equation with interlacing indices.

2 Solutions of the system (1.21) in explicit form

Suppose that $x_{n_0} = 0$ for some $n_0 \geq -3$. Then from the second equation in (1.21) it follows that $y_{n_0+1} = 0$. If $y_{n_0+1} = 0$, then from the third equation in (1.21) it follows that $z_{n_0+2} = 0$, and consequently $f z_{n_0+2} + e y_{n_0+1} = 0$, from which it follows that z_{n_0+4} is not defined. Assume that $y_{n_1} = 0$ for some $n_1 \geq -3$. Then from the third equation in (1.21) it follows that $z_{n_1+1} = 0$. If $z_{n_1+1} = 0$, then from the first equation in (1.21) it follows that $x_{n_1+2} = 0$, and consequently $b x_{n_1+2} + a z_{n_1+1} = 0$, from which it follows that x_{n_1+4} is not defined. Suppose that $z_{n_2} = 0$ for some $n_2 \geq -3$. Then from the first equation in (1.21) it follows that $x_{n_2+1} = 0$. If $x_{n_2+1} = 0$, then from the second equation in (1.21) it follows that $y_{n_2+2} = 0$, and consequently $d y_{n_2+2} + c x_{n_2+1} = 0$, from which it follows that y_{n_2+4} is not defined. This means that the set

$$\left\{\vec{\mathbb{F}} : x_{-j} = 0 \text{ or } y_{-j} = 0 \text{ or } z_{-j} = 0,\ j \in \{1, 2, 3\}\right\},$$

is a subset of the forbidden set of solutions to system (1.21), where
$\vec{\mathbb{F}} = (x_{-3}, x_{-2}, x_{-1}, y_{-3}, y_{-2}, y_{-1}, z_{-3}, z_{-2}, z_{-1})$. Thus, for every well-defined solution of system (1.21), we get that $x_n y_n z_n \neq 0$, $n \geq -3$, if and only if $x_{-j} y_{-j} z_{-j} \neq 0$, $j \in \{1, 2, 3\}$.

2.1 Particular Cases of System (1.21)

Now, we will examine the solutions in 15 different cases depending on whether the parameters are zero or non-zero.

2.1.1 Case $b = d = f = 0$, $ace \neq 0$.

In this case, system (1.21) reduces to the following system

$$x_n = \frac{z_{n-1}}{a},\ y_n = \frac{x_{n-1}}{c},\ z_n = \frac{y_{n-1}}{e},\ n \in \mathbb{N}_0. \tag{2.1}$$

From (2.1), we get

$$x_n = \frac{x_{n-3}}{ace},\ y_n = \frac{y_{n-3}}{ace},\ z_n = \frac{z_{n-3}}{ace}, n \geq 2, \tag{2.2}$$

which are homogeneous linear third-order difference equations with constant coefficient. Equations in (2.2) are equations with interlacing indices of order three. Hence, the sequences

$$x_m^{(j)} = x_{3m+j},\ y_m^{(j)} = y_{3m+j},\ z_m^{(j)} = z_{3m+j},\ m \geq -1,\ j \in \{2,3,4\},$$

are solutions of the first-order difference equation

$$p_m = \frac{p_{m-1}}{ace},\ m \in \mathbb{N}_0. \tag{2.3}$$

From (1.8), the solution of difference equation (2.3)

$$p_m = \frac{p_{-1}}{(ace)^{m+1}},\ m \in \mathbb{N}_0. \tag{2.4}$$

From (2.4), we can write the solutions of equations in (2.2) as in the following form

$$x_{3m+j} = \frac{x_{j-3}}{(ace)^{m+1}},\ y_{3m+j} = \frac{y_{j-3}}{(ace)^{m+1}},\ z_{3m+j} = \frac{z_{j-3}}{(ace)^{m+1}}, \tag{2.5}$$

for $m \in \mathbb{N}_0$ and $j \in \{2,3,4\}$.

2.1.2 Case $a = c = e = 0$, $bdf \neq 0$.

In this case, system (1.21) is expressed as

$$x_n = \frac{z_{n-1}z_{n-3}}{bx_{n-2}},\ y_n = \frac{x_{n-1}x_{n-3}}{dy_{n-2}},\ z_n = \frac{y_{n-1}y_{n-3}}{fz_{n-2}},\ n \in \mathbb{N}_0. \tag{2.6}$$

From (2.6), we get

$$bx_nx_{n-2} = z_{n-1}z_{n-3},\ dy_ny_{n-2} = x_{n-1}x_{n-3},\ fz_nz_{n-2} = y_{n-1}y_{n-3},\ n \in \mathbb{N}_0. \tag{2.7}$$

From which it follows that

$$x_nx_{n-2} = \frac{x_{n-3}x_{n-5}}{bdf},\ y_ny_{n-2} = \frac{y_{n-3}y_{n-5}}{bdf},\ z_nz_{n-2} = \frac{z_{n-3}z_{n-5}}{bdf},\ n \geq 2. \tag{2.8}$$

By employing the change of variables

$$k_n = x_nx_{n-2},\ \widehat{k}_n = y_ny_{n-2},\ \widetilde{k}_n = z_nz_{n-2},\ n \geq -1, \tag{2.9}$$

equations in (2.8) are transformed into the following equations

$$k_n = \frac{k_{n-3}}{bdf},\ \widehat{k}_n = \frac{\widehat{k}_{n-3}}{bdf},\ \widetilde{k}_n = \frac{\widetilde{k}_{n-3}}{bdf},\ n \geq 2, \tag{2.10}$$

which are homogeneous linear third-order difference equations with constant coefficient. Equations in (2.10) are equations with interlacing indices of order three. Hence, the sequences

$$k_m^{(j)} = k_{3m+j},\ \widehat{k}_m^{(j)} = \widehat{k}_{3m+j},\ \widetilde{k}_m^{(j)} = \widetilde{k}_{3m+j},\ m \geq -1,\ j \in \{2,3,4\},$$

are solutions of the first-order difference equation

$$r_m = \frac{r_{m-1}}{bdf},\ m \in \mathbb{N}_0.$$

From (1.8), the general solution is

$$r_m = \frac{r_{-1}}{(bdf)^{m+1}}, \ m \in \mathbb{N}_0. \tag{2.11}$$

From (2.11), we can write the solutions of equations in (2.10),

$$k_{3m+j} = \frac{k_{j-3}}{(bdf)^{m+1}}, \ \widehat{k}_{3m+j} = \frac{\widehat{k}_{j-3}}{(bdf)^{m+1}}, \ \widetilde{k}_{3m+j} = \frac{\widetilde{k}_{j-3}}{(bdf)^{m+1}}, \tag{2.12}$$

for $m \in \mathbb{N}_0$, $j \in \{2,3,4\}$. From (2.9), we have that

$$x_n = \frac{k_n}{k_{n-2}} \frac{k_{n-4}}{k_{n-6}} \frac{k_{n-8}}{k_{n-10}} x_{n-12}, \ n \geq 9, \tag{2.13}$$

$$y_n = \frac{\widehat{k}_n}{\widehat{k}_{n-2}} \frac{\widehat{k}_{n-4}}{\widehat{k}_{n-6}} \frac{\widehat{k}_{n-8}}{\widehat{k}_{n-10}} y_{n-12}, \ n \geq 9, \tag{2.14}$$

$$z_n = \frac{\widetilde{k}_n}{\widetilde{k}_{n-2}} \frac{\widetilde{k}_{n-4}}{\widetilde{k}_{n-6}} \frac{\widetilde{k}_{n-8}}{\widetilde{k}_{n-10}} z_{n-12}, \ n \geq 9. \tag{2.15}$$

From (2.13)-(2.15), we get

$$x_{12m+l} = \frac{k_{12m+l}}{k_{12m+l-2}} \frac{k_{12m+l-4}}{k_{12m+l-6}} \frac{k_{12m+l-8}}{k_{12m+l-10}} x_{12(m-1)+l},$$

$$y_{12m+l} = \frac{\widehat{k}_{12m+l}}{\widehat{k}_{12m+l-2}} \frac{\widehat{k}_{12m+l-4}}{\widehat{k}_{12m+l-6}} \frac{\widehat{k}_{12m+l-8}}{\widehat{k}_{12m+l-10}} y_{12(m-1)+l},$$

and

$$z_{12m+l} = \frac{\widetilde{k}_{12m+l}}{\widetilde{k}_{12m+l-2}} \frac{\widetilde{k}_{12m+l-4}}{\widetilde{k}_{12m+l-6}} \frac{\widetilde{k}_{12m+l-8}}{\widetilde{k}_{12m+l-10}} z_{12(m-1)+l},$$

where $m \in \mathbb{N}_0$ and $l = \overline{9,20}$, from which it follows that

$$x_{12m+3i+p-2} = x_{3i+p-14} \prod_{s=1}^{m+1} \frac{k_{12s+3i+p-2}}{k_{12s+3i+p-4}} \frac{k_{12s+3i+p-6}}{k_{12s+3i+p-8}} \frac{k_{12s+3i+p-10}}{k_{12s+3i+p-12}}, \tag{2.16}$$

$$y_{12m+3i+p-2} = y_{3i+p-14} \prod_{s=1}^{m+1} \frac{\widehat{k}_{12s+3i+p-2}}{\widehat{k}_{12s+3i+p-4}} \frac{\widehat{k}_{12s+3i+p-6}}{\widehat{k}_{12s+3i+p-8}} \frac{\widehat{k}_{12s+3i+p-10}}{\widehat{k}_{12s+3i+p-12}}, \tag{2.17}$$

$$z_{12m+3i+p-2} = z_{3i+p-14} \prod_{s=1}^{m+1} \frac{\widetilde{k}_{12s+3i+p-2}}{\widetilde{k}_{12s+3i+p-4}} \frac{\widetilde{k}_{12s+3i+p-6}}{\widetilde{k}_{12s+3i+p-8}} \frac{\widetilde{k}_{12s+3i+p-10}}{\widetilde{k}_{12s+3i+p-12}}, \tag{2.18}$$

where $m \in \mathbb{N}_0$, $p \in \{2,3,4\}$ and $i \in \{3,4,5,6\}$.
By applying solutions (2.12) in (2.16)-(2.18), after some basic calculation, we get

$$x_{12m+3i+j} = \frac{x_{3i+j-12}}{(bdf)^{2m+2}}, y_{12m+3i+j} = \frac{y_{3i+j-12}}{(bdf)^{2m+2}}, z_{12m+3i+j} = \frac{z_{3i+j-12}}{(bdf)^{2m+2}}, \tag{2.19}$$

for $m \in \mathbb{N}_0$, $j \in \{0,1,2\}$ and $i \in \{3,4,5,6\}$.

2.1.3 Case $b = 0$, $acdef \neq 0$.

In this case, system (1.21) becomes

$$x_n = \frac{z_{n-1}}{a}, \ y_n = \frac{x_{n-1}x_{n-3}}{dy_{n-2} + cx_{n-3}}, \ z_n = \frac{y_{n-1}y_{n-3}}{fz_{n-2} + ey_{n-3}}, \ n \in \mathbb{N}_0. \tag{2.20}$$

Employing the first equation in system (2.20) in the second equation in system (2.20), we get

$$y_n = \frac{z_{n-2}z_{n-4}}{a^2dy_{n-2} + acz_{n-4}}, \ n \geq 3.$$

We consider the following two-dimensional system

$$y_n=\frac{z_{n-2}z_{n-4}}{a^2dy_{n-2}+acz_{n-4}},\ n\geq 3,\ z_n=\frac{y_{n-1}y_{n-3}}{fz_{n-2}+ey_{n-3}},\ n\in\mathbb{N}_0. \tag{2.21}$$

The system (2.21) can be written in the form

$$\frac{y_n}{z_{n-2}}=\frac{1}{a^2d\frac{y_{n-2}}{z_{n-4}}+ac},\ n\geq 3,\ \frac{z_n}{y_{n-1}}=\frac{1}{f\frac{z_{n-2}}{y_{n-3}}+e},\ n\in\mathbb{N}_0. \tag{2.22}$$

Now, we may use the change of variables

$$\frac{y_n}{z_{n-2}}=t_n,\ n\geq 1,\ \frac{z_n}{y_{n-1}}=\widehat{t}_n,\ n\geq -2. \tag{2.23}$$

and transform system (2.22) into the following equations

$$t_n=\frac{1}{a^2dt_{n-2}+ac},\ n\geq 3,\ \widehat{t}_n=\frac{1}{f\widehat{t}_{n-2}+e},\ n\in\mathbb{N}_0. \tag{2.24}$$

Equations in (2.24) are equations with interlacing indices of order two. Hence, the sequences

$$\begin{aligned} t_m^{(i)}&=t_{2m+i},\ m\in\mathbb{N}_0,\ i\in\{1,2\},\\ \widehat{t}_m^{(j)}&=\widehat{t}_{2m+j},\ m\geq -1,\ j\in\{0,1\},\end{aligned}$$

are solutions to the following difference equations

$$t_m^{(i)}=\frac{1}{a^2dt_{m-1}^{(i)}+ac},\ m\in\mathbb{N},\ i\in\{1,2\},\ \ \widehat{t}_m^{(j)}=\frac{1}{f\widehat{t}_{m-1}^{(j)}+e},\ m\in\mathbb{N}_0,\ j\in\{0,1\}. \tag{2.25}$$

The solutions of difference equations in (2.25)

$$t_{2m+i}=\begin{cases}\frac{c}{ad}\frac{\left(\lambda_1\frac{adt_i+c}{c}-R_1\right)\lambda_1^m-\left(\lambda_2\frac{adt_i+c}{c}-R_1\right)\lambda_2^m}{\left(\lambda_1\frac{adt_i+c}{c}-R_1\right)\lambda_1^{m-1}-\left(\lambda_2\frac{adt_i+c}{c}-R_1\right)\lambda_2^{m-1}}-\frac{c}{ad}, & \text{if } R_1\neq\frac14,\\ \frac{c}{ad}\frac{2\frac{adt_i+c}{c}+\left(2\frac{adt_i+c}{c}-1\right)m}{4\frac{adt_i+c}{c}+\left(4\frac{adt_i+c}{c}-2\right)(m-1)}-\frac{c}{ad}, & \text{if } R_1=\frac14,\end{cases}\quad m\in\mathbb{N}_0, \tag{2.26}$$

$$\widehat{t}_{2m+j}=\begin{cases}\frac{e}{f}\frac{\left(\lambda_3\frac{f\widehat{t}_{j-2}+e}{e}-R_2\right)\lambda_3^{m+1}-\left(\lambda_4\frac{f\widehat{t}_{j-2}+e}{e}-R_2\right)\lambda_4^{m+1}}{\left(\lambda_3\frac{f\widehat{t}_{j-2}+e}{e}-R_2\right)\lambda_3^{m}-\left(\lambda_4\frac{f\widehat{t}_{j-2}+e}{e}-R_2\right)\lambda_4^{m}}-\frac{e}{f}, & \text{if } R_2\neq\frac14,\\ \frac{e}{f}\frac{2\frac{f\widehat{t}_{j-2}+e}{e}+\left(2\frac{f\widehat{t}_{j-2}+e}{e}-1\right)(m+1)}{4\frac{f\widehat{t}_{j-2}+e}{e}+\left(4\frac{f\widehat{t}_{j-2}+e}{e}-2\right)m}-\frac{e}{f}, & \text{if } R_2=\frac14,\end{cases}\quad m\in\mathbb{N}_0, \tag{2.27}$$

for $i\in\{1,2\}$, $j\in\{0,1\}$, where $\lambda_1=\frac{1+\sqrt{1-4R_1}}{2}$, $\lambda_2=\frac{1-\sqrt{1-4R_1}}{2}$, $\lambda_3=\frac{1+\sqrt{1-4R_2}}{2}$, $\lambda_4=\frac{1-\sqrt{1-4R_2}}{2}$, $R_1=\frac{-d}{c^2}$ and $R_2=\frac{-f}{e^2}$.
From (2.23), we have that

$$y_n=t_n\widehat{t}_{n-2}t_{n-3}\widehat{t}_{n-5}y_{n-6},\ n\geq 4, \tag{2.28}$$

$$z_n=\widehat{t}_nt_{n-1}\widehat{t}_{n-3}t_{n-4}z_{n-6},\ n\geq 5. \tag{2.29}$$

From the first equation in system (2.20) and (2.28)-(2.29), we have

$$x_{6m+2\widehat{j}_2+\widehat{i}_2+1}=\frac{1}{a}z_{2\widehat{j}_2+\widehat{i}_2}\prod_{k=1}^{m}\widehat{t}_{6k+2\widehat{j}_2+\widehat{i}_2}t_{6k+2\widehat{j}_2+\widehat{i}_2-1}\widehat{t}_{6k+2\widehat{j}_2+\widehat{i}_2-3}t_{6k+2\widehat{j}_2+\widehat{i}_2-4}, \tag{2.30}$$

$$y_{6m+2\widehat{j}_1+\widehat{i}_1}=y_{2\widehat{j}_1+\widehat{i}_1}\prod_{k=1}^{m}t_{6k+2\widehat{j}_1+\widehat{i}_1}\widehat{t}_{6k+2\widehat{j}_1+\widehat{i}_1-2}t_{6k+2\widehat{j}_1+\widehat{i}_1-3}\widehat{t}_{6k+2\widehat{j}_1+\widehat{i}_1-5}, \tag{2.31}$$

$$z_{6m+2\widehat{j}_2+\widehat{i}_2}=z_{2\widehat{j}_2+\widehat{i}_2}\prod_{k=1}^{m}\widehat{t}_{6k+2\widehat{j}_2+\widehat{i}_2}t_{6k+2\widehat{j}_2+\widehat{i}_2-1}\widehat{t}_{6k+2\widehat{j}_2+\widehat{i}_2-3}t_{6k+2\widehat{j}_2+\widehat{i}_2-4}, \tag{2.32}$$

where $m\in\mathbb{N}_0$, $\widehat{j}_1\in\{-1,0,1\}$, $\widehat{i}_1\in\{0,1\}$, $\widehat{j}_2\in\{0,1,2\}$ and $\widehat{i}_2\in\{-1,0\}$.
By applying (2.26)-(2.27) in (2.30)-(2.32), after some basic calculation, the solutions in explicit form of system (2.20) can be found.

2.1.4 Case $d=0$, $abcef\neq 0$.

In this case, system (1.21) is equivalent to the system

$$x_n=\frac{z_{n-1}z_{n-3}}{bx_{n-2}+az_{n-3}},\ y_n=\frac{x_{n-1}}{c},\ z_n=\frac{y_{n-1}y_{n-3}}{fz_{n-2}+ey_{n-3}},\ n\in\mathbb{N}_0, \tag{2.33}$$

which is an analogue of the system (2.20). By interchanging variables y_n, z_n, x_n, c, e, f, a, b, instead of x_n, y_n, z_n, a, c, d, e, f, respectively, the system in (2.20) is transformed into (2.33). So, by interchanging s_n, $\widehat{s}_n$ instead of t_n, $\widehat{t}_n$, the formulas in (2.26)-(2.27), they are transformed into the following formulas

$$s_{2m+i}=\begin{cases}\frac{e}{cf}\frac{\left(\lambda_3\frac{cfs_i+e}{e}-R_2\right)\lambda_3^m-\left(\lambda_4\frac{cfs_i+e}{e}-R_2\right)\lambda_4^m}{\left(\lambda_3\frac{cfs_i+e}{e}-R_2\right)\lambda_3^{m-1}-\left(\lambda_4\frac{cfs_i+e}{e}-R_2\right)\lambda_4^{m-1}}-\frac{e}{cf}, & \text{if } R_2\neq\frac{1}{4},\\ \frac{e}{cf}\frac{2\frac{cfs_i+e}{e}+\left(2\frac{cfs_{i-2}+e}{e}-1\right)m}{4\frac{cfs_{i-2}+e}{e}+\left(4\frac{cfs_i+e}{e}-2\right)(m-1)}-\frac{e}{cf}, & \text{if } R_2=\frac{1}{4},\end{cases}\quad m\in\mathbb{N}_0, \tag{2.34}$$

$$\widehat{s}_{2m+j}=\begin{cases}\frac{a}{b}\frac{\left(\lambda_5\frac{b\widehat{s}_{j-2}+a}{a}-R_3\right)\lambda_5^{m+1}-\left(\lambda_6\frac{b\widehat{s}_{j-2}+a}{a}-R_3\right)\lambda_6^{m+1}}{\left(\lambda_5\frac{b\widehat{s}_{j-2}+a}{a}-R_3\right)\lambda_5^{m}-\left(\lambda_6\frac{b\widehat{s}_{j-2}+a}{a}-R_3\right)\lambda_6^{m}}-\frac{a}{b}, & \text{if } R_3\neq\frac{1}{4},\\ \frac{a}{b}\frac{2\frac{b\widehat{s}_{j-2}+a}{a}+\left(2\frac{b\widehat{s}_{j-2}+a}{a}-1\right)(m+1)}{4\frac{b\widehat{s}_{j-2}+a}{a}+\left(4\frac{b\widehat{s}_{j-2}+a}{a}-2\right)m}-\frac{a}{b}, & \text{if } R_3=\frac{1}{4},\end{cases}\quad m\in\mathbb{N}_0, \tag{2.35}$$

for $i\in\{1,2\}$, $j\in\{0,1\}$, where $\lambda_3=\frac{1+\sqrt{1-4R_2}}{2}$, $\lambda_4=\frac{1-\sqrt{1-4R_2}}{2}$, $\lambda_5=\frac{1+\sqrt{1-4R_3}}{2}$, $\lambda_6=\frac{1-\sqrt{1-4R_3}}{2}$, $R_2=\frac{-f}{e^2}$ and $R_3=\frac{-b}{a^2}$. Then, (2.30)-(2.32) is transformed into the following formulas

$$x_{6m+2\widehat{j}_2+\widehat{i}_2}=x_{2\widehat{j}_2+\widehat{i}_2}\prod_{k=1}^{m}\widehat{s}_{6k+2\widehat{j}_2+\widehat{i}_2}s_{6k+2\widehat{j}_2+\widehat{i}_2-1}\widehat{s}_{6k+2\widehat{j}_2+\widehat{i}_2-3}s_{6k+2\widehat{j}_2+\widehat{i}_2-4}, \tag{2.36}$$

$$y_{6m+2\widehat{j}_2+\widehat{i}_2+1}=\frac{1}{c}x_{2\widehat{j}_2+\widehat{i}_2}\prod_{k=1}^{m}\widehat{s}_{6k+2\widehat{j}_2+\widehat{i}_2}s_{6k+2\widehat{j}_2+\widehat{i}_2-1}\widehat{s}_{6k+2\widehat{j}_2+\widehat{i}_2-3}s_{6k+2\widehat{j}_2+\widehat{i}_2-4}, \tag{2.37}$$

$$z_{6m+2\widehat{j}_1+\widehat{i}_1}=z_{2\widehat{j}_1+\widehat{i}_1}\prod_{k=1}^{m}s_{6k+2\widehat{j}_1+\widehat{i}_1}\widehat{s}_{6k+2\widehat{j}_1+\widehat{i}_1-2}s_{6k+2\widehat{j}_1+\widehat{i}_1-3}\widehat{s}_{6k+2\widehat{j}_1+\widehat{i}_1-5}, \tag{2.38}$$

where $m\in\mathbb{N}_0$, $\widehat{j}_1\in\{-1,0,1\}$, $\widehat{i}_1\in\{0,1\}$, $\widehat{j}_2\in\{0,1,2\}$ and $\widehat{i}_2\in\{-1,0\}$.
By applying (2.34)-(2.35) in (2.36)-(2.38), after some basic calculation, the solutions in explicit form of system (2.33) can be found.

2.1.5 Case $f=0$, $abcde\neq 0$.

In this case, we obtain the system

$$x_n=\frac{z_{n-1}z_{n-3}}{bx_{n-2}+az_{n-3}},\ y_n=\frac{x_{n-1}x_{n-3}}{dy_{n-2}+cx_{n-3}},\ z_n=\frac{y_{n-1}}{e},\ n\in\mathbb{N}_0, \tag{2.39}$$

which is an analogue of the system (2.33). By interchanging variables y_n, z_n, x_n, c, d, e, a, b, instead of x_n, y_n, z_n, a, b, c, e, f, respectively, the system in (2.33) is transformed into (2.39). So, by interchanging l_n, $\widehat{l}_n$ instead of s_n, $\widehat{s}_n$, the formulas in (2.34)-(2.35), they are transformed into the following formulas

$$l_{2m+i}=\begin{cases}\frac{a}{eb}\frac{\left(\lambda_5\frac{ebl_i+a}{a}-R_3\right)\lambda_5^m-\left(\lambda_6\frac{ebl_i+a}{a}-R_3\right)\lambda_6^m}{\left(\lambda_5\frac{ebl_i+a}{a}-R_3\right)\lambda_5^{m-1}-\left(\lambda_6\frac{ebl_i+a}{a}-R_3\right)\lambda_6^{m-1}}-\frac{a}{eb}, & \text{if } R_3\neq\frac{1}{4},\\ \frac{a}{eb}\frac{2\frac{ebl_i+a}{a}+\left(2\frac{ebl_i+a}{a}-1\right)m}{4\frac{ebl_i+a}{a}+\left(4\frac{ebl_i+a}{a}-2\right)(m-1)}-\frac{a}{eb}, & \text{if } R_3=\frac{1}{4},\end{cases}\quad m\in\mathbb{N}_0, \tag{2.40}$$

$$\widehat{l}_{2m+j}=\begin{cases}\frac{c}{d}\frac{\left(\lambda_1\frac{d\widehat{l}_{j-2}+c}{c}-R_1\right)\lambda_1^{m+1}-\left(\lambda_2\frac{d\widehat{l}_{j-2}+c}{c}-R_1\right)\lambda_2^{m+1}}{\left(\lambda_1\frac{d\widehat{l}_{j-2}+c}{c}-R_1\right)\lambda_1^{m}-\left(\lambda_2\frac{d\widehat{l}_{j-2}+c}{c}-R_1\right)\lambda_2^{m}}-\frac{c}{d}, & \text{if } R_1\neq\frac{1}{4},\\ \frac{c}{d}\frac{2\frac{d\widehat{l}_{j-2}+c}{c}+\left(2\frac{d\widehat{l}_{j-2}+c}{c}-1\right)(m+1)}{4\frac{d\widehat{l}_{j-2}+c}{c}+\left(4\frac{d\widehat{l}_{j-2}+c}{c}-2\right)m}-\frac{c}{d}, & \text{if } R_1=\frac{1}{4},\end{cases}\quad m\in\mathbb{N}_0, \tag{2.41}$$

for $i\in\{1,2\}$, $j\in\{0,1\}$, where $\lambda_5=\frac{1+\sqrt{1-4R_3}}{2}$, $\lambda_6=\frac{1-\sqrt{1-4R_3}}{2}$, $\lambda_1=\frac{1+\sqrt{1-4R_1}}{2}$, $\lambda_2=\frac{1-\sqrt{1-4R_1}}{2}$, $R_3=\frac{-b}{a^2}$ and $R_1=\frac{-d}{c^2}$. Then, (2.36)-(2.38) is transformed into the following formulas

$$x_{6m+2\widehat{j}_1+\widehat{i}_1}=x_{2\widehat{j}_1+\widehat{i}_1}\prod_{k=1}^{m} l_{6k+2\widehat{j}_1+\widehat{i}_1}\widehat{l}_{6k+2\widehat{j}_1+\widehat{i}_1-2}l_{6k+2\widehat{j}_1+\widehat{i}_1-3}\widehat{l}_{6k+2\widehat{j}_1+\widehat{i}_1-5}, \tag{2.42}$$

$$y_{6m+2\widehat{j}_2+\widehat{i}_2}=y_{2\widehat{j}_2+\widehat{i}_2}\prod_{k=1}^{m}\widehat{l}_{6k+2\widehat{j}_2+\widehat{i}_2}l_{6k+2\widehat{j}_2+\widehat{i}_2-1}\widehat{l}_{6k+2\widehat{j}_2+\widehat{i}_2-3}l_{6k+2\widehat{j}_2+\widehat{i}_2-4}, \tag{2.43}$$

$$z_{6m+2\widehat{j}_2+\widehat{i}_2+1}=\frac{1}{e}y_{2\widehat{j}_2+\widehat{i}_2}\prod_{k=1}^{m}\widehat{l}_{6k+2\widehat{j}_2+\widehat{i}_2}l_{6k+2\widehat{j}_2+\widehat{i}_2-1}\widehat{l}_{6k+2\widehat{j}_2+\widehat{i}_2-3}l_{6k+2\widehat{j}_2+\widehat{i}_2-4}, \tag{2.44}$$

where $m\in\mathbb{N}_0$, $\widehat{j}_1\in\{-1,0,1\}$, $\widehat{i}_1\in\{0,1\}$, $\widehat{j}_2\in\{0,1,2\}$ and $\widehat{i}_2\in\{-1,0\}$.
By applying (2.40)-(2.41) in (2.42)-(2.44), after some basic calculation, the solutions in explicit form of system (2.39) can be found.

2.1.6 Case $a=0$, $bcdef\neq 0$.

In this case, system (1.21) becomes

$$x_n=\frac{z_{n-1}z_{n-3}}{bx_{n-2}},\ y_n=\frac{x_{n-1}x_{n-3}}{dy_{n-2}+cx_{n-3}},\ z_n=\frac{y_{n-1}y_{n-3}}{fz_{n-2}+ey_{n-3}},\ n\in\mathbb{N}_0. \tag{2.45}$$

System (2.45) can be written in the following form

$$\frac{x_n}{z_{n-1}}=\frac{1}{b\frac{x_{n-2}}{z_{n-3}}},\ \frac{y_n}{x_{n-1}}=\frac{1}{d\frac{y_{n-2}}{x_{n-3}}+c},\ \frac{z_n}{y_{n-1}}=\frac{1}{f\frac{z_{n-2}}{y_{n-3}}+e},\ n\in\mathbb{N}_0. \tag{2.46}$$

Next, by employing the change of variables

$$\widehat{u}_n=\frac{x_n}{z_{n-1}},\ \widehat{v}_n=\frac{y_n}{x_{n-1}},\ \widehat{w}_n=\frac{z_n}{y_{n-1}},\ n\geq -2, \tag{2.47}$$

and transform (2.46) into the following equations

$$\widehat{u}_n=\frac{1}{b\widehat{u}_{n-2}}=\widehat{u}_{n-4}, n\geq 2,\ \widehat{v}_n=\frac{1}{d\widehat{v}_{n-2}+c}, n\in\mathbb{N}_0,\ \widehat{w}_n=\frac{1}{f\widehat{w}_{n-2}+e}, n\in\mathbb{N}_0, \tag{2.48}$$

which means that $(\widehat{u}_n)_{n\geq -2}$ are four-periodic, that is,

$$\widehat{u}_{4n+\widetilde{i}}=\widehat{u}_{\widetilde{i}}, \tag{2.49}$$

where $n\in\mathbb{N}_0$ and $\widetilde{i}=\overline{-2,1}$ and the second and the third equations in (2.48) are equations with interlacing indices of order two. Hence, the sequences

$$\widehat{v}_m^{(l)}=\frac{1}{d\widehat{v}_{m-1}^{(l)}+c},\ \widehat{w}_m^{(l)}=\frac{1}{f\widehat{w}_{m-1}^{(l)}+e},\ m\in\mathbb{N}_0, \tag{2.50}$$

for $l\in\{0,1\}$. From equalities in (1.17), the solutions of difference equations in (2.50)

$$\widehat{v}_{2m+l}=\begin{cases}\frac{c}{d}\frac{\left(\lambda_1\frac{d\widehat{v}_{l-2}+c}{c}-R_1\right)\lambda_1^{m+1}-\left(\lambda_2\frac{d\widehat{v}_{l-2}+c}{c}-R_1\right)\lambda_2^{m+1}}{\left(\lambda_1\frac{d\widehat{v}_{l-2}+c}{c}-R_1\right)\lambda_1^{m}-\left(\lambda_2\frac{d\widehat{v}_{l-2}+c}{c}-R_1\right)\lambda_2^{m}}-\frac{c}{d}, & \text{if } R_1\neq\frac14,\\ \frac{c}{d}\frac{2\frac{d\widehat{v}_{l-2}+c}{c}+\left(2\frac{d\widehat{v}_{l-2}+c}{c}-1\right)(m+1)}{4\frac{d\widehat{v}_{l-2}+c}{c}+\left(4\frac{d\widehat{v}_{l-2}+c}{c}-2\right)m}-\frac{c}{d}, & \text{if } R_1=\frac14,\end{cases}\quad m\in\mathbb{N}_0, \tag{2.51}$$

$$\widehat{w}_{2m+l}=\begin{cases}\frac{e}{f}\frac{\left(\lambda_3\frac{f\widehat{w}_{l-2}+e}{e}-R_2\right)\lambda_3^{m+1}-\left(\lambda_4\frac{f\widehat{w}_{l-2}+e}{e}-R_2\right)\lambda_4^{m+1}}{\left(\lambda_3\frac{f\widehat{w}_{l-2}+e}{e}-R_2\right)\lambda_3^{m}-\left(\lambda_4\frac{f\widehat{w}_{l-2}+e}{e}-R_2\right)\lambda_4^{m}}-\frac{e}{f}, & \text{if } R_2\neq\frac14,\\ \frac{e}{f}\frac{2\frac{f\widehat{w}_{l-2}+e}{e}+\left(2\frac{f\widehat{w}_{l-2}+e}{e}-1\right)(m+1)}{4\frac{f\widehat{w}_{l-2}+e}{e}+\left(4\frac{f\widehat{w}_{l-2}+e}{e}-2\right)m}-\frac{e}{f}, & \text{if } R_2=\frac14,\end{cases}\quad m\in\mathbb{N}_0, \tag{2.52}$$

for $l \in \{0,1\}$, where $\lambda_1 = \frac{1+\sqrt{1-4R_1}}{2}$, $\lambda_2 = \frac{1-\sqrt{1-4R_1}}{2}$, $\lambda_3 = \frac{1+\sqrt{1-4R_2}}{2}$, $\lambda_4 = \frac{1-\sqrt{1-4R_2}}{2}$, $R_1 = \frac{-d}{c^2}$ and $R_2 = \frac{-f}{e^2}$.
From (2.47), we get

$$x_n = \widehat{u}_n\widehat{w}_{n-1}\widehat{v}_{n-2}\widehat{u}_{n-3}\widehat{w}_{n-4}\widehat{v}_{n-5}\widehat{u}_{n-6}\widehat{w}_{n-7}\widehat{v}_{n-8}\widehat{u}_{n-9}\widehat{w}_{n-10}\widehat{v}_{n-11}x_{n-12},\ n \geq 9, \tag{2.53}$$

$$y_n = \widehat{v}_n\widehat{u}_{n-1}\widehat{w}_{n-2}\widehat{v}_{n-3}\widehat{u}_{n-4}\widehat{w}_{n-5}\widehat{v}_{n-6}\widehat{u}_{n-7}\widehat{w}_{n-8}\widehat{v}_{n-9}\widehat{u}_{n-10}\widehat{w}_{n-11}y_{n-12},\ n \geq 9, \tag{2.54}$$

$$z_n = \widehat{w}_n\widehat{v}_{n-1}\widehat{u}_{n-2}\widehat{w}_{n-3}\widehat{v}_{n-4}\widehat{u}_{n-5}\widehat{w}_{n-6}\widehat{v}_{n-7}\widehat{u}_{n-8}\widehat{w}_{n-9}\widehat{v}_{n-10}\widehat{u}_{n-11}z_{n-12},\ n \geq 9. \tag{2.55}$$

From (2.53)-(2.55) we have

$$\begin{aligned} x_{12m+4j+i} = x_{4j+i}\prod_{k=1}^{m} & \widehat{u}_{12k+4j+i}\widehat{w}_{12k+4j+i-1}\widehat{v}_{12k+4j+i-2}\widehat{u}_{12k+4j+i-3} \\ & \times\widehat{w}_{12k+4j+i-4}\widehat{v}_{12k+4j+i-5}\widehat{u}_{12k+4j+i-6}\widehat{w}_{12k+4j+i-7} \\ & \times\widehat{v}_{12k+4j+i-8}\widehat{u}_{12k+4j+i-9}\widehat{w}_{12k+4j+i-10}\widehat{v}_{12k+4j+i-11}, \end{aligned} \tag{2.56}$$

$$\begin{aligned} y_{12m+4j+i} = y_{4j+i}\prod_{k=1}^{m} & \widehat{v}_{12k+4j+i}\widehat{u}_{12k+4j+i-1}\widehat{w}_{12k+4j+i-2}\widehat{v}_{12k+4j+i-3} \\ & \times\widehat{u}_{12k+4j+i-4}\widehat{w}_{12k+4j+i-5}\widehat{v}_{12k+4j+i-6}\widehat{u}_{12k+4j+i-7} \\ & \times\widehat{w}_{12k+4j+i-8}\widehat{v}_{12k+4j+i-9}\widehat{u}_{12k+4j+i-10}\widehat{w}_{12k+4j+i-11}, \end{aligned} \tag{2.57}$$

$$\begin{aligned} z_{12m+4j+i} = z_{4j+i}\prod_{k=1}^{m} & \widehat{w}_{12k+4j+i}\widehat{v}_{12k+4j+i-1}\widehat{u}_{12k+4j+i-2}\widehat{w}_{12k+4j+i-3} \\ & \times\widehat{v}_{12k+4j+i-4}\widehat{u}_{12k+4j+i-5}\widehat{w}_{12k+4j+i-6}\widehat{v}_{12k+4j+i-7} \\ & \times\widehat{u}_{12k+4j+i-8}\widehat{w}_{12k+4j+i-9}\widehat{v}_{12k+4j+i-10}\widehat{u}_{12k+4j+i-11}, \end{aligned} \tag{2.58}$$

where $m \in \mathbb{N}_0$, $j = \overline{-1,1}$ and $i = \overline{1,4}$.
By applying (2.49), (2.51), (2.52) in (2.56)-(2.58), after some basic calculation, the solutions in explicit form of system (2.45) can be found.

2.1.7 Case $c = 0$, $abdef \neq 0$.

In this case, system (1.21) is equivalent to the system

$$x_n = \frac{z_{n-1}z_{n-3}}{bx_{n-2} + az_{n-3}},\ y_n = \frac{x_{n-1}x_{n-3}}{dy_{n-2}},\ z_n = \frac{y_{n-1}y_{n-3}}{fz_{n-2} + ey_{n-3}},\ n \in \mathbb{N}_0, \tag{2.59}$$

which is an analogue of the system (2.45). By interchanging variables y_n, z_n, x_n, d, e, f, a, b, instead of x_n, y_n, z_n, b, c, d, e, f, respectively, the system in (2.45) is transformed into (2.59). So, by interchanging $\widehat{v}_n$ instead of $\widehat{u}_n$ the formulas in (2.49), it is transformed into the following formula

$$\widehat{v}_{4n+\widetilde{i}} = \widehat{v}_{\widetilde{i}}, \tag{2.60}$$

where $n \in \mathbb{N}_0$ and $\widetilde{i} = \overline{-2,1}$. From (2.35), (2.52) and (2.60) we have

$$\begin{aligned} x_{12m+4j+i} = x_{4j+i}\prod_{k=1}^{m} & \widehat{s}_{12k+4j+i}\widehat{w}_{12k+4j+i-1}\widehat{v}_{12k+4j+i-2}\widehat{s}_{12k+4j+i-3} \\ & \times\widehat{w}_{12k+4j+i-4}\widehat{v}_{12k+4j+i-5}\widehat{s}_{12k+4j+i-6}\widehat{w}_{12k+4j+i-7} \\ & \times\widehat{v}_{12k+4j+i-8}\widehat{s}_{12k+4j+i-9}\widehat{w}_{12k+4j+i-10}\widehat{v}_{12k+4j+i-11}, \end{aligned} \tag{2.61}$$

$$\begin{aligned} y_{12m+4j+i} = y_{4j+i}\prod_{k=1}^{m} & \widehat{v}_{12k+4j+i}\widehat{s}_{12k+4j+i-1}\widehat{w}_{12k+4j+i-2}\widehat{v}_{12k+4j+i-3} \\ & \times\widehat{s}_{12k+4j+i-4}\widehat{w}_{12k+4j+i-5}\widehat{v}_{12k+4j+i-6}\widehat{s}_{12k+4j+i-7} \\ & \times\widehat{w}_{12k+4j+i-8}\widehat{v}_{12k+4j+i-9}\widehat{s}_{12k+4j+i-10}\widehat{w}_{12k+4j+i-11}, \end{aligned} \tag{2.62}$$

$$z_{12m+4j+i} = z_{4j+i}\prod_{k=1}^{m}\widehat{w}_{12k+4j+i}\widehat{v}_{12k+4j+i-1}\widehat{s}_{12k+4j+i-2}\widehat{w}_{12k+4j+i-3}\\ \times\widehat{v}_{12k+4j+i-4}\widehat{s}_{12k+4j+i-5}\widehat{w}_{12k+4j+i-6}\widehat{v}_{12k+4j+i-7}\\ \times\widehat{s}_{12k+4j+i-8}\widehat{w}_{12k+4j+i-9}\widehat{v}_{12k+4j+i-10}\widehat{s}_{12k+4j+i-11}, \tag{2.63}$$

where $m \in \mathbb{N}_0$, $j = \overline{-1,1}$ and $i = \overline{1,4}$.
By applying (2.35), (2.52), (2.60), in (2.61)-(2.63) after some basic calculation, the solutions in explicit form of system (2.59) can be found.

2.1.8 Case $e = 0$, $abcdf \neq 0$.

In this case, system (1.21) can be written in the form

$$x_n = \frac{z_{n-1}z_{n-3}}{bx_{n-2} + az_{n-3}},\ y_n = \frac{x_{n-1}x_{n-3}}{dy_{n-2} + cx_{n-3}},\ z_n = \frac{y_{n-1}y_{n-3}}{fz_{n-2}},\ n \in \mathbb{N}_0, \tag{2.64}$$

which is an analogue of the system (2.59). By interchanging variables y_n, z_n, x_n, c, d, f, a, b, instead of x_n, y_n, z_n, a, b, d, e, f, respectively, the system in (2.59) is transformed into (2.64). So, by interchanging $\widehat{w}_n$ instead of $\widehat{v}_n$ the formulas in (2.60), it is transformed into the following formula

$$\widehat{w}_{4n+\widetilde{i}} = \widehat{w}_{\widetilde{i}}, \tag{2.65}$$

where $n \in \mathbb{N}_0$ and $\widetilde{i} = \overline{-2,1}$. From (2.35), (2.51) and (2.65) we have

$$x_{12m+4j+i} = x_{4j+i}\prod_{k=1}^{m}\widehat{s}_{12k+4j+i}\widehat{w}_{12k+4j+i-1}\widehat{v}_{12k+4j+i-2}\widehat{s}_{12k+4j+i-3}\\ \times\widehat{w}_{12k+4j+i-4}\widehat{v}_{12k+4j+i-5}\widehat{s}_{12k+4j+i-6}\widehat{w}_{12k+4j+i-7}\\ \times\widehat{v}_{12k+4j+i-8}\widehat{s}_{12k+4j+i-9}\widehat{w}_{12k+4j+i-10}\widehat{v}_{12k+4j+i-11}, \tag{2.66}$$

$$y_{12m+4j+i} = y_{4j+i}\prod_{k=1}^{m}\widehat{v}_{12k+4j+i}\widehat{s}_{12k+4j+i-1}\widehat{w}_{12k+4j+i-2}\widehat{v}_{12k+4j+i-3}\\ \times\widehat{s}_{12k+4j+i-4}\widehat{w}_{12k+4j+i-5}\widehat{v}_{12k+4j+i-6}\widehat{s}_{12k+4j+i-7}\\ \times\widehat{w}_{12k+4j+i-8}\widehat{v}_{12k+4j+i-9}\widehat{s}_{12k+4j+i-10}\widehat{w}_{12k+4j+i-11}, \tag{2.67}$$

$$z_{12m+4j+i} = z_{4j+i}\prod_{k=1}^{m}\widehat{w}_{12k+4j+i}\widehat{v}_{12k+4j+i-1}\widehat{s}_{12k+4j+i-2}\widehat{w}_{12k+4j+i-3}\\ \times\widehat{v}_{12k+4j+i-4}\widehat{s}_{12k+4j+i-5}\widehat{w}_{12k+4j+i-6}\widehat{v}_{12k+4j+i-7}\\ \times\widehat{s}_{12k+4j+i-8}\widehat{w}_{12k+4j+i-9}\widehat{v}_{12k+4j+i-10}\widehat{s}_{12k+4j+i-11}, \tag{2.68}$$

where $m \in \mathbb{N}_0$, $j = \overline{-1,1}$ and $i = \overline{1,4}$.
By applying (2.35), (2.51), (2.65), in (2.66)-(2.68) after some basic calculation, the solutions in explicit form of system (2.64) can be found.

2.1.9 Case $a = c = 0$, $bdef \neq 0$.

In this case, system (1.21) reduces to the following system

$$x_n = \frac{z_{n-1}z_{n-3}}{bx_{n-2}},\ y_n = \frac{x_{n-1}x_{n-3}}{dy_{n-2}},\ z_n = \frac{y_{n-1}y_{n-3}}{fz_{n-2} + ey_{n-3}},\ n \in \mathbb{N}_0. \tag{2.69}$$

System (2.69) can be written in the form

$$\frac{x_n}{z_{n-1}} = \frac{1}{b\frac{x_{n-2}}{z_{n-3}}},\ \frac{y_n}{x_{n-1}} = \frac{1}{d\frac{y_{n-2}}{x_{n-3}}},\ \frac{z_n}{y_{n-1}} = \frac{1}{f\frac{z_{n-2}}{y_{n-3}} + e},\ n \in \mathbb{N}_0.$$

Next, by employing the change of variables

$$\widetilde{u}_n = \frac{x_n}{z_{n-1}},\ \widetilde{v}_n = \frac{y_n}{x_{n-1}},\ \widetilde{w}_n = \frac{z_n}{y_{n-1}},\ n \geq -2, \tag{2.70}$$

and transform (2.70) into the following equations

$$\widetilde{u}_n = \frac{1}{b\widetilde{u}_{n-2}} = \widetilde{u}_{n-4},\ n \geq 2,\ \widetilde{v}_n = \frac{1}{d\widetilde{v}_{n-2}} = \widetilde{v}_{n-4},\ n \geq 2,\ \widetilde{w}_n = \frac{1}{f\widetilde{w}_{n-2}+e},\ n \in \mathbb{N}_0, \tag{2.71}$$

which means that $(\widetilde{u}_n)_{n\geq -2}$, $(\widetilde{v}_n)_{n\geq -2}$ are four-periodic, that is,

$$\widetilde{u}_{4n+\widetilde{i}} = \widetilde{u}_{\widetilde{i}},\ \widetilde{v}_{4n+\widetilde{i}} = \widetilde{v}_{\widetilde{i}}, \tag{2.72}$$

where $n \in \mathbb{N}_0$ and $\widetilde{i} = \overline{-2,1}$ and the third equation in (2.71) is equation with interlacing indices of order two. Hence, the sequence

$$\widetilde{w}_m^{(l)} = \frac{1}{f\widetilde{w}_{m-1}^{(l)}+e},\ m \in \mathbb{N}_0, \tag{2.73}$$

for $l \in \{0,1\}$. From equalities in (2.27), the solution of difference equation in (2.73)

$$\widetilde{w}_{2m+l} = \begin{cases} \frac{e}{f}\frac{\left(\lambda_3\frac{f\widetilde{w}_{l-2}+e}{e}-R_2\right)\lambda_3^{m+1}-\left(\lambda_4\frac{f\widetilde{w}_{l-2}+e}{e}-R_2\right)\lambda_4^{m+1}}{\left(\lambda_3\frac{f\widetilde{w}_{l-2}+e}{e}-R_2\right)\lambda_3^{m}-\left(\lambda_4\frac{f\widetilde{w}_{l-2}+e}{e}-R_2\right)\lambda_4^{m}} - \frac{e}{f}, & \text{if } R_2 \neq \frac{1}{4}, \\ \frac{e}{f}\frac{2\frac{f\widetilde{w}_{l-2}+e}{e}+\left(2\frac{f\widetilde{w}_{l-2}+e}{e}-1\right)(m+1)}{4\frac{f\widetilde{w}_{l-2}+e}{e}+\left(4\frac{f\widetilde{w}_{l-2}+e}{e}-2\right)m} - \frac{e}{f}, & \text{if } R_2 = \frac{1}{4}, \end{cases} \quad m \in \mathbb{N}_0, \tag{2.74}$$

for $l \in \{0,1\}$, where $\lambda_3 = \frac{1+\sqrt{1-4R_2}}{2}$, $\lambda_4 = \frac{1-\sqrt{1-4R_2}}{2}$ and $R_2 = \frac{-f}{e^2}$.
From (2.70), we get

$$x_n = \widetilde{u}_n\widetilde{w}_{n-1}\widetilde{v}_{n-2}\widetilde{u}_{n-3}\widetilde{w}_{n-4}\widetilde{v}_{n-5}\widetilde{u}_{n-6}\widetilde{w}_{n-7}\widetilde{v}_{n-8}\widetilde{u}_{n-9}\widetilde{w}_{n-10}\widetilde{v}_{n-11}x_{n-12},\ n \geq 9, \tag{2.75}$$

$$y_n = \widetilde{v}_n\widetilde{u}_{n-1}\widetilde{w}_{n-2}\widetilde{v}_{n-3}\widetilde{u}_{n-4}\widetilde{w}_{n-5}\widetilde{v}_{n-6}\widetilde{u}_{n-7}\widetilde{w}_{n-8}\widetilde{v}_{n-9}\widetilde{u}_{n-10}\widetilde{w}_{n-11}y_{n-12},\ n \geq 9, \tag{2.76}$$

$$z_n = \widetilde{w}_n\widetilde{v}_{n-1}\widetilde{u}_{n-2}\widetilde{w}_{n-3}\widetilde{v}_{n-4}\widetilde{u}_{n-5}\widetilde{w}_{n-6}\widetilde{v}_{n-7}\widetilde{u}_{n-8}\widetilde{w}_{n-9}\widetilde{v}_{n-10}\widetilde{u}_{n-11}z_{n-12},\ n \geq 9. \tag{2.77}$$

From (2.75)-(2.77) we have

$$\begin{aligned} x_{12m+4j+i} = x_{4j+i-12}\prod_{k=0}^{m} & \widetilde{u}_{12k+4j+i}\widetilde{w}_{12k+4j+i-1}\widetilde{v}_{12k+4j+i-2}\widetilde{u}_{12k+4j+i-3} \\ & \times\widetilde{w}_{12k+4j+i-4}\widetilde{v}_{12k+4j+i-5}\widetilde{u}_{12k+4j+i-6}\widetilde{w}_{12k+4j+i-7} \\ & \times\widetilde{v}_{12k+4j+i-8}\widetilde{u}_{12k+4j+i-9}\widetilde{w}_{12k+4j+i-10}\widetilde{v}_{12k+4j+i-11}, \end{aligned} \tag{2.78}$$

$$\begin{aligned} y_{12m+4j+i} = y_{4j+i-12}\prod_{k=0}^{m} & \widetilde{v}_{12k+4j+i}\widetilde{u}_{12k+4j+i-1}\widetilde{w}_{12k+4j+i-2}\widetilde{v}_{12k+4j+i-3} \\ & \times\widetilde{u}_{12k+4j+i-4}\widetilde{w}_{12k+4j+i-5}\widetilde{v}_{12k+4j+i-6}\widetilde{u}_{12k+4j+i-7} \\ & \times\widetilde{w}_{12k+4j+i-8}\widetilde{v}_{12k+4j+i-9}\widetilde{u}_{12k+4j+i-10}\widetilde{w}_{12k+4j+i-11}, \end{aligned} \tag{2.79}$$

$$\begin{aligned} z_{12m+4j+i} = z_{4j+i-12}\prod_{k=0}^{m} & \widetilde{w}_{12k+4j+i}\widetilde{v}_{12k+4j+i-1}\widetilde{u}_{12k+4j+i-2}\widetilde{w}_{12k+4j+i-3} \\ & \times\widetilde{v}_{12k+4j+i-4}\widetilde{u}_{12k+4j+i-5}\widetilde{w}_{12k+4j+i-6}\widetilde{v}_{12k+4j+i-7} \\ & \times\widetilde{u}_{12k+4j+i-8}\widetilde{w}_{12k+4j+i-9}\widetilde{v}_{12k+4j+i-10}\widetilde{u}_{12k+4j+i-11}, \end{aligned} \tag{2.80}$$

where $m \in \mathbb{N}_0$, $j \in \{2,3,4\}$ and $i = \overline{1,4}$.
By applying (2.72), (2.74), in (2.78)-(2.80) after some basic calculation, the solutions in explicit form of system (2.69) can be found.

2.1.10 Case $c = e = 0$, $abdf \neq 0$.

In this case, system (1.21) is expressed as

$$x_n = \frac{z_{n-1}z_{n-3}}{bx_{n-2} + az_{n-3}},\ y_n = \frac{x_{n-1}x_{n-3}}{dy_{n-2}},\ z_n = \frac{y_{n-1}y_{n-3}}{fz_{n-2}},\ n \in \mathbb{N}_0, \tag{2.81}$$

which is an analogue of the system (2.69). By interchanging variables y_n, z_n, x_n, d, f, a, b, instead of x_n, y_n, z_n, b, d, e, f, respectively, the system in (2.69) is transformed into (2.81). So, by interchanging $\widetilde{v}_n$, $\widetilde{w}_n$, $\widetilde{u}_n$ instead of $\widetilde{u}_n$, $\widetilde{v}_n$, $\widetilde{w}_n$ the formulas in (2.72),(2.74), they are transformed into the following formulas

$$\widetilde{v}_{4n+\widetilde{i}} = \widetilde{v}_{\widetilde{i}},\ \widetilde{w}_{4n+\widetilde{i}} = \widetilde{w}_{\widetilde{i}}, \tag{2.82}$$

where $n \in \mathbb{N}_0$ and $\widetilde{i} = \overline{-2,1}$ and

$$\widetilde{u}_{2m+l} = \begin{cases} \frac{a}{b}\,\frac{\left(\lambda_5\frac{b\widetilde{u}_{l-2}+a}{a}-R_3\right)\lambda_5^{m+1}-\left(\lambda_6\frac{b\widetilde{u}_{l-2}+a}{a}-R_3\right)\lambda_6^{m+1}}{\left(\lambda_5\frac{b\widetilde{u}_{l-2}+a}{a}-R_3\right)\lambda_5^{m}-\left(\lambda_6\frac{b\widetilde{u}_{l-2}+a}{a}-R_3\right)\lambda_6^{m}} - \frac{a}{b}, & \text{if } R_3 \neq \frac{1}{4}, \\ \frac{a}{b}\,\frac{2\frac{b\widetilde{u}_{l-2}+a}{a}+\left(2\frac{b\widetilde{u}_{l-2}+a}{a}-1\right)(m+1)}{4\frac{b\widetilde{u}_{l-2}+a}{a}+\left(4\frac{b\widetilde{u}_{l-2}+a}{a}-2\right)m} - \frac{a}{b}, & \text{if } R_3 = \frac{1}{4}, \end{cases} \quad m \in \mathbb{N}_0, \tag{2.83}$$

for $l \in \{0, 1\}$, where $\lambda_5 = \frac{1+\sqrt{1-4R_3}}{2}$, $\lambda_6 = \frac{1-\sqrt{1-4R_3}}{2}$ and $R_3 = \frac{-b}{a^2}$.
Putting (2.82), (2.83), in (2.78)-(2.80) after some basic calculation, we get solutions of system (2.81).

2.1.11 Case $a = e = 0$, $bcdf \neq 0$.

In this case, system (1.21) can be written in the form

$$x_n = \frac{z_{n-1}z_{n-3}}{bx_{n-2}},\ y_n = \frac{x_{n-1}x_{n-3}}{dy_{n-2} + cz_{n-3}},\ z_n = \frac{y_{n-1}y_{n-3}}{fz_{n-2}},\ n \in \mathbb{N}_0, \tag{2.84}$$

which is an analogue of the system (2.81). By interchanging variables y_n, z_n, x_n, c, d, f, b, instead of x_n, y_n, z_n, a, b, d, f, respectively, system in (2.81) is transformed into (2.84). So, by interchanging $\widetilde{v}_n$, $\widetilde{w}_n$, $\widetilde{u}_n$ instead of $\widetilde{u}_n$, $\widetilde{v}_n$, $\widetilde{w}_n$ the formulas in (2.82)-(2.83), they are transformed into the following formulas

$$\widetilde{w}_{4n+\widetilde{i}} = \widetilde{w}_{\widetilde{i}},\ \widetilde{u}_{4n+\widetilde{i}} = \widetilde{u}_{\widetilde{i}}, \tag{2.85}$$

where $n \in \mathbb{N}_0$ and $\widetilde{i} = \overline{-2,1}$ and

$$\widetilde{v}_{2m+l} = \begin{cases} \frac{c}{d}\,\frac{\left(\lambda_1\frac{d\widetilde{v}_{l-2}+c}{c}-R_1\right)\lambda_1^{m+1}-\left(\lambda_2\frac{d\widetilde{v}_{l-2}+c}{c}-R_1\right)\lambda_2^{m+1}}{\left(\lambda_1\frac{d\widetilde{v}_{l-2}+c}{c}-R_1\right)\lambda_1^{m}-\left(\lambda_2\frac{d\widetilde{v}_{l-2}+c}{c}-R_1\right)\lambda_2^{m}} - \frac{c}{d}, & \text{if } R_1 \neq \frac{1}{4}, \\ \frac{c}{d}\,\frac{2\frac{d\widetilde{v}_{j-2}+c}{c}+\left(2\frac{d\widetilde{v}_{j-2}+c}{c}-1\right)(m+1)}{4\frac{d\widetilde{v}_{j-2}+c}{c}+\left(4\frac{d\widetilde{v}_{j-2}+c}{c}-2\right)m} - \frac{c}{d}, & \text{if } R_1 = \frac{1}{4}, \end{cases} \quad m \in \mathbb{N}_0, \tag{2.86}$$

for $l \in \{0, 1\}$, where $\lambda_1 = \frac{1+\sqrt{1-4R_1}}{2}$, $\lambda_2 = \frac{1-\sqrt{1-4R_1}}{2}$ and $R_1 = \frac{-d}{c^2}$.
Putting (2.85), (2.86), in (2.78)-(2.80) after some basic calculation, we get solutions of system (2.84).

2.1.12 Case $b = d = 0$, $acef \neq 0$.

In this case, system (1.21) becomes

$$x_n = \frac{z_{n-1}}{a},\ y_n = \frac{x_{n-1}}{c},\ z_n = \frac{y_{n-1}y_{n-3}}{fz_{n-2} + ey_{n-3}},\ n \in \mathbb{N}_0. \tag{2.87}$$

Employing the first equation in system (2.87) in the second equation in system (2.87), the second equation in system (2.87) in the third equation in system (2.87), we get

$$x_n = \frac{z_{n-1}}{a},\ n \in \mathbb{N}_0,\ y_n = \frac{z_{n-2}}{ac},\ n \geq 1,\ z_n = \frac{z_{n-3}z_{n-5}}{a^2c^2fz_{n-2} + acez_{n-5}},\ n \geq 4. \tag{2.88}$$

We consider the following equation

$$z_n = \frac{z_{n-3}z_{n-5}}{a^2c^2fz_{n-2} + acez_{n-5}},\ n \geq 4. \tag{2.89}$$

The equation (2.89) can be written in the form

$$\frac{z_n}{z_{n-3}} = \frac{1}{a^2c^2f\frac{z_{n-2}}{z_{n-5}} + ace},\ n \geq 4. \tag{2.90}$$

Now, we may use the change of variables

$$\frac{z_n}{z_{n-3}} = \widehat{z}_n,\ n \geq 2, \tag{2.91}$$

and transform equation (2.90) into the following equation

$$\widehat{z}_n = \frac{1}{a^2c^2f\widehat{z}_{n-2} + ace},\ n \geq 4. \tag{2.92}$$

Equation (2.92) is equation with interlacing indices of order two. Hence, for $m \in \mathbb{N}_0$, the sequence

$$\widehat{z}_m^{(i)} = \widehat{z}_{2m+i},\ i \in \{2,3\},$$

is solution to the following difference equation

$$\widehat{z}_m^{(i)} = \frac{1}{a^2c^2f\widehat{z}_{m-1}^{(i)} + ace}, \tag{2.93}$$

for $m \in \mathbb{N}$, $i \in \{2,3\}$. From equalities in (1.17), the solution of difference equation in (2.93)

$$\widehat{z}_{2m+i} = \begin{cases} \frac{e}{acf}\frac{\left(\lambda_3\frac{acf\widehat{z}_i+e}{e}-R_2\right)\lambda_3^m-\left(\lambda_4\frac{acf\widehat{z}_i+e}{e}-R_2\right)\lambda_4^m}{\left(\lambda_3\frac{acf\widehat{z}_i+e}{e}-R_2\right)\lambda_3^{m-1}-\left(\lambda_4\frac{acf\widehat{z}_i+e}{e}-R_2\right)\lambda_4^{m-1}} - \frac{e}{acf}, & \text{if } R_2 \neq \frac{1}{4}, \\ \frac{e}{acf}\frac{1+\left(2\frac{acf\widehat{z}_i+e}{e}-1\right)(m+1)}{2+\left(4\frac{acf\widehat{z}_i+e}{e}-2\right)m} - \frac{e}{acf}, & \text{if } R_2 = \frac{1}{4}, \end{cases} \quad m \in \mathbb{N}_0, \tag{2.94}$$

for $i \in \{2,3\}$, where $\lambda_3 = \frac{1+\sqrt{1-4R_2}}{2}$, $\lambda_4 = \frac{1-\sqrt{1-4R_2}}{2}$ and $R_2 = \frac{-f}{e^2}$. From (2.91), we have that

$$z_n = \widehat{z}_n\widehat{z}_{n-3}z_{n-6},\ n \geq 5. \tag{2.95}$$

From the first equation in system (2.88), the second equation in system (2.88), (2.94) and(2.95), we have

$$x_{6m+2j+\widetilde{i}+1} = \frac{1}{a}z_{2j+\widetilde{i}-6}\prod_{k=0}^{m}\widehat{z}_{6k+2j+\widetilde{i}}\widehat{z}_{6k+2j+\widetilde{i}-3}, \tag{2.96}$$

$$y_{6m+2j+\widetilde{i}+2} = \frac{1}{ac}z_{2j+\widetilde{i}-6}\prod_{k=0}^{m}\widehat{z}_{6k+2j+\widetilde{i}}\widehat{z}_{6k+2j+\widetilde{i}-3}, \tag{2.97}$$

$$z_{6m+2j+\widetilde{i}} = z_{2j+\widetilde{i}-6}\prod_{k=0}^{m}\widehat{z}_{6k+2j+\widetilde{i}}\widehat{z}_{6k+2j+\widetilde{i}-3}, \tag{2.98}$$

where $m \in \mathbb{N}_0$, $j \in \{1,2,3\}$ and $\widetilde{i} \in \{3,4\}$.
By applying (2.94) in (2.96)-(2.98), after some basic calculation, the solutions in explicit form of system (2.87) can be found.

2.1.13 Case $d = f = 0$, $abce \neq 0$.

In this case, system (1.21) reduces to the following system

$$x_n = \frac{z_{n-1}z_{n-3}}{bx_{n-2} + az_{n-3}},\ y_n = \frac{x_{n-1}}{c},\ z_n = \frac{y_{n-1}}{e},\ n \in \mathbb{N}_0. \tag{2.99}$$

which is an analogue of system (2.87). By interchanging variables y_n, z_n, x_n, c, e, a, b, instead of x_n, y_n, z_n, a, c, e, f, respectively, system in (2.87) is transformed into (2.99). So, by interchanging $\widetilde{x}_{2m+i}$ instead of $\widetilde{z}_{2m+i}$ the formulas in (2.91) and (2.94), they are transformed into the following formulas

$$\frac{x_n}{x_{n-3}} = \widehat{x}_n,\ n \geq 2, \tag{2.100}$$

and

$$\widehat{x}_{2m+i} = \begin{cases} \frac{a}{ceb}\frac{\left(\lambda_5\frac{ceb\widehat{x}_i+a}{a}-R_3\right)\lambda_5^m-\left(\lambda_6\frac{ceb\widehat{x}_i+a}{a}-R_3\right)\lambda_6^m}{\left(\lambda_5\frac{ceb\widehat{x}_i+a}{a}-R_3\right)\lambda_5^{m-1}\left(\lambda_6\frac{ceb\widehat{x}_i+a}{a}-R_3\right)\lambda_6^{m-1}} - \frac{a}{ceb}, & \text{if } R_3 \neq \frac{1}{4}, \\ \frac{a}{ceb}\frac{1+\left(2\frac{ceb\widehat{x}_i+a}{a}-1\right)(m+1)}{2+\left(4\frac{ceb\widehat{x}_i+a}{a}-2\right)m} - \frac{a}{ceb}, & \text{if } R_3 = \frac{1}{4}, \end{cases} \quad m \in \mathbb{N}_0, \tag{2.101}$$

for $i \in \{2,3\}$, where $\lambda_5 = \frac{1+\sqrt{1-4R_3}}{2}$, $\lambda_6 = \frac{1-\sqrt{1-4R_3}}{2}$ and $R_3 = \frac{-b}{a^2}$. From (2.100), we have that

$$x_n = \widehat{x}_n\widehat{x}_{n-3}x_{n-6}, \ n \geq 5. \tag{2.102}$$

From the second equation in system (2.99), the third equation in system (2.99), (2.101) and(2.102), we have

$$x_{6m+2j+\widetilde{i}} = x_{2j+\widetilde{i}-6}\prod_{k=0}^{m}\widehat{x}_{6k+2j+\widetilde{i}}\widehat{x}_{6k+2j+\widetilde{i}-3}, \tag{2.103}$$

$$y_{6m+2j+\widetilde{i}+1} = \frac{1}{c}x_{2j+\widetilde{i}-6}\prod_{k=0}^{m}\widehat{x}_{6k+2j+\widetilde{i}}\widehat{x}_{6k+2j+\widetilde{i}-3}, \tag{2.104}$$

$$z_{6m+2j+\widetilde{i}+2} = \frac{1}{ce}x_{2j+\widetilde{i}-6}\prod_{k=0}^{m}\widehat{x}_{6k+2j+\widetilde{i}}\widehat{x}_{6k+2j+\widetilde{i}-3}, \tag{2.105}$$

where $m \in \mathbb{N}_0$, $j \in \{1,2,3\}$ and $\widetilde{i} \in \{3,4\}$.
Putting (2.101) in (2.103)-(2.105) after some basic calculation, we get solutions of system (2.99).

2.1.14 Case $b = f = 0$, $acde \neq 0$.

In this case, we obtain the following system

$$x_n = \frac{z_{n-1}}{a}, \ y_n = \frac{x_{n-1}x_{n-3}}{dy_{n-2}+cx_{n-3}}, \ z_n = \frac{y_{n-1}}{e}, \ n \in \mathbb{N}_0. \tag{2.106}$$

which is an analogue of system (2.99). By interchanging variables y_n, z_n, x_n, c, d, e, a, instead of x_n, y_n, z_n, a, b, c, e, respectively, system in (2.99) is transformed into (2.106). So, by interchanging $\widetilde{y}_{2m+i}$ instead of $\widetilde{x}_{2m+i}$ the formulas in (2.100) and (2.101), they are transformed into the following formulas

$$\frac{y_n}{y_{n-3}} = \widehat{y}_n, \ n \geq 2, \tag{2.107}$$

and

$$\widehat{y}_{2m+i} = \begin{cases} \frac{c}{ade}\frac{\left(\lambda_1\frac{ade\widehat{y}_i+c}{c}-R_1\right)\lambda_1^m-\left(\lambda_2\frac{ade\widehat{y}_i+c}{c}-R_1\right)\lambda_2^m}{\left(\lambda_1\frac{ade\widehat{y}_i+c}{c}-R_1\right)\lambda_1^{m-1}\left(\lambda_2\frac{ade\widehat{y}_i+c}{c}-R_1\right)\lambda_2^{m-1}} - \frac{c}{ade}, & \text{if } R_1 \neq \frac{1}{4}, \\ \frac{c}{ade}\frac{1+\left(2\frac{ade\widehat{y}_i+c}{c}-1\right)(m+1)}{2+\left(4\frac{ade\widehat{y}_i+c}{c}-2\right)m} - \frac{c}{ade}, & \text{if } R_1 = \frac{1}{4}, \end{cases} \quad m \in \mathbb{N}_0, \tag{2.108}$$

for $i \in \{2,3\}$, where $\lambda_1 = \frac{1+\sqrt{1-4R_1}}{2}$, $\lambda_2 = \frac{1-\sqrt{1-4R_1}}{2}$ and $R_1 = \frac{-d}{c^2}$. From (2.107), we have that

$$y_n = \widehat{y}_n\widehat{y}_{n-3}y_{n-6}, \ n \geq 5. \tag{2.109}$$

From the first equation in system (2.106), the third equation in system (2.106), (2.108) and(2.109), we have

$$x_{6m+2j+\widetilde{i}+2} = \frac{1}{ae}y_{2j+\widetilde{i}-6}\prod_{k=0}^{m}\widehat{y}_{6k+2j+\widetilde{i}}\widehat{y}_{6k+2j+\widetilde{i}-3}, \tag{2.110}$$

$$y_{6m+2j+\widetilde{i}} = y_{2j+\widetilde{i}-6}\prod_{k=0}^{m}\widehat{y}_{6k+2j+\widetilde{i}}\widehat{y}_{6k+2j+\widetilde{i}-3}, \tag{2.111}$$

$$z_{6m+2j+\widetilde{i}+1} = \frac{1}{e}y_{2j+\widetilde{i}-6}\prod_{k=0}^{m}\widehat{y}_{6k+2j+\widetilde{i}}\widehat{y}_{6k+2j+\widetilde{i}-3}, \tag{2.112}$$

where $m \in \mathbb{N}_0$, $j \in \{1,2,3\}$ and $\widetilde{i} \in \{3,4\}$.
Putting (2.108) in (2.110)-(2.112) after some basic calculation, we get solutions of system (2.106).

2.1.15 Case $abcdef \neq 0$.

In this case, system (1.21) can be written in the form

$$\frac{z_{n-1}}{x_n} = \frac{bx_{n-2} + az_{n-3}}{z_{n-3}}, \ \frac{x_{n-1}}{y_n} = \frac{dy_{n-2} + cx_{n-3}}{x_{n-3}}, \ \frac{y_{n-1}}{z_n} = \frac{fz_{n-2} + ey_{n-3}}{y_{n-3}}, \tag{2.113}$$

for $n \in \mathbb{N}_0$. Next, by employing the change of variables

$$u_n = \frac{z_{n-1}}{x_n}, \ v_n = \frac{x_{n-1}}{y_n}, \ w_n = \frac{y_{n-1}}{z_n}, \ n \geq -2, \tag{2.114}$$

system (2.113) is transformed into the following system

$$u_n = \frac{au_{n-2} + b}{u_{n-2}}, \ v_n = \frac{cv_{n-2} + d}{v_{n-2}}, \ w_n = \frac{ew_{n-2} + f}{w_{n-2}}, \ n \in \mathbb{N}_0. \tag{2.115}$$

Equations in (2.115) are equations with interlacing indices of order two. Hence, the sequences

$$u_m^{(i)} = u_{2m+i}, \ v_m^{(i)} = v_{2m+i}, \ w_m^{(i)} = w_{2m+i}, \ m \geq -1, \ i \in \{0, 1\},$$

are solutions to the following difference equations

$$u_m^{(i)} = \frac{au_{m-1}^{(i)} + b}{u_{m-1}^{(i)}}, \ v_m^{(i)} = \frac{cv_{m-1}^{(i)} + d}{v_{m-1}^{(i)}}, \ w_m^{(i)} = \frac{ew_{m-1}^{(i)} + f}{w_{m-1}^{(i)}}, \tag{2.116}$$

for $m \in \mathbb{N}_0$, $i \in \{0, 1\}$. The solutions of difference equations in (2.116)

$$u_{m-1}^{(i)} = \begin{cases} \dfrac{\left(\widehat{\lambda}_2 - u_{-1}^{(i)}\right)\widehat{\lambda}_1^{m+1} + \left(u_{-1}^{(i)} - \widehat{\lambda}_1\right)\widehat{\lambda}_2^{m+1}}{\left(\widehat{\lambda}_2 - u_{-1}^{(i)}\right)\widehat{\lambda}_1^{m} + \left(u_{-1}^{(i)} - \widehat{\lambda}_1\right)\widehat{\lambda}_2^{m}}, & a^2 + 4b \neq 0, \\ \dfrac{\left(u_{-1}^{(i)}(m+1) - \widehat{\lambda}_1 m\right)\widehat{\lambda}_1^{m}}{\left(u_{-1}^{(i)} m - \widehat{\lambda}_1 (m-1)\right)\widehat{\lambda}_1^{m-1}}, & a^2 + 4b = 0, \end{cases} \quad m \in \mathbb{N}_0,$$

$$v_{m-1}^{(i)} = \begin{cases} \dfrac{\left(\widehat{\lambda}_4 - v_{-1}^{(i)}\right)\widehat{\lambda}_3^{m+1} + \left(v_{-1}^{(i)} - \widehat{\lambda}_3\right)\widehat{\lambda}_4^{m+1}}{\left(\widehat{\lambda}_4 - v_{-1}^{(i)}\right)\widehat{\lambda}_3^{m} + \left(v_{-1}^{(i)} - \widehat{\lambda}_3\right)\widehat{\lambda}_4^{m}}, & c^2 + 4d \neq 0, \\ \dfrac{\left(v_{-1}^{(i)}(m+1) - \widehat{\lambda}_3 m\right)\widehat{\lambda}_3^{m}}{\left(v_{-1}^{(i)} m - \widehat{\lambda}_3 (m-1)\right)\widehat{\lambda}_3^{m-1}}, & c^2 + 4d = 0, \end{cases} \quad m \in \mathbb{N}_0,$$

$$w_{m-1}^{(i)} = \begin{cases} \dfrac{\left(\widehat{\lambda}_6 - w_{-1}^{(i)}\right)\widehat{\lambda}_5^{m+1} + \left(w_{-1}^{(i)} - \widehat{\lambda}_5\right)\widehat{\lambda}_6^{m+1}}{\left(\widehat{\lambda}_6 - w_{-1}^{(i)}\right)\widehat{\lambda}_5^{m} + \left(w_{-1}^{(i)} - \widehat{\lambda}_5\right)\widehat{\lambda}_6^{m}}, & e^2 + 4f \neq 0, \\ \dfrac{\left(w_{-1}^{(i)}(m+1) - \widehat{\lambda}_5 m\right)\widehat{\lambda}_5^{m}}{\left(w_{-1}^{(i)} m - \widehat{\lambda}_5 (m-1)\right)\widehat{\lambda}_5^{m-1}}, & e^2 + 4f = 0, \end{cases} \quad m \in \mathbb{N}_0,$$

for $i \in \{0, 1\}$, where $\widehat{\lambda}_1 = \frac{a+\sqrt{a^2+4b}}{2}$, $\widehat{\lambda}_2 = \frac{a-\sqrt{a^2+4b}}{2}$, $\widehat{\lambda}_3 = \frac{c+\sqrt{c^2+4d}}{2}$, $\widehat{\lambda}_4 = \frac{c-\sqrt{c^2+4d}}{2}$, $\widehat{\lambda}_5 = \frac{e+\sqrt{e^2+4f}}{2}$ and $\widehat{\lambda}_6 = \frac{e-\sqrt{e^2+4f}}{2}$. and consequently

$$u_{2(m-1)+i} = \frac{\left(\widehat{\lambda}_2 - \frac{z_{i-3}}{x_{i-2}}\right)\widehat{\lambda}_1^{m+1} + \left(\frac{z_{i-3}}{x_{i-2}} - \widehat{\lambda}_1\right)\widehat{\lambda}_2^{m+1}}{\left(\widehat{\lambda}_2 - \frac{z_{i-3}}{x_{i-2}}\right)\widehat{\lambda}_1^{m} + \left(\frac{z_{i-3}}{x_{i-2}} - \widehat{\lambda}_1\right)\widehat{\lambda}_2^{m}}, \ m \in \mathbb{N}_0, \tag{2.117}$$

when $a^2 + 4b \neq 0$, for $i \in \{0, 1\}$,

$$v_{2(m-1)+i} = \frac{\left(\widehat{\lambda}_4 - \frac{x_{i-3}}{y_{i-2}}\right)\widehat{\lambda}_3^{m+1} + \left(\frac{x_{i-3}}{y_{i-2}} - \widehat{\lambda}_3\right)\widehat{\lambda}_4^{m+1}}{\left(\widehat{\lambda}_4 - \frac{x_{i-3}}{y_{i-2}}\right)\widehat{\lambda}_3^{m} + \left(\frac{x_{i-3}}{y_{i-2}} - \widehat{\lambda}_3\right)\widehat{\lambda}_4^{m}}, \ m \in \mathbb{N}_0, \tag{2.118}$$

when $c^2 + 4d \neq 0$, for $i \in \{0, 1\}$,

$$w_{2(m-1)+i} = \frac{\left(\widehat{\lambda}_6 - \frac{y_{i-3}}{z_{i-2}}\right)\widehat{\lambda}_5^{m+1} + \left(\frac{y_{i-3}}{z_{i-2}} - \widehat{\lambda}_5\right)\widehat{\lambda}_6^{m+1}}{\left(\widehat{\lambda}_6 - \frac{y_{i-3}}{z_{i-2}}\right)\widehat{\lambda}_5^{m} + \left(\frac{y_{i-3}}{z_{i-2}} - \widehat{\lambda}_5\right)\widehat{\lambda}_6^{m}}, \ m \in \mathbb{N}_0, \tag{2.119}$$

when $e^2+4f\neq 0$, for $i\in\{0,1\}$ and

$$u_{2(m-1)+i}=\frac{\left(\frac{z_{i-3}}{x_{i-2}}(m+1)-\widehat{\lambda}_1 m\right)\widehat{\lambda}_1^m}{\left(\frac{z_{i-3}}{x_{i-2}}m-\widehat{\lambda}_1(m-1)\right)\widehat{\lambda}_1^{m-1}},\ m\in\mathbb{N}_0, \tag{2.120}$$

when $a^2+4b=0$, for $i\in\{0,1\}$,

$$v_{2(m-1)+i}=\frac{\left(\frac{x_{i-3}}{y_{i-2}}(m+1)-\widehat{\lambda}_3 m\right)\widehat{\lambda}_3^m}{\left(\frac{x_{i-3}}{y_{i-2}}m-\widehat{\lambda}_3(m-1)\right)\widehat{\lambda}_3^{m-1}},\ m\in\mathbb{N}_0, \tag{2.121}$$

when $c^2+4d=0$, for $i\in\{0,1\}$,

$$w_{2(m-1)+i}=\frac{\left(\frac{y_{i-3}}{z_{i-2}}(m+1)-\widehat{\lambda}_5 m\right)\widehat{\lambda}_5^m}{\left(\frac{y_{i-3}}{z_{i-2}}m-\widehat{\lambda}_5(m-1)\right)\widehat{\lambda}_5^{m-1}},\ m\in\mathbb{N}_0, \tag{2.122}$$

when $e^2+4f=0$, for $i\in\{0,1\}$.
From (2.114), we have that

$$x_n=\frac{x_{n-6}}{u_n w_{n-1} v_{n-2} u_{n-3} w_{n-4} v_{n-5}},\ n\geq 3, \tag{2.123}$$

$$y_n=\frac{y_{n-6}}{v_n u_{n-1} w_{n-2} v_{n-3} u_{n-4} w_{n-5}},\ n\geq 3, \tag{2.124}$$

$$z_n=\frac{z_{n-6}}{w_n v_{n-1} u_{n-2} w_{n-3} v_{n-4} u_{n-5}},\ n\geq 3. \tag{2.125}$$

From (2.123)-(2.125), we have

$$x_{6m+l}=\frac{x_{6(m-1)+l}}{u_{6m+l}w_{6m+l-1}v_{6m+l-2}u_{6m+l-3}w_{6m+l-4}v_{6m+l-5}},$$

$$y_{6m+l}=\frac{y_{6(m-1)+l}}{v_{6m+l}u_{6m+l-1}w_{6m+l-2}v_{6m+l-3}u_{6m+l-4}w_{6m+l-5}},$$

and

$$z_{6m+l}=\frac{z_{6(m-1)+l}}{w_{6m+l}v_{6m+l-1}u_{6m+l-2}w_{6m+l-3}v_{6m+l-4}u_{6m+l-5}},$$

where $m\in\mathbb{N}$ and $l=\overline{-3,2}$, from which it follows that

$$x_{6m+2j+i-1}=\frac{x_{2j+i-1}}{\prod_{k=1}^m u_{6k+2j+i-1}w_{6k+2j+i-2}v_{6k+2j+i-3}u_{6k+2j+i-4}w_{6k+2j+i-5}v_{6k+2j+i-6}}, \tag{2.126}$$

$$y_{6m+2j+i-1}=\frac{y_{2j+i-1}}{\prod_{k=1}^m v_{6k+2j+i-1}u_{6k+2j+i-2}w_{6k+2j+i-3}v_{6k+2j+i-4}u_{6k+2j+i-5}w_{6k+2j+i-6}}, \tag{2.127}$$

$$z_{6m+2j+i-1}=\frac{z_{2j+i-1}}{\prod_{k=1}^m w_{6k+2j+i-1}v_{6k+2j+i-2}u_{6k+2j+i-3}w_{6k+2j+i-4}v_{6k+2j+i-5}u_{6k+2j+i-6}}, \tag{2.128}$$

where $m \in \mathbb{N}_0$, $j \in \{-1,0,1\}$ and $i \in \{0,1\}$.
By applying (2.117)-(2.119) in (2.126)-(2.128), after some basic calculation, we obtain

$$\begin{aligned} x_{6m+2j+i-1} &= x_{2j+i-1}\Bigg(\prod_{k=1}^{m} \frac{\left(\widehat{\lambda}_2-\frac{z_{i_2-3}}{x_{i_2-2}}\right)\widehat{\lambda}_1^{3k+j+i+1}+\left(\frac{z_{i_2-3}}{x_{i_2-2}}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{3k+j+i+1}}{\left(\widehat{\lambda}_2-\frac{z_{i_2-3}}{x_{i_2-2}}\right)\widehat{\lambda}_1^{3k+j+i}+\left(\frac{z_{i_2-3}}{x_{i_2-2}}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{3k+j+i}} \\ &\times \frac{\left(\widehat{\lambda}_6-\frac{y_{i-3}}{z_{i-2}}\right)\widehat{\lambda}_5^{3k+j+1}+\left(\frac{y_{i-3}}{z_{i-2}}-\widehat{\lambda}_5\right)\widehat{\lambda}_6^{3k+j+1}}{\left(\widehat{\lambda}_6-\frac{y_{i-3}}{z_{i-2}}\right)\widehat{\lambda}_5^{3k+j}+\left(\frac{y_{i-3}}{z_{i-2}}-\widehat{\lambda}_5\right)\widehat{\lambda}_6^{3k+j}} \\ &\times \frac{\left(\widehat{\lambda}_4-\frac{x_{i_2-3}}{y_{i_2-2}}\right)\widehat{\lambda}_3^{3k+j+i}+\left(\frac{x_{i_2-3}}{y_{i_2-2}}-\widehat{\lambda}_3\right)\widehat{\lambda}_4^{3k+j+i}}{\left(\widehat{\lambda}_4-\frac{x_{i_2-3}}{y_{i_2-2}}\right)\widehat{\lambda}_3^{3k+j+i-1}+\left(\frac{x_{i_2-3}}{y_{i_2-2}}-\widehat{\lambda}_3\right)\widehat{\lambda}_4^{3k+j+i-1}} \\ &\times \frac{\left(\widehat{\lambda}_2-\frac{z_{i-3}}{x_{i-2}}\right)\widehat{\lambda}_1^{3k+j}+\left(\frac{z_{i-3}}{x_{i-2}}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{3k+j}}{\left(\widehat{\lambda}_2-\frac{z_{i-3}}{x_{i-2}}\right)\widehat{\lambda}_1^{3k+j-1}+\left(\frac{z_{i-3}}{x_{i-2}}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{3k+j-1}} \\ &\times \frac{\left(\widehat{\lambda}_6-\frac{y_{i_2-3}}{z_{i_2-2}}\right)\widehat{\lambda}_5^{3k+j+i-1}+\left(\frac{y_{i_2-3}}{z_{i_2-2}}-\widehat{\lambda}_5\right)\widehat{\lambda}_6^{3k+j+i-1}}{\left(\widehat{\lambda}_6-\frac{y_{i_2-3}}{z_{i_2-2}}\right)\widehat{\lambda}_5^{3k+j+i-2}+\left(\frac{y_{i_2-3}}{z_{i_2-2}}-\widehat{\lambda}_5\right)\widehat{\lambda}_6^{3k+j+i-2}} \\ &\times \frac{\left(\widehat{\lambda}_4-\frac{x_{i-3}}{y_{i-2}}\right)\widehat{\lambda}_3^{3k+j-1}+\left(\frac{x_{i-3}}{y_{i-2}}-\widehat{\lambda}_3\right)\widehat{\lambda}_4^{3k+j-1}}{\left(\widehat{\lambda}_4-\frac{x_{i-3}}{y_{i-2}}\right)\widehat{\lambda}_3^{3k+j-2}+\left(\frac{x_{i-3}}{y_{i-2}}-\widehat{\lambda}_3\right)\widehat{\lambda}_4^{3k+j-2}}\Bigg)^{-1}, \end{aligned} \tag{2.129}$$

$$\begin{aligned} y_{6m+2j+i-1} &= y_{2j+i-1}\Bigg(\prod_{k=1}^{m} \frac{\left(\widehat{\lambda}_4-\frac{x_{i_2-3}}{y_{i_2-2}}\right)\widehat{\lambda}_3^{3k+j+i+1}+\left(\frac{x_{i_2-3}}{y_{i_2-2}}-\widehat{\lambda}_3\right)\widehat{\lambda}_4^{3k+j+i+1}}{\left(\widehat{\lambda}_4-\frac{x_{i_2-3}}{y_{i_2-2}}\right)\widehat{\lambda}_3^{3k+j+i}+\left(\frac{x_{i_2-3}}{y_{i_2-2}}-\widehat{\lambda}_3\right)\widehat{\lambda}_4^{3k+j+i}} \\ &\times \frac{\left(\widehat{\lambda}_2-\frac{z_{i-3}}{x_{i-2}}\right)\widehat{\lambda}_1^{3k+j+1}+\left(\frac{z_{i-3}}{x_{i-2}}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{3k+j+1}}{\left(\widehat{\lambda}_2-\frac{z_{i-3}}{x_{i-2}}\right)\widehat{\lambda}_1^{3k+j}+\left(\frac{z_{i-3}}{x_{i-2}}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{3k+j}} \\ &\times \frac{\left(\widehat{\lambda}_6-\frac{y_{i_2-3}}{z_{i_2-2}}\right)\widehat{\lambda}_5^{3k+j+i}+\left(\frac{y_{i_2-3}}{z_{i_2-2}}-\widehat{\lambda}_5\right)\widehat{\lambda}_6^{3k+j+i}}{\left(\widehat{\lambda}_6-\frac{y_{i_2-3}}{z_{i_2-2}}\right)\widehat{\lambda}_5^{3k+j+i-1}+\left(\frac{y_{i_2-3}}{z_{i_2-2}}-\widehat{\lambda}_5\right)\widehat{\lambda}_6^{3k+j+i-1}} \\ &\times \frac{\left(\widehat{\lambda}_4-\frac{x_{i-3}}{y_{i-2}}\right)\widehat{\lambda}_3^{3k+j}+\left(\frac{x_{i-3}}{y_{i-2}}-\widehat{\lambda}_3\right)\widehat{\lambda}_4^{3k+j}}{\left(\widehat{\lambda}_4-\frac{x_{i-3}}{y_{i-2}}\right)\widehat{\lambda}_3^{3k+j-1}+\left(\frac{x_{i-3}}{y_{i-2}}-\widehat{\lambda}_3\right)\widehat{\lambda}_4^{3k+j-1}} \\ &\times \frac{\left(\widehat{\lambda}_2-\frac{z_{i_2-3}}{x_{i_2-2}}\right)\widehat{\lambda}_1^{3k+j+i-1}+\left(\frac{z_{i_2-3}}{x_{i_2-2}}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{3k+j+i-1}}{\left(\widehat{\lambda}_2-\frac{z_{i_2-3}}{x_{i_2-2}}\right)\widehat{\lambda}_1^{3k+j+i-2}+\left(\frac{z_{i_2-3}}{x_{i_2-2}}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{3k+j+i-2}} \\ &\times \frac{\left(\widehat{\lambda}_6-\frac{y_{i-3}}{z_{i-2}}\right)\widehat{\lambda}_5^{3k+j-1}+\left(\frac{y_{i-3}}{z_{i-2}}-\widehat{\lambda}_5\right)\widehat{\lambda}_6^{3k+j-1}}{\left(\widehat{\lambda}_6-\frac{y_{i-3}}{z_{i-2}}\right)\widehat{\lambda}_5^{3k+j-2}+\left(\frac{y_{i-3}}{z_{i-2}}-\widehat{\lambda}_5\right)\widehat{\lambda}_6^{3k+j-2}}\Bigg)^{-1}, \end{aligned} \tag{2.130}$$

$$
\begin{aligned}
z_{6m+2j+i-1} &= z_{2j+i-1}\Bigg(\prod_{k=1}^{m} \frac{\left(\widehat{\lambda}_6-\frac{y_{i_2-3}}{z_{i_2-2}}\right)\widehat{\lambda}_5^{3k+j+i+1}+\left(\frac{y_{i_2-3}}{z_{i_2-2}}-\widehat{\lambda}_5\right)\widehat{\lambda}_6^{3k+j+i+1}}{\left(\widehat{\lambda}_6-\frac{y_{i_2-3}}{z_{i_2-2}}\right)\widehat{\lambda}_5^{3k+j+i}+\left(\frac{y_{i_2-3}}{z_{i_2-2}}-\widehat{\lambda}_5\right)\widehat{\lambda}_6^{3k+j+i}}\\
&\times \frac{\left(\widehat{\lambda}_4-\frac{x_{i-3}}{y_{i-2}}\right)\widehat{\lambda}_3^{3k+j+1}+\left(\frac{x_{i-3}}{y_{i-2}}-\widehat{\lambda}_3\right)\widehat{\lambda}_4^{3k+j+1}}{\left(\widehat{\lambda}_4-\frac{x_{i-3}}{y_{i-2}}\right)\widehat{\lambda}_3^{3k+j}+\left(\frac{x_{i-3}}{y_{i-2}}-\widehat{\lambda}_3\right)\widehat{\lambda}_4^{3k+j}}\\
&\times \frac{\left(\widehat{\lambda}_2-\frac{z_{i_2-3}}{x_{i_2-2}}\right)\widehat{\lambda}_1^{3k+j+i}+\left(\frac{z_{i_2-3}}{x_{i_2-2}}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{3k+j+i}}{\left(\widehat{\lambda}_2-\frac{z_{i_2-3}}{x_{i_2-2}}\right)\widehat{\lambda}_1^{3k+j+i-1}+\left(\frac{z_{i_2-3}}{x_{i_2-2}}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{3k+j+i-1}}\\
&\times \frac{\left(\widehat{\lambda}_6-\frac{y_{i-3}}{z_{i-2}}\right)\widehat{\lambda}_5^{3k+j}+\left(\frac{y_{i-3}}{z_{i-2}}-\widehat{\lambda}_5\right)\widehat{\lambda}_6^{3k+j}}{\left(\widehat{\lambda}_6-\frac{y_{i-3}}{z_{i-2}}\right)\widehat{\lambda}_5^{3k+j-1}+\left(\frac{y_{i-3}}{z_{i-2}}-\widehat{\lambda}_5\right)\widehat{\lambda}_6^{3k+j-1}}\\
&\times \frac{\left(\widehat{\lambda}_4-\frac{x_{i_2-3}}{y_{i_2-2}}\right)\widehat{\lambda}_3^{3k+j+i-1}+\left(\frac{x_{i_2-3}}{y_{i_2-2}}-\widehat{\lambda}_3\right)\widehat{\lambda}_4^{3k+j+i-1}}{\left(\widehat{\lambda}_4-\frac{x_{i_2-3}}{y_{i_2-2}}\right)\widehat{\lambda}_3^{3k+j+i-2}+\left(\frac{x_{i_2-3}}{y_{i_2-2}}-\widehat{\lambda}_3\right)\widehat{\lambda}_4^{3k+j+i-2}}\\
&\times \frac{\left(\widehat{\lambda}_2-\frac{z_{i-3}}{x_{i-2}}\right)\widehat{\lambda}_1^{3k+j-1}+\left(\frac{z_{i-3}}{x_{i-2}}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{3k+j-1}}{\left(\widehat{\lambda}_2-\frac{z_{i-3}}{x_{i-2}}\right)\widehat{\lambda}_1^{3k+j-2}+\left(\frac{z_{i-3}}{x_{i-2}}-\widehat{\lambda}_1\right)\widehat{\lambda}_2^{3k+j-2}}\Bigg)^{-1},
\end{aligned}
\tag{2.131}
$$

if $a^2+4b\neq 0$, $c^2+4d\neq 0$, $e^2+4f\neq 0$, for $m\in\mathbb{N}_0$, $j\in\{-1,0,1\}$, $i\in\{0,1\}$ and $i_2:=\begin{cases}1, & i+1\equiv 1 \text{ (mod2)}\\ 0, & i+1\equiv 0 \text{ (mod2)}\end{cases}$. Similarly, by using (2.120)-(2.122) in (2.126)-(2.128), after some basic calculation, we obtain

$$
\begin{aligned}
x_{6m+2j+i-1} = x_{2j+i-1}\Bigg(\prod_{k=1}^{m} &\frac{\left(\frac{z_{i_2-3}}{x_{i_2-2}}(3k+j+i+1)-\widehat{\lambda}_1(3k+j+i)\right)\widehat{\lambda}_1^{3k+j+i}}{\left(\frac{z_{i_2-3}}{x_{i_2-2}}(3k+j+i)-\widehat{\lambda}_1(3k+j+i-1)\right)\widehat{\lambda}_1^{3k+j+i-1}}\\
&\times \frac{\left(\frac{y_{i-3}}{z_{i-2}}(3k+j+1)-\widehat{\lambda}_5(3k+j)\right)\widehat{\lambda}_5^{3k+j}}{\left(\frac{y_{i-3}}{z_{i-2}}(3k+j)-\widehat{\lambda}_5(3k+j-1)\right)\widehat{\lambda}_5^{3k+j-1}}\\
&\times \frac{\left(\frac{x_{i_2-3}}{y_{i_2-2}}(3k+j+i)-\widehat{\lambda}_3(3k+j+i-1)\right)\widehat{\lambda}_3^{3k+j+i-1}}{\left(\frac{x_{i_2-3}}{y_{i_2-2}}(3k+j+i-1)-\widehat{\lambda}_3(3k+j+i-2)\right)\widehat{\lambda}_3^{3k+j+i-2}}\\
&\times \frac{\left(\frac{z_{i-3}}{x_{i-2}}(3k+j)-\widehat{\lambda}_1(3k+j-1)\right)\widehat{\lambda}_1^{3k+j-1}}{\left(\frac{z_{i-3}}{x_{i-2}}(3k+j-1)-\widehat{\lambda}_1(3k+j-2)\right)\widehat{\lambda}_1^{3k+j-2}}\\
&\times \frac{\left(\frac{y_{i_2-3}}{z_{i_2-2}}(3k+j+i-1)-\widehat{\lambda}_5(3k+j+i-2)\right)\widehat{\lambda}_5^{3k+j+i-2}}{\left(\frac{y_{i_2-3}}{z_{i_2-2}}(3k+j+i-2)-\widehat{\lambda}_5(3k+j+i-3)\right)\widehat{\lambda}_5^{3k+j+i-3}}\\
&\times \frac{\left(\frac{x_{i-3}}{y_{i-2}}(3k+j-1)-\widehat{\lambda}_3(3k+j-2)\right)\widehat{\lambda}_3^{3k+j-2}}{\left(\frac{x_{i-3}}{y_{i-2}}(3k+j-2)-\widehat{\lambda}_3(3k+j-3)\right)\widehat{\lambda}_3^{3k+j-3}}\Bigg)^{-1},
\end{aligned}
\tag{2.132}
$$

$$y_{6m+2j+i-1} = y_{2j+i-1}\Bigg(\prod_{k=1}^{m} \frac{\left(\frac{x_{i_2-3}}{y_{i_2-2}}(3k+j+i+1) - \widehat{\lambda}_3(3k+j+i)\right)\widehat{\lambda}_3^{3k+j+i}}{\left(\frac{x_{i_2-3}}{y_{i_2-2}}(3k+j+i) - \widehat{\lambda}_3(3k+j+i-1)\right)\widehat{\lambda}_3^{3k+j+i-1}}$$
$$\times \frac{\left(\frac{z_{i-3}}{x_{i-2}}(3k+j+1) - \widehat{\lambda}_1(3k+j)\right)\widehat{\lambda}_1^{3k+j}}{\left(\frac{z_{i-3}}{x_{i-2}}(3k+j) - \widehat{\lambda}_1(3k+j-1)\right)\widehat{\lambda}_1^{3k+j-1}}$$
$$\times \frac{\left(\frac{y_{i_2-3}}{z_{i_2-2}}(3k+j+i) - \widehat{\lambda}_5(3k+j+i-1)\right)\widehat{\lambda}_5^{3k+j+i-1}}{\left(\frac{y_{i_2-3}}{z_{i_2-2}}(3k+j+i-1) - \widehat{\lambda}_5(3k+j+i-2)\right)\widehat{\lambda}_5^{3k+j+i-2}} \tag{2.133}$$
$$\times \frac{\left(\frac{x_{i-3}}{y_{i-2}}(3k+j) - \widehat{\lambda}_3(3k+j-1)\right)\widehat{\lambda}_3^{3k+j-1}}{\left(\frac{x_{i-3}}{y_{i-2}}(3k+j-1) - \widehat{\lambda}_3(3k+j-2)\right)\widehat{\lambda}_3^{3k+j-2}}$$
$$\times \frac{\left(\frac{z_{i_2-3}}{x_{i_2-2}}(3k+j+i-1) - \widehat{\lambda}_1(3k+j+i-2)\right)\widehat{\lambda}_1^{3k+j+i-2}}{\left(\frac{z_{i_2-3}}{x_{i_2-2}}(3k+j+i-2) - \widehat{\lambda}_1(3k+j+i-3)\right)\widehat{\lambda}_1^{3k+j+i-3}}$$
$$\times \frac{\left(\frac{y_{i-3}}{z_{i-2}}(3k+j-1) - \widehat{\lambda}_5(3k+j-2)\right)\widehat{\lambda}_5^{3k+j-2}}{\left(\frac{y_{i-3}}{z_{i-2}}(3k+j-2) - \widehat{\lambda}_5(3k+j-3)\right)\widehat{\lambda}_5^{3k+j-3}}\Bigg)^{-1},$$

$$z_{6m+2j+i-1} = z_{2j+i-1}\Bigg(\prod_{k=1}^{m} \frac{\left(\frac{y_{i_2-3}}{z_{i_2-2}}(3k+j+i+1) - \widehat{\lambda}_5(3k+j+i)\right)\widehat{\lambda}_5^{3k+j+i}}{\left(\frac{y_{i_2-3}}{z_{i_2-2}}(3k+j+i) - \widehat{\lambda}_5(3k+j+i-1)\right)\widehat{\lambda}_5^{3k+j+i-1}}$$
$$\times \frac{\left(\frac{x_{i-3}}{y_{i-2}}(3k+j+1) - \widehat{\lambda}_3(3k+j)\right)\widehat{\lambda}_3^{3k+j}}{\left(\frac{x_{i-3}}{y_{i-2}}(3k+j) - \widehat{\lambda}_3(3k+j-1)\right)\widehat{\lambda}_3^{3k+j-1}}$$
$$\times \frac{\left(\frac{z_{i_2-3}}{x_{i_2-2}}(3k+j+i) - \widehat{\lambda}_1(3k+j+i-1)\right)\widehat{\lambda}_1^{3k+j+i-1}}{\left(\frac{z_{i_2-3}}{x_{i_2-2}}(3k+j+i-1) - \widehat{\lambda}_1(3k+j+i-2)\right)\widehat{\lambda}_1^{3k+j+i-2}} \tag{2.134}$$
$$\times \frac{\left(\frac{y_{i-3}}{z_{i-2}}(3k+j) - \widehat{\lambda}_5(3k+j-1)\right)\widehat{\lambda}_5^{3k+j-1}}{\left(\frac{y_{i-3}}{z_{i-2}}(3k+j-1) - \widehat{\lambda}_5(3k+j-2)\right)\widehat{\lambda}_5^{3k+j-2}}$$
$$\times \frac{\left(\frac{x_{i_2-3}}{y_{i_2-2}}(3k+j+i-1) - \widehat{\lambda}_3(3k+j+i-2)\right)\widehat{\lambda}_3^{3k+j+i-2}}{\left(\frac{x_{i_2-3}}{y_{i_2-2}}(3k+j+i-2) - \widehat{\lambda}_3(3k+j+i-3)\right)\widehat{\lambda}_3^{3k+j+i-3}}$$
$$\times \frac{\left(\frac{z_{i-3}}{x_{i-2}}(3k+j-1) - \widehat{\lambda}_1(3k+j-2)\right)\widehat{\lambda}_1^{3k+j-2}}{\left(\frac{z_{i-3}}{x_{i-2}}(3k+j-2) - \widehat{\lambda}_1(3k+j-3)\right)\widehat{\lambda}_1^{3k+j-3}}\Bigg)^{-1},$$

if $a^2+4b=0$, $c^2+4d=0$, $e^2+4f=0$, for $m\in\mathbb{N}_0$, $j\in\{-1,0,1\}$, $i\in\{0,1\}$ and
$i_2 := \begin{cases} 1, & i+1\equiv 1 \pmod 2 \\ 0, & i+1\equiv 0 \pmod 2 \end{cases}$. We say that the following result holds from the above calculations.

Theorem 2.1. Suppose that a, b, c, d, e, f and the initial values x_{-i}, y_{-i}, $i\in\{1,2,3\}$, are real numbers. Then, the following statements hold.

a) If $b=d=f=0$ and $ace\neq 0$, then the general solutions of system (1.21) is given by formulas in (2.5).
b) If $a=c=e=0$ and $bdf\neq 0$, then the general solutions of system (1.21) is given by formulas in (2.19).
c) If $b=0$ and $acdef\neq 0$, then the general solutions of system (1.21) is given by formulas in (2.30)-(2.32).
d) If $d=0$ and $abcef\neq 0$, then the general solutions of system (1.21) is given by formulas in (2.36)-(2.38).
e) If $f=0$ and $abcde\neq 0$, then the general solutions of system (1.21) is given by formulas in (2.42)-(2.44).

f) If $a=0$ and $bcdef\neq 0$, then the general solutions of system (1.21) is given by formulas in (2.56)-(2.58).
g) If $c=0$ and $abdef\neq 0$, then the general solutions of system (1.21) is given by formulas in (2.61)-(2.63).
h) If $e=0$ and $abcdf\neq 0$, then the general solutions of system (1.21) is given by formulas in (2.66)-(2.68).
i) If $a=c=0$ and $bdef\neq 0$, then the general solutions of system (1.21) is given by formulas in (2.78)-(2.80).
j) If $c=e=0$ and $abdf\neq 0$, then the general solutions of system (1.21) is given by formulas in (2.78)-(2.80).
k) If $a=e=0$ and $bcdf\neq 0$, then the general solutions of system (1.21) is given by formulas in (2.78)-(2.80).
l) If $b=d=0$ and $acef\neq 0$, then the general solutions of system (1.21) is given by formulas in (2.96)-(2.98).
m) If $d=f=0$ and $abce\neq 0$, then the general solutions of system (1.21) is given by formulas in (2.103)-(2.105).
n) If $b=f=0$ and $acde\neq 0$, then the general solutions of system (1.21) is given by formulas in (2.110)-(2.112).
o) If $abcdef\neq 0$, $a^2+4b\neq 0$, $c^2+4d\neq 0$ and $e^2+4f\neq 0$, then the general solutions of system (1.21) is given by formulas in (2.129)-(2.131).
p) If $abcdef\neq 0$, $a^2+4b=0$, $c^2+4d=0$ and $e^2+4f=0$, then the general solutions of system (1.21) is given by formulas in (2.132)-(2.134).

By the following theorem, we characterize the forbidden set of the initial values for system (1.21).

Theorem 2.2. The forbidden set of the initial values for system (1.21) is given by the set

$$\begin{aligned}\mathbb{F}=&\bigcup_{m\in\mathbb{N}_0}\bigcup_{i=0}^{1}\Big\{\frac{z_{i-3}}{x_{i-2}}=\widehat{f}^{-m-1}\left(-\frac{b}{a}\right),\quad \frac{x_{i-3}}{y_{i-2}}=g^{-m-1}\left(-\frac{d}{c}\right),\\ &\frac{y_{i-3}}{z_{i-2}}=h^{-m-1}\left(-\frac{f}{e}\right)\Big\}\bigcup\bigcup_{j=1}^{3}\Big\{\left(\vec{x}_{-(3,1)},\vec{y}_{-(3,1)},\vec{z}_{-(3,1)}\right)\in\mathbb{R}^9:\\ &\qquad x_{-j}=0 \text{ or } y_{-j}=0 \text{ or } z_{-j}=0\Big\},\end{aligned}\tag{2.135}$$

where $\vec{x}_{-(3,1)}=(x_{-3},x_{-2},x_{-1})$, $\vec{y}_{-(3,1)}=(y_{-3},y_{-2},y_{-1})$, $\vec{z}_{-(3,1)}=(z_{-3},z_{-2},z_{-1})$.

Proof . At the beginning of Section 2, we have acquired that the set

$$\bigcup_{j=1}^{3}\Big\{\left(\vec{x}_{-(3,1)},\vec{y}_{-(3,1)},\vec{z}_{-(3,1)}\right)\in\mathbb{R}^9:\ x_{-j}=0 \text{ or } y_{-j}=0 \text{ or } z_{-j}=0\Big\},$$

where $\vec{x}_{-(3,1)}=(x_{-3},x_{-2},x_{-1})$, $\vec{y}_{-(3,1)}=(y_{-3},y_{-2},y_{-1})$, $\vec{z}_{-(3,1)}=(z_{-3},z_{-2},z_{-1})$, belongs to the forbidden set of the initial values for system (1.21). If $x_{-j}\neq 0$, $y_{-j}\neq 0$ and $z_{-j}\neq 0$, $j\in\{1,2,3\}$, then system (1.21) is undefined if and only if

$$bx_{n-2}+az_{n-3}=0,\ dy_{n-2}+cx_{n-3}=0,\ fz_{n-2}+ey_{n-3}=0,$$

for $n\in\mathbb{N}_0$. By taking into account the change of variables (2.114), we can write the corresponding conditions

$$u_{n-2}=-\frac{b}{a},\ v_{n-2}=-\frac{d}{c} \text{ and } w_{n-2}=-\frac{f}{e},\ n\in\mathbb{N}_0.\tag{2.136}$$

Thus, we can define the forbidden set of the initial values for system (1.21) by using equations in (2.115). We know that following statements

$$u_{2m+i}=\widehat{f}^{m+1}\left(u_{i-2}\right)\tag{2.137}$$
$$v_{2m+i}=g^{m+1}\left(v_{i-2}\right)\tag{2.138}$$
$$w_{2m+i}=h^{m+1}\left(w_{i-2}\right)\tag{2.139}$$

where $m\in\mathbb{N}_0$, $i\in\{0,1\}$, $\widehat{f}(x)=\frac{ax+b}{x}$, $g(x)=\frac{cx+d}{x}$ and $h(x)=\frac{ex+f}{x}$, characterize the solutions of equations in (2.115). By using the conditions (2.136) and the statements (2.137)-(2.139), we get

$$u_{i-2}=\widehat{f}^{-m-1}\left(-\frac{b}{a}\right),\tag{2.140}$$

$$v_{i-2} = g^{-m-1}\left(-\frac{d}{c}\right), \tag{2.141}$$

$$w_{i-2} = h^{-m-1}\left(-\frac{f}{e}\right), \tag{2.142}$$

where $m \in \mathbb{N}_0$, $i \in \{0,1\}$ and $abcdef \neq 0$. This means that if one of the conditions in (2.140)-(2.142) holds, then $m-$th iteration or $(m+1)-$th iteration in system (1.21) cannot be calculated. Consequently, desired result follows from (2.135). □

3 An application

In this section, we will derive the solution forms of system (1.21) with $a = b = f = 1$, $c = 3$, $d = -1$, $e = 2$ that is, we get the following system

$$x_n = \frac{z_{n-1}z_{n-3}}{x_{n-2} + z_{n-3}},\ y_n = \frac{x_{n-1}x_{n-3}}{-y_{n-2} + 3x_{n-3}},\ z_n = \frac{y_{n-1}y_{n-3}}{z_{n-2} + 2y_{n-3}},\ n \in \mathbb{N}_0. \tag{3.1}$$

From (2.116), we get

$$u_m^{(i)} = \frac{u_{m-1}^{(i)} + 1}{u_{m-1}^{(i)}},\quad v_m^{(i)} = \frac{3v_{m-1}^{(i)} - 1}{v_{m-1}^{(i)}},\quad w_m^{(i)} = \frac{2w_{m-1}^{(i)} + 1}{w_{m-1}^{(i)}}, \tag{3.2}$$

for $m \in \mathbb{N}_0$, $i \in \{0,1\}$. It is well-known that the substitutions

$$u_{m-1}^{(i)} = \frac{r_m}{r_{m-1}},\ v_{m-1}^{(i)} = \frac{s_m}{s_{m-1}},\ w_{m-1}^{(i)} = \frac{t_m}{t_{m-1}},\ m \in \mathbb{N}_0,$$

transforms equations in (3.2) into the following second order linear difference equations,

$$r_{m+1} - r_m - r_{m-1} = 0,\ m \in \mathbb{N}_0, \tag{3.3}$$

$$s_{m+1} - 3s_m + s_{m-1} = 0,\ m \in \mathbb{N}_0, \tag{3.4}$$

$$t_{m+1} - 2t_m - t_{m-1} = 0,\ m \in \mathbb{N}_0. \tag{3.5}$$

It can be clearly obtained from the roots λ_1 and λ_2 of characteristic equation of (3.3) as the form $\lambda^2 - \lambda - 1 = 0$, where $\lambda_1 = \frac{1+\sqrt{5}}{2} = \alpha$ and $\lambda_2 = \frac{1-\sqrt{5}}{2} = \beta$. On the other hand, taking into account $\alpha\beta = -1$ and the Binet Formula for Fibonacci numbers, which is defined by $F_m = \frac{\alpha^m - \beta^m}{\alpha - \beta}$, $F_0 = 0$, $F_1 = 1$, then we can rewrite the equation (2.117)

$$\begin{aligned} u_{2(m-1)+i} &= \frac{-x_{i-2}\alpha^m - z_{i-3}\alpha^{m+1} + z_{i-3}\beta^{m+1} + x_{i-2}\beta^m}{-x_{i-2}\alpha^{m-1} - z_{i-3}\alpha^m + z_{i-3}\beta^m + x_{i-2}\beta^{m-1}} \\ &= \frac{x_{i-2}F_m + z_{i-3}F_{m+1}}{x_{i-2}F_{m-1} + z_{i-3}F_m}, \end{aligned} \tag{3.6}$$

where $m \in \mathbb{N}_0$, $i \in \{0,1\}$ and F_m is mth Fibonacci number.
It can be clearly obtained from the roots λ_3 and λ_4 of characteristic equation of (3.4) as the form $\lambda^2 - 3\lambda + 1 = 0$, where $\lambda_3 = \frac{3+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \alpha^2$ and $\lambda_2 = \frac{3-\sqrt{5}}{2} = \left(\frac{1-\sqrt{5}}{2}\right)^2 = \beta^2$. On the other hand, taking into account $\alpha\beta = -1$, $\alpha^2\beta^2 = 1$ and the Binet Formula for Fibonacci numbers, then we can rewrite the equation (2.118)

$$\begin{aligned} v_{2(m-1)+i} &= \frac{y_{i-2}\alpha^{2m} - x_{i-3}\alpha^{2m+2} + x_{i-3}\beta^{2m+2} - y_{i-2}\beta^{2m}}{y_{i-2}\alpha^{2m-2} - x_{i-3}\alpha^{2m} + x_{i-3}\beta^{2m} - y_{i-2}\beta^{2m-2}} \\ &= \frac{y_{i-2}F_{2m} - x_{i-3}F_{2m+2}}{y_{i-2}F_{2m-2} - x_{i-3}F_{2m}}. \end{aligned} \tag{3.7}$$

where $m \in \mathbb{N}_0$, $i \in \{0,1\}$ and F_m is mth Fibonacci number.
It can be clearly obtained from the roots λ_5 and λ_6 of characteristic equation of (3.5) as the form $\lambda^2 - 2\lambda - 1 = 0$,

where $\lambda_5 = 1+\sqrt{2} = \gamma$ and $\lambda_6 = 1-\sqrt{2} = \delta$. On the other hand, taking into account $\gamma\delta = -1$ and the Binet Formula for Pell numbers, which is defined by $P_m = \frac{\gamma^m-\delta^m}{\gamma-\delta}$, $P_0 = 0$, $P_1 = 1$, then we can rewrite the equation (2.119)

$$\begin{aligned} w_{2(m-1)+i} &= \frac{-z_{i-2}\gamma^m - y_{i-3}\gamma^{m+1} + y_{i-3}\delta^{m+1} + z_{i-2}\delta^m}{-z_{i-2}\gamma^{m-1} - y_{i-3}\gamma^m + y_{i-3}\delta^m + z_{i-2}\delta^{m-1}} \\ &= \frac{z_{i-2}P_m + y_{i-3}P_{m+1}}{z_{i-2}P_{m-1} + y_{i-3}P_m}. \end{aligned} \tag{3.8}$$

where $m \in \mathbb{N}_0$, $i \in \{0,1\}$ and P_m is mth Pell number.
By substituting the formulas in (3.6)-(3.8) into (2.126)-(2.128) and changing indices, we have the following results.

Theorem 3.1. Assume that $(x_n, y_n, z_n)_{n\geq -3}$ is a well-defined solution of system (3.1). Then the following results are true.

$$\begin{aligned} x_{6m+2j+i-1} = x_{2j+i-1}\Bigg(\prod_{k=1}^{m} & \frac{x_{i_2-2}F_{3k+j+i} + z_{i_2-3}F_{3k+j+i+1}}{x_{i_2-2}F_{3k+j+i-1} + z_{i_2-3}F_{3k+j+i}} \frac{z_{i-2}P_{3k+j} + y_{i-3}P_{3k+j+1}}{z_{i-2}P_{3k+j-1} + y_{i-3}P_{3k+j}} \\ \times & \frac{y_{i_2-2}F_{6k+2j+2i-2} - x_{i_2-3}F_{6k+2j+2i}}{y_{i_2-2}F_{6k+2j+2i-4} - x_{i_2-3}F_{6k+2j+2i-2}} \frac{x_{i-2}F_{3k+j-1} + z_{i-3}F_{3k+j}}{x_{i-2}F_{3k+j-2} + z_{i-3}F_{3k+j-1}} \\ \times & \frac{z_{i_2-2}P_{3k+j+i-2} + y_{i_2-3}P_{3k+j+i-1}}{z_{i_2-2}P_{3k+j+i-3} + y_{i_2-3}P_{3k+j+i-2}} \frac{y_{i-2}F_{6k+2j-4} - x_{i-3}F_{6k+2j-2}}{y_{i-2}F_{6k+2j-6} - x_{i-3}F_{6k+2j-4}}\Bigg)^{-1}, \end{aligned}$$

$$\begin{aligned} y_{6m+2j+i-1} = y_{2j+i-1}\Bigg(\prod_{k=1}^{m} & \frac{y_{i_2-2}F_{6k+2j+2i} - x_{i_2-3}F_{6k+2j+2i+2}}{y_{i_2-2}F_{6k+2j+2i-2} - x_{i_2-3}F_{6k+2j+2i}} \frac{x_{i-2}F_{3k+j} + z_{i-3}F_{3k+j+1}}{x_{i-2}F_{3k+j-1} + z_{i-3}F_{3k+j}} \\ \times & \frac{z_{i_2-2}P_{3k+j+i-1} + y_{i_2-3}P_{3k+j+i}}{z_{i_2-2}P_{3k+j+i-2} + y_{i_2-3}P_{3k+j+i-1}} \frac{y_{i-2}F_{6k+2j-2} - x_{i-3}F_{6k+2j}}{y_{i-2}F_{6k+2j-4} - x_{i-3}F_{6k+2j-2}} \\ \times & \frac{x_{i_2-2}F_{3k+j+i-2} + z_{i_2-3}F_{3k+j+i-1}}{x_{i_2-2}F_{3k+j+i-3} + z_{i_2-3}F_{3k+j+i-2}} \frac{z_{i-2}P_{3k+j-2} + y_{i-3}P_{3k+j-1}}{z_{i-2}P_{3k+j-3} + y_{i-3}P_{3k+j-2}}\Bigg)^{-1}, \end{aligned}$$

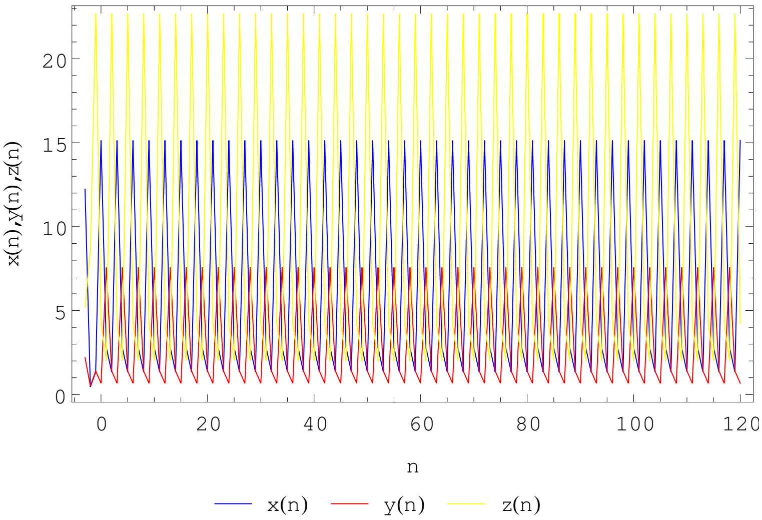
$$\begin{aligned} z_{6m+2j+i-1} = z_{2j+i-1}\Bigg(\prod_{k=1}^{m} & \frac{z_{i_2-2}P_{3k+j+i} + y_{i_2-3}P_{3k+j+i+1}}{z_{i_2-2}P_{3k+j+i-1} + y_{i_2-3}P_{3k+j+i}} \frac{y_{i-2}F_{6k+2j} - x_{i-3}F_{6k+2j+2}}{y_{i-2}F_{6k+2j-2} - x_{i-3}F_{6k+2j}} \\ \times & \frac{x_{i_2-2}F_{3k+j+i-1} + z_{i_2-3}F_{3k+j+i}}{x_{i_2-2}F_{3k+j+i-2} + z_{i_2-3}F_{3k+j+i-1}} \frac{z_{i-2}P_{3k+j-1} + y_{i-3}P_{3k+j}}{z_{i-2}P_{3k+j-2} + y_{i-3}P_{3k+j-1}} \\ \times & \frac{y_{i_2-2}F_{6k+2j+2i-4} - x_{i_2-3}F_{6k+2j+2i-2}}{y_{i_2-2}F_{6k+2j+2i-6} - x_{i_2-3}F_{6k+2j+2i-4}} \frac{x_{i-2}F_{3k+j-2} + z_{i-3}F_{3k+j-1}}{x_{i-2}F_{3k+j-3} + z_{i-3}F_{3k+j-2}}\Bigg)^{-1}, \end{aligned}$$

for $m \in \mathbb{N}_0$, $j \in \{-1,0,1\}$, $i \in \{0,1\}$ and $i_2 := \begin{cases} 1, & i+1 \equiv 1 \ (\text{mod}2) \\ 0, & i+1 \equiv 0 \ (\text{mod}2) \end{cases}$.

4 Numerical Examples

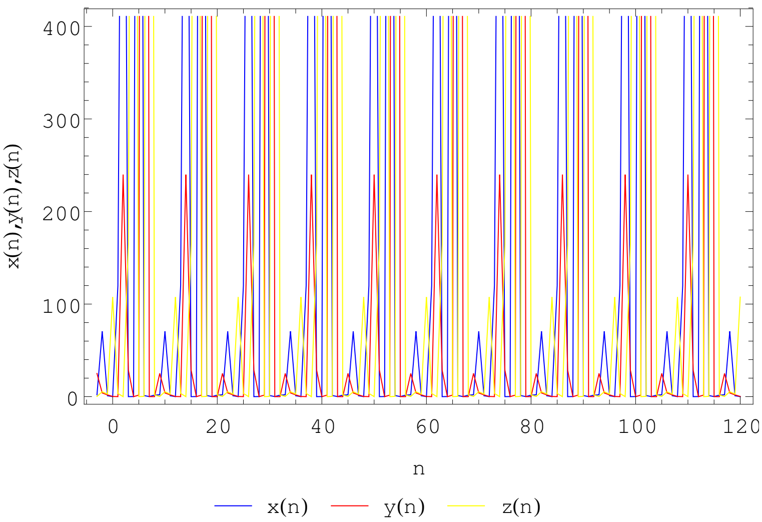
To support our theoretical results, we present numerical examples for the solutions of system (1.21) regard to the different values of a, b, c, d, e and f.

Example 4.1. Consider system (1.21) with the initial values $x_{-3} = 2.3$, $x_{-2} = 3.5$, $x_{-1} = 1.4$, $y_{-3} = 2.25$, $y_{-2} = 4.9$, $y_{-1} = 12.9$, $z_{-3} = 5.2$, $z_{-2} = 8.6$, $z_{-1} = 22.7$, and the parameters, $a = 1.5$, $b = 0$, $c = 2$, $d = 0$, $e = \frac{1}{3}$, $f = 0$ the solutions are represented as in the following figures.



In this case equations in (2.5) are satisfied. Hence, the solutions of system (1.21) has a periodic solution with period three.

Example 4.2. Consider system (1.21) with the initial values $x_{-3} = 2$, $x_{-2} = 70.6$, $x_{-1} = 0.99$, $y_{-3} = 25$, $y_{-2} = 4$, $y_{-1} = 1.9$, $z_{-3} = 0.23$, $z_{-2} = 5.3$, $z_{-1} = 2.74$, and the parameters, $a = 0$, $b = 4.8$, $c = 0$, $d = 2.5$, $e = 0$, $f = \frac{1}{12}$ the solutions are represented as in the following figures.



In this case equations in (2.19) are satisfied. Hence, the solutions of system (1.21) has a periodic solution with period twelve.

5 Conclusion

In this study, we consider the following three-dimensional system of difference equations

$$x_n = \frac{z_{n-1}z_{n-3}}{bx_{n-2} + az_{n-3}},\ y_n = \frac{x_{n-1}x_{n-3}}{dy_{n-2} + cx_{n-3}},\ z_n = \frac{y_{n-1}y_{n-3}}{fz_{n-2} + ey_{n-3}},\ n \in \mathbb{N}_0,$$

where the parameters a, b, c, d, e, f and the initial values x_{-i}, y_{-i}, z_{-i}, $i \in \{1, 2, 3\}$, are real numbers. We have obtained solutions of above system in closed form according to the special cases of the parameters. In addition, the forbidden set of the initial values for aforementioned system is obtained. Finally, an application and numerical examples to support our results are given.

We will give the following important open problem for system of difference equations theory researchers.

Open Problem: System (1.21) can extend to the following $p-$dimensional system of difference equations

$$\begin{aligned} x_n^{(1)} &= \frac{x_{n-1}^{(3)} x_{n-3}^{(3)}}{b^{(1)} x_{n-2}^{(1)} + a^{(1)} x_{n-3}^{(3)}}, \\ x_n^{(2)} &= \frac{x_{n-1}^{(4)} x_{n-3}^{(4)}}{b^{(2)} x_{n-2}^{(2)} + a^{(2)} x_{n-3}^{(4)}}, \\ &\vdots \\ x_n^{(p-1)} &= \frac{x_{n-1}^{(1)} x_{n-3}^{(1)}}{b^{(p-1)} x_{n-2}^{(p-1)} + a^{(p-1)} x_{n-3}^{(1)}}, \\ x_n^{(p)} &= \frac{x_{n-1}^{(2)} x_{n-3}^{(2)}}{b^{(p)} x_{n-2}^{(p)} + a^{(p)} x_{n-3}^{(2)}}, \end{aligned} \tag{5.1}$$

for $n \in \mathbb{N}_0$, where the parameters $a^{(j)}$, $b^{(j)}$ for $j = \overline{1,p}$ and the initial values $x_{-i}^{(j)}$, $i \in \{1,2,3\}$, $j = \overline{1,p}$, are real numbers. Can system (5.1) be solved? If the $p-$dimensional system (5.1) can be solved, how will solutions obtain according to the special cases of the parameters? How will the forbidden set of the initial values for system (5.1) get?

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