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Mixed nonlocal boundary value problem for implicit fractional differential equation involving both retarded and advanced arguments

Nedjemeddine Derdar

Department of Mathematics, Higher Normal School of Technological Education-Skikda, Algeria

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Abstract

In this paper, we investigate the existence and uniqueness of solutions for nonlinear implicit Hilfer-Hadamard fractional differential equations involving both retarded and advanced arguments and nonlocal mixed boundary conditions. We also use the Banach contraction mapping principle and Schaefer's fixed point theorem to show the existence and uniqueness of solutions. The results obtained here extend the work of Benchohra et al. [10] and Haoues et al. [18]. An example is also given to illustrate the results.

Keywords: Hilfer–Hadamard fractional derivative, Implicit fractional differential equations, retarded argument, advanced argument, Fixed point theorems, Existence and uniqueness
2020 MSC: 26A33, 34A08, 34A12, 34K32

1 Introduction

Fractional differential equations with different type of initial conditions have recently gained a lot importance and attention due to their application in various fields such as physics, mechanics, bioengineering, biology, chemistry, economics, viscoelasticity, acoustics, optics, robotics, control theory and electronics. For details, see [20], [24], [25], [27], [33], and the references therein.
Fractional differential equations involving fractal derivatives of Riemann-Liouville and Caputo have been widely studied by many researchers over the course of a decade. Recently, the authors studied the fractional differential equations of Hadamard-type, Caputo-Hadamard-type, Hilfer-Hadamard-type, Hilfer–Katugampola-type and Erdeyl–Kober-type, etc., see [1]-[5], [7]-[9], [11], [12], [15], [22], [26], [31], [32], [34], [37].

The fractional derivative due to Hadamard was introduced in 1892, which in different from earlier derivatives such as Riemann–Liouville and Caputo-type fractional derivative. That is, the integral kernel contains a logarithmic function of an arbitrary exponent. A detailed description of Hadamard's fractional and integral derivative can be found in the references [14], [16], [21], [23].

Note that Hilfer studied applications of a generalized fractional operator having the Riemann–Liouville and the Caputo derivatives as specific cases. Later, the modification of Hilfer fractional derivative resulted in the concept of the Hilfer–Hadamard derivative. And to see some of the basic properties of the Hilfer-Hadamard fractional derivative that

Email address: nedjm1972@yahoo.fr (Nedjemeddine Derdar)

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were used in this work, we direct the reader to the references [16], [19], [20], [24]]. Moreover, the existence results for Hilfer–Hadamard fractional differential equations of order in (0, 1] were studied by several researchers, for instance, see [[6], [28], [30], [32], [35], [36]. To the best of our knowledge, only a few results are available in the literature concerning boundary value problems for Hilfer–Hadamard fractional differential equations of order in (1, 2].

The motivating thing about this work is that all the articles published about fractional differential equations with retarded and advanced arguments put the initial condition between the retarded and advanced fields equal to zero, see [6], [7], [10], [13], [18], while I put this condition not equal to zero in my work.

In 2018, M. Benchohra et al. [10] have investigated the existence and uniqueness of solutions for a class of problem for nonlinear implicit fractional differential equations of Hadamard type involving both retarded and advanced arguments of the form:

$$\begin{cases} D^{\alpha}y(t) = f\Big(t, y_t, D^{\alpha}y(t)\Big), \text{ for each } t \in [1,e], \\ \\ y(t) = \chi(t),\ 1-r \le t \le 1,\ r>0 \\ \\ y(t) = \Psi(t),\ e \le t \le e+h,\ h>0 \end{cases}$$

where D^{α} is the Hadamard fractional derivative of order $1 < \alpha \le 2$. By employing the Schauder fixed point theorem and the Banach contraction mapping principle, the authors obtained existence and uniqueness results. In 2020, M. Haoues et al. [18], applied the tools of the fixed-point theory (the Banach contraction mapping principle and the Krasnoselskii theorem) to study the existence and uniqueness of solutions for nonlinear retarded and advanced implicit Hadamard fractional differential equations with nonlocal conditions:

$$\begin{cases} D^{\alpha}y(t) = f\Big(t, y_t, D^{\alpha}y(t)\Big), \text{ for each } t \in [1,e], \\ \\ y(t) + (H_1 y)(t) = \chi(t),\ 1-r \le t \le 1,\ r>0 \\ \\ y(t) + (H_2 y)(t) = \Psi(t),\ e \le t \le e+h,\ h>0 \end{cases}$$

where $D^{\alpha}$is the Hadamard fractional derivative of order $1 < \alpha \le 2$, $f : J \times C([-r,h],\mathbb{R}) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given continuous function, $H_1 : C([1-r,e+h],\mathbb{R}) \longrightarrow C([1-r,1],\mathbb{R})$ and $H_2 : C([1-r,e+h],\mathbb{R}) \longrightarrow C([e,e+h],\mathbb{R})$ are given continuous mappings, $\chi \in C([1-r,1],\mathbb{R})$ and $\Psi \in C([e,e+h],\mathbb{R})$.

In 2021, B. Ahmed et al. [6], have discussed the existence and uniqueness of solutions for a Hilfer–Hadamard fractional differential equation, supplemented with mixed nonlocal (multi-point, fractional integral multi-order and fractional derivative multi-order) boundary conditions:

$$\begin{cases} {}_H^H D_1^{\alpha,\beta} x(t) = f(t, x(t)),\ t \in J := [1,T], \\ \\ x(1) = 0,\quad x(T) = \sum_{j=1}^{m} \eta_j x(\zeta_j) + \sum_{i=1}^{n} \xi_i {}_H I_1^{\phi_i} x(\theta_i) + \sum_{k=1}^{r} \lambda_k {}_H D_1^{\omega_k} x(\mu_k), \end{cases}$$

where ${}_H^H D_1^{\alpha,\beta}$ denotes the Hilfer–Hadamard fractional derivative operator of order, $\alpha \in (1,2]$ and type $\beta \in [0,1]$, $f : [1,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given continuous function, ${}_H I^{\phi_i}$ is the Hadamard fractional integral operator of order $\phi_i > 0$. In this paper, we consider the implicit Hilfer-Hadamard fractional differential equations with involving both retarded and advanced arguments and nonlocal mixed boundary conditions

$$\begin{cases} {}_H^H D^{\alpha,\beta} x(t) = f(t, x_t, {}_H^H D^{\alpha,\beta} x(t)),\ t \in J := (1,e), & (1.1) \\ x(1^+) = 0,\ a_H I^{\delta} x(\eta) + b {}_H^H D^{1,1} x(e^-) = c,\ \gamma = \alpha + \beta - \alpha\beta, & (1.2) \\ x(t) + g_1(x)(t) = \phi(t),\ 1-r_1 \le t \le 1,\ r_1 > 0, & (1.3) \\ x(t) + g_2(x)(t) = \psi(t),\ e \le t \le e + r_2,\ r_2 > 0, & (1.4) \end{cases}$$

where ${}_H^H D^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivatives of order $1 < \alpha \le 2$, ${}_H I^{\delta}$ is the standard Hadamard fractional integral of order $\delta > 0$, $f : J \times C([-r_1, r_2],\mathbb{R}) \times \mathbb{R} \longrightarrow \mathbb{R}$, $g_1 : C([1-r_1, e+r_2],\mathbb{R}) \longrightarrow C([1-r_1,1],\mathbb{R})$ and $g_2 : C([1-r_1, e+r_2],\mathbb{R}) \longrightarrow C([e, e+r_2],\mathbb{R})$ are given functions, $\phi \in C([1-r_1,1],\mathbb{R})$ and $\psi \in C([e,e+r_2],\mathbb{R})$, a, b and c are real constants, and $\eta \in (1,e)$. For any function x define on $[1-r_1, e+r_2]$ and any $1 \le t \le e$, $x_t(\theta) = x(t+\theta)$ for $-r_1 \le \theta \le r_2$.

This paper is organized as follows. In section 2, we introduce some notations and lemmas, and state some preliminaries results needed in later section 2. In Section 3, we give two results, one based on Banach fixed point Theorem

3.1 and another one based on Schauder's fixed point Theorem 3.2. In Section 4, we illustrate the results obtained with an example.

2 Preliminaries

In this section, we introduce some notations and defenitions of Hilfer-Hadamard type fractional calculus. Let now $[a,b]$, $(-\infty < a < b < +\infty)$ be interval finite. By $C([a,b],\ \mathbb{R})$ be the Banach space of all continuous functions from $[a,b]$ into $\mathbb{R}$ with the norm

$$\|y(t)\|_{[a,b]} = \sup\{|y(t)| :\ a \le t \le b\}.$$

Let $AC([a,b],\ \mathbb{R})$ be the space of functions $g : [a,b] \longrightarrow \mathbb{R}$ that are absolutely continuous. Let $\delta = t\frac{d}{dt},\ \ \delta^n = \delta(\delta^{n-1})$, we consider the set of functions:

$$AC^n_\delta([a,b],\ \mathbb{R}) = \{g : [a,b] \longrightarrow \mathbb{R} : \delta^{n-1}g(t) \in AC([a,b],\ \mathbb{R})\}.$$

Definition 2.1. (Hadamard fractional integral [24]) Let $f : [a,\infty] \longrightarrow \mathbb{R}$. Then The Hadamard fractional integral of order $\alpha > 0$ is defined as follows:

$$_HI^\alpha_a f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s)\frac{ds}{s},\quad t > a \tag{2.1}$$

provided that the integral exists, where $\log(\cdot) = \log_e(\cdot)$.

Definition 2.2. (Hadamard fractional derivative [24]) For a function $f : [a,\infty] \longrightarrow \mathbb{R}$ the Hadamard fractional derivative of order $\alpha > 0$ is defined as follows:

$$_HD^\alpha_a f(t) = \delta^n\Big({}_HI^{n-\alpha}_a f\Big)(t),\ t > a,\ n-1 < \alpha < n,\ n = [\alpha]+1, \tag{2.2}$$

where $\delta^n = (t\frac{d}{dt})^n$, and $[\alpha]$ denotes the integer part of the real number α.

Definition 2.3. (Hilfer-Hadamard fractional derivative [[19], [20]]). Let $f \in L^1(a,b)$ and $n-1 < \alpha < n$, $0 \le \beta \le 1$. we define the Hilfer-Hadamard fractional derivative of order α and type β for f as follows:

$$\begin{aligned}
{}^H_HD^{\alpha,\beta}f(t) &= \Big({}_HI^{\beta(n-\alpha)}\delta^n I^{(n-\alpha)(1-\beta)}f\Big)(t)\\
&= \Big({}_HI^{\beta(n-\alpha)}\delta^n {}_HI^{\beta(n-\gamma)}f\Big)(t)\\
&= \Big({}_HI^{\beta(n-\alpha)}{}_HD^{\gamma}f\Big)(t),\quad \gamma = \alpha + n\beta - \alpha\beta,
\end{aligned}$$

where ${}_HI^{(\cdot)}$ and ${}_HD^{(\cdot)}$ are defined by (2.1) and (2.2), respectively.

Lemma 2.4. (See [24]) If $\alpha > 0$, $\beta > 0$ and $0 < a < b < \infty$, then

(i) $\Big({}_HI^\alpha_{a^+}(\log\frac{t}{a})^{\beta-1}\Big)(x) = \dfrac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\log\frac{x}{a})^{\beta+\alpha-1}$,

(ii) $\Big({}_HD^\alpha_{a^+}(\log\frac{t}{a})^{\beta-1}\Big)(x) = \dfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\log\frac{x}{a})^{\beta-\alpha-1}$.

In particular, if $\beta = 1$ and $0 < \alpha < 1$, then the following is the case:

$$D^\alpha_{a^+}(1) = \frac{1}{\Gamma(1-\alpha)}(\log\frac{x}{a})^{-\alpha} \neq 0.$$

Lemma 2.5. (See [21]) Let $g \in AC^n_\delta([a,b],\ \mathbb{R})$ and $\alpha > 0$, $n \in \mathbb{N}$. Then

$${}_HI^\alpha_1({}_HD^\alpha_1 g)(t) = g(t) - \sum_{k=0}^{n-1}\frac{\delta^{(k)}g(a)}{k!}(\log\frac{t}{a})^k.$$

Lemma 2.6. (See [24]) For all $\mu > 0$ and $\nu > -1$, then

$$\frac{1}{\Gamma(\mu)}\int_a^t \Big(\log\frac{t}{s}\Big)^{\mu-1}(\log s)^{\nu}ds = \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)}(\log t)^{\mu+\nu}.$$

Theorem 2.7. ([16], [24]) Let $\alpha > 0$, $0 \le \beta \le 1$, $\gamma = \alpha + n\beta - \alpha\beta$, $n = [\alpha] + 1$ and $0 < a < b < \infty$, if $f \in L^1(a, b)$ and $({}_HI_a^{n-\alpha}f)(t) \in AC_\delta^n[a, b]$, then

$$\begin{aligned}({}_HI_a^{\alpha}{}_H^HD_a^{\alpha}f)(t) &= ({}_HI_a^{\gamma}{}_H^HD_a^{\gamma,\beta}f)(t)\\ &= f(t) - \sum_{j=0}^{n-1}\frac{(\delta^{(n-j-1)}({}_HI_a^{n-\gamma}f)(a)}{\Gamma(\gamma-j)}(\log\frac{t}{a})^{\gamma-j-1},\end{aligned}$$

observe that $\Gamma(\gamma - j)$ exists for all $j = 1, 2, \dots, n - 1$ for $\gamma \in [\alpha, n]$.

Theorem 2.8. ([39])(Banach's fixed point Theorem). Let (X, d) be a nonempty complete metric space with $T : X \longrightarrow X$ is a contraction mapping. Then map T has a fixed point $x^* \in X$ such that $Tx^* = x^*$.

Theorem 2.9. ([17])(Schaefer's fixed point Theorem). Let X be a Banach space, let $T : X \longrightarrow X$ be a completely continuous operator, and let the set $D = \{x \in X : x = \lambda Tx,\ 0 < \lambda \le 1\}$ be bounded. Then T has a fixed point in X.

3 Existence and Uniqueness Results

In the case $n = [\alpha] + 1 = 2$, we have $\gamma = \alpha + (2 - \alpha)\beta$.

Definition 3.1. A function $x \in C^2(1 - r_1, e + r_2], \mathbb{R})$ is said to be a solution of the problem (1.1)-(1.4) if x satisfies the equation ${}_H^HD^{\alpha,\beta}x(t) = f(t, x_t, {}_H^HD^{\alpha,\beta}x(t))$, and satisfies the conditions $x(1^+) = 0$, $a_HI^{\gamma}x(\eta) + b_H^HD^{\alpha,\beta}x(e^-) = c$, on J, $x(t) + g_1(x)(t) = \phi(t)$, on $[1 - r_1, 1]$ and $x(t) + g_2(x)(t) = \psi(t)$ on $[e, e + r_2]$.

To prove the existence of solutions to the problem (1)-(2), we need the following auxiliary lemma.

Lemma 3.2. Let h be a continuous function. Then the linear problem

$$\begin{cases} {}_H^HD^{\alpha,\beta}x(t) = h(t),\ t \in J := (0, 1) & (3.1)\\ x(1^+) = 0,\ a_HI^{\delta}x(\eta) + b_H^HD^{1,1}x(e^-) = c, & (3.2)\\ x(t) + g_1(x)(t) = \phi(t),\ 1 - r_1 \le t \le 1, & (3.3)\\ x(t) + g_2(x)(t) = \psi(t),\ e \le t \le e + r_2, & (3.4)\end{cases}$$

has a unique solution which is given by

$$x(t) = \begin{cases} \phi(t) - g_1(x)(t),\ \text{if } t \in [1 - r_1, 1],\\ {}_HI^{\alpha}h(t) + \dfrac{(\log t)^{\gamma-1}}{\Lambda}\Big[c - a_HI^{\alpha+\gamma}h(\eta) - b_HI^{\alpha-1}h(e^-)\Big],\ \text{if } t \in J := (1, e),\\ \psi(t) - g_2(x)(t),\ \text{if } t \in [e, e + r_2],\end{cases} \tag{3.5}$$

where

$$\Lambda = \frac{a\Gamma(\gamma)}{\Gamma(\gamma+\delta)}(\log\eta)^{\gamma+\delta-1} + b(\gamma - 1) \ne 0. \tag{3.6}$$

Proof . Applying the Hadamard fractional integral of order α to both sides of (3.1) and then using

$$x(t) - \frac{\delta({}_HI_{1^+}^{2-\gamma}x)(1)}{\Gamma(\gamma)}(\log t)^{\gamma-1} - \frac{({}_HI_{1^+}^{2-\gamma}x)(1)}{\Gamma(\gamma)}(\log t)^{\gamma-2} = {}_HI_1^{\alpha}h(t), \tag{3.7}$$

which can be rewritten as follows:

$$x(t) = c_0(\log t)^{\gamma-1} + c_1(\log t)^{\gamma-2} + {}_HI^{\alpha}h(t), \tag{3.8}$$

where c_0 and c_1 are arbitrary constants.
Using the first bondary condition ($x(1^+) = 0$) in (3.7), yields $c_1 = 0$, since $\gamma \in [\alpha, 2]$.
In consequence, (3.8) takes the following form:

$$x(t) = c_0(\log t)^{\gamma-1} + {}_HI^{\alpha}h(t)$$

Using Lemma 2.4, we can write

$${}_HI^{\delta}x(\eta) = \frac{c_0\Gamma(\gamma)}{\Gamma(\gamma+\delta)}(\log\eta)^{\gamma+\delta-1} + {}_HI^{\alpha+\gamma}h(\eta),$$

and

$${}^H_HD^{1,1}x(e^-)(t) = (\gamma-1)c_0 + {}_HI^{\alpha-1}h(e^-).$$

Using the second bondary condition ($a_HI^{\delta}x(\eta) + b_HD^{1,1}x(e^-) = c$), we get

$$a\Big[\frac{c_0\Gamma(\gamma)}{\Gamma(\gamma+\delta)}(\log\eta)^{\gamma+\delta-1} + {}_HI^{\alpha+\delta}h(\eta)\Big] + b\Big[(\gamma-1)c_0 + {}_HI^{\alpha-1}h(e^-)\Big] = c,$$

thus,

$$\Big[\frac{a\Gamma(\gamma)}{\Gamma(\gamma+\delta)}(\log\eta)^{\gamma+\delta-1} + b(\gamma-1)\Big]c_0 + a_HI^{\alpha+\delta}h(\eta) + b_HI^{\alpha-1}h(e^-) = c.$$

Consequently,

$$c_0 = \frac{c}{\Lambda} - \frac{a}{\Lambda}{}_HI^{\alpha+\delta}h(\eta) - \frac{b}{\Lambda}{}_HI^{\alpha-1}h(e^-),$$

where

$$\Lambda = \frac{a\Gamma(\gamma)}{\Gamma(\gamma+\delta)}(\log\eta)^{\gamma+\delta-1} + b(\gamma-1).$$

Finally, substituting the values of c_0 and c_1 in (3.8), we obtain (3.5). □
We assume the following conditions to prove the existence of a solution of problem (1.1)-(1.4):

(H1) The function $f : J \times C([-r_1, r_2], \mathbb{R}) \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.

(H2) There exists constants $L_1 \in \mathbb{R}_+$, and $L_2 \in (0, 1)$ such that

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \leq L_1\|u - \bar{u}\|_{[-r_1,r_2]} + L_2|v - \bar{v}|,$$

for any u, $\bar{u} \in C([-r_1, r_2], \mathbb{R})$ and v, $\bar{v} \in \mathbb{R}$ for a.e., $t \in J$.

(H3) There exists constants $L_3, L_4 \in (0, 1)$ such that

$$\|g_1(x_1) - g_1(x_2))\|_{[1-r_1,1]} \leq L_3\|x_1 - x_2\|_{[1-r_1,e+r_2]},$$

and

$$\|g_2(x_1) - g_2(x_2))\|_{[e,e+r_2]} \leq L_4\|x_1 - x_2\|_{[1-r_1,e+r_2]},$$

for any x_1, $x_2 \in C([1 - r_1, e + r_2], \mathbb{R})$.

Our first result is based on the Banach contraction mapping principle.
Let the constant ρ be such that

$$\rho \geq \max\{\Omega, L_3, L_4\},$$

where

$$\Omega := \frac{L_1}{(1-L_2)}\left(\frac{1}{\Gamma(\alpha+1)} + \frac{|a|(\log\eta)^{\alpha+\delta}}{|\Lambda|\Gamma(\alpha+\delta+1)} + \frac{|b|}{|\Lambda|\Gamma(\alpha+1)}\right),$$

and

$$\Lambda = \frac{a\Gamma(\gamma)}{\Gamma(\gamma+\delta)}(\log\eta)^{\gamma+\delta-1} + b(\gamma-1).$$

Theorem 3.3. If the hypotheses (H1)-(H3) are satisfied, and if

$$\rho < 1, \tag{3.9}$$

then there exists a unique solution for problem (1.1)-(1.4).

Proof . Transform the problem (1.1)-(1.4) into a fixed point problem. Consider the operator $\mathcal{F} : C([1-r_1, e+r_2], \mathbb{R}) \longrightarrow C([1-r_1, e+r_2],\ \mathbb{R})$ defined by

$$(\mathcal{F}x)(t) = \begin{cases} \phi(t) - g_1(x)(t),\ \text{if } t \in [1-r_1, 1], \\ {}_H I^{\alpha}\sigma_x(t) + \dfrac{(\log t)^{\gamma-1}}{\Lambda}\Big[c - a\,{}_H I^{\alpha+\gamma}\sigma_x(\eta) - b\,{}_H I^{\alpha-1}\sigma_x(e^-)\Big],\ \text{if } t \in J := (1, e), \\ \psi(t) - g_2(x)(t),\ \text{if } t \in [e, e+r_2], \end{cases} \tag{3.10}$$

where

$$\sigma_x(t) = f\Big(t, x_t, {}_H^H D^{\alpha,\beta} x(t)\Big).$$

Clearly, the fixed points of $\mathcal{F}$ are solutions of problem (1.1)-(1.4). Let $x,\ y \in C([1-r_1, e+r_2],\ \mathbb{R})$. If $t \in [1-r_1, 1]$ and by (H3), then

$$\begin{aligned} \Big|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\Big| &= |g_1(x)(t) - g_1(y)(t)| \\ &\le \|g_1(x) - g_1(y)\|_{[1-r_1,1]} \\ &\le L_3\|x - y\|_{[1-r_1,e+r_2]} \\ &\le \rho\|x - y\|_{[1-r_1,e+r_2]}. \end{aligned} \tag{3.11}$$

If $t \in [e, e+r_2]$ and by (H3), then

$$\begin{aligned} \Big|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\Big| &= |g_2(x)(t) - g_2(y)(t)| \\ &\le \|g_2(x) - g_2(y)\|_{[e,e+r_2]} \\ &\le L_4\|x - y\|_{[1-r_1,e+r_2]} \\ &\le \rho\|x - y\|_{[1-r_1,e+r_2]}. \end{aligned} \tag{3.12}$$

For $t \in J$, we have

$$\begin{aligned} \Big|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\Big| \le \frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}|\sigma_x(s) - \sigma_y(s)|\frac{ds}{s} + \frac{|a|\log t}{|\Lambda|\Gamma(\alpha+\delta)}\int_1^{\eta}(\log\frac{\eta}{s})^{\alpha+\delta-1}|\sigma_x(s) - \sigma_y(s)|\frac{ds}{s} \\ + \frac{|b|}{|\Lambda|\Gamma(\alpha)}\int_1^{e^-}(\log\frac{e^-}{s})^{\alpha-1}|\sigma_x(s) - \sigma_y(s)|\frac{ds}{s}, \end{aligned} \tag{3.13}$$

where $\sigma_x,\ \sigma_y \in C(J,\ \mathbb{R})$ such that

$$\sigma_x(t) = f(t, x_t, \sigma_x(t)),$$

and

$$\sigma_y(t) = f(t, y_t, \sigma_y(t)).$$

By (H2), we have

$$\begin{aligned} |\sigma_x(t) - \sigma_y(t)| &= |f(t, x_t, \sigma_x(t)) - f(t, y_t, \sigma_y(t))| \\ &\le L_1\|x_t - y_t\|_{[-r_1,r_2]} + L_2|\sigma_x(t) - \sigma_y(t)|, \end{aligned}$$

then

$$|\sigma_x(t) - \sigma_y(t)| \le \frac{L_1}{1-L_2}\|x_t - y_t\|_{[-r_1,r_2]}. \tag{3.14}$$

By replacing (3.14) in the inequality (3.13), we get

$$\begin{aligned}|(\mathcal{F}x)(t)-(\mathcal{F}y)(t)| &\leq \frac{L_1}{(1-L_2)\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}\|x_s-y_s\|_{[-r_1,r_2]}\frac{ds}{s}\\ &+\frac{|a|L_1\log t}{|\Lambda|(1-L_2)\Gamma(\alpha+\delta)}\int_1^\eta(\log\frac{\eta}{s})^{\alpha+\delta-1}\|x_s-y_s\|_{[-r_1,r_2]}\frac{ds}{s}\\ &+\frac{|b|L_1\log t}{|\Lambda|(1-L_2)\Gamma(\alpha)}\int_1^{e^-}(\log\frac{e^-}{s})^{\alpha-1}\|x_s-y_s\|_{[-r_1,r_2]}\frac{ds}{s}\\ &\leq \frac{L_1}{(1-L_2)}\left(\frac{1}{\Gamma(\alpha+1)}+\frac{|a|(\log\eta)^{\alpha+\delta}}{|\Lambda|\Gamma(\alpha+\delta+1)}+\frac{|b|}{|\Lambda|\Gamma(\alpha+1)}\right)\|x_s-y_s\|_{[-r_1,r_2]}\\ &\leq \Omega\|x-y\|_{[1-r_1,e+r_2]}. \end{aligned} \tag{3.15}$$

Thus

$$\|(\mathcal{F}x)-(\mathcal{F}y)\|_{[1-r_1,e+r_2]}\leq \rho\|x-y\|_{[1-r_1,e+r_2]}. \tag{3.16}$$

Consequently by (3.9), $\mathcal{F}$ is a contraction. As a consequence of Banach fixed point theorem, we deduce that $\mathcal{F}$ has a fixed point which is a solution of the problem (1.1)-(1.4). □

The second result is based on Schaefer's fixed point theorem.

(H4) There exists constants $M>0$ such that

$$|f(t,u,v)|\leq M,$$

for any $u\in C([-r_1,r_2],\mathbb{R})$, $v\in\mathbb{R}$ for a.e., $t\in J$.

(H5) There exists constants $M_1>0$ and $M_2>0$ such that

$$\|g_1(u)\|_{[1-r_1,1]}\leq M_1 \text{ and } \|g_2(u)\|_{[e,e+r_2]}\leq M_2,$$

for any $u\in C([1-r_1,e+r_2],\mathbb{R})$.

Theorem 3.4. Assume that conditions (H1), (H3), (H4) and (H5) hold. Then the problem (1.1)-(1.4) has at least one solution.

Proof . We show that operator $\mathcal{F}$ defined in (3.9) has at least one fixed point in $C([1-r_1,e+r_2],\mathbb{R})$. The proof is divided into four steps:

Step 1: The operator $\mathcal{F}$ is continuous

Let $\{x_n\}$ be a sequence such that $x_n\longrightarrow x$ in $C([1-r_1,e+r_2],\mathbb{R})$. If $t\in[1-r_1,1]$, and by (3.11), then

$$\Big|\mathcal{F}(x_n)(t)-\mathcal{F}(x)(t)\Big|\leq L_3\|x_n-x\|_{[1-r_1,e+r_2]}.$$

Thus

$$\Big|\mathcal{F}(x_n)(t)-\mathcal{F}(x)(t)\Big|\longrightarrow 0 \text{ as } n\longrightarrow+\infty.$$

If $t\in[e,e+r_2]$, and by (3.12), then

$$\Big|\mathcal{F}(x_n)(t)-\mathcal{F}(x)(t)\Big|\leq L_4\|x_n-x\|_{[1-r_1,e+r_2]}.$$

Thus

$$\Big|\mathcal{F}(x_n)(t)-\mathcal{F}(x)(t)\Big|\longrightarrow 0 \text{ as } n\longrightarrow+\infty.$$

For $t\in J$, by (3.15), we have

$$\Big|\mathcal{F}(x_n)(t)-\mathcal{F}(x)(t)\Big|\leq \Omega\|x_n-x\|_{[1-r_1,e+r_2]}.$$

Since σ is a continuous (i.e f is continuous), then by the Lebesgue dominated convergence theorem, we have

$$\Big|\mathcal{F}(x_n)(t)-\mathcal{F}(x)\Big|\longrightarrow 0 \text{ as } n\longrightarrow+\infty.$$

Consequently, $\mathcal{F}$ is continuous.
Step 2: The operator $\mathcal{F}$ maps bounded sets into bounded sets in $C([1-r_1, e+r_2], \mathbb{R})$.
For $\eta > 0$, there exists a constant $l > 0$, such that $l := \max(l_1, l_2, l_3)$, where

$$l_1 := \|\phi\|_{[1-r_1,1]} + M_1,$$

$$l_2 := \|\psi\|_{[e,e+r_2]} + M_2,$$

and

$$l_3 := M\left[\frac{1}{\Gamma(\alpha+1)} + \frac{|b|(\log\eta)^{\alpha+\delta}}{|\Lambda|\Gamma(\alpha+\delta+1)} + \frac{|b|}{|\Lambda|\Gamma(\alpha+1)}\right] + \frac{|c|}{|\Lambda|},$$

for each $x \in B_\eta = \{x \in C([1-r_1, e+r_2], \mathbb{R}) : \ \|x\|_{[1-r_1,e+r_2]} \leq \eta\}$, we have $\|(\mathcal{F}x)\|_{[1-r_1,e+r_2]} \leq l$.
Indeed, for any $t \in [1-r_1, 1]$, $x \in B_\eta$, and by (H5), we have

$$\begin{aligned} |(\mathcal{F}x)(t)| &\leq \|\phi\|_{[1-r_1,1]} + \|g_1(x)\|_{[1-r_1,1]} \\ &\leq \|\phi\|_{[1-r_1,1]} + M_1 := l_1 \leq l, \end{aligned} \tag{3.17}$$

for any $t \in [e, e+r_2]$, $x \in B_\eta$, and by (H5), we have

$$\begin{aligned} |(\mathcal{F}x)(t)| &\leq \|\psi\|_{[e,e+r_2]} + \|g_2(x)\|_{[e,e+r_2]} \\ &\leq \|\psi\|_{[e,e+r_2]} + M_2 := l_2 \leq l, \end{aligned} \tag{3.18}$$

for $t \in J$, we have

$$\begin{aligned} \Big|\mathcal{F}(x)(t)\Big| \leq& \frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}|\sigma_x(s)|\frac{ds}{s} + \frac{|a|}{|\Lambda|\Gamma(\alpha+\delta)}\int_1^\eta (\log\frac{\eta}{s})^{\alpha+\delta-1}|\sigma_x(s)|\frac{ds}{s} \\ &+ \frac{|b|}{|\Lambda|\Gamma(\alpha)}\int_1^{e^-} (\log\frac{e^-}{s})^{\alpha-1}|\sigma_x(s)|\frac{ds}{s} + \frac{|c|(\log t)^{\gamma-1}}{|\Lambda|}, \end{aligned}$$

where $\sigma_x \in C(J, \mathbb{R})$ is such that

$$\sigma_x(t) = f(t, x(t), \sigma_x(t)).$$

From (H4), for $t \in J$, we have

$$\Big|\mathcal{F}(x)(t)\Big| \leq M\left[\frac{1}{\Gamma(\alpha+1)} + \frac{|b|(\log\eta)^{\alpha+\delta}}{|\Lambda|\Gamma(\alpha+\delta+1)} + \frac{|b|}{|\Lambda|\Gamma(\alpha+1)}\right] + \frac{|c|}{|\Lambda|} := l_3 \leq l. \tag{3.19}$$

Step 3: The operator $\mathcal{F}$ maps bounded sets into equicontinuous sets of $C([1-r_1, e+r_2], \mathbb{R})$.
Let $t_1,\ t_2 \in J,\ \ t_1 < t_2$ and let B_η be a bounded set of $C([1-r_1, e+r_2], \mathbb{R})$ as in step 2, and let $x \in B_\eta$. Then

$$\begin{aligned} \Big|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)\Big| &= \Big|\frac{1}{\Gamma(\alpha)}\int_1^{t_1} (\log\frac{t_2}{s})^{\alpha-1}\sigma_x(s)\frac{ds}{s} - \frac{1}{\Gamma(\alpha)}\int_1^{t_2} (\log\frac{t_1}{s})^{\alpha-1}\sigma_x(s)\frac{ds}{s}\Big| \\ &= \Big|\frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left[(\log\frac{t_2}{s})^{\alpha-1} - (\log\frac{t_1}{s})^{\alpha-1}\right]\sigma_x(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} (\log\frac{t_2}{s})^{\alpha-1}\sigma_x(s)\frac{ds}{s}\Big| \\ &\leq \frac{M}{\Gamma(\alpha)}\Big|\int_1^{t_1}\left[(\log\frac{t_2}{s})^{\alpha-1} - (\log\frac{t_1}{s})^{\alpha-1}\right]\frac{ds}{s}\Big| + \frac{M}{\Gamma(\alpha)}\Big|\int_{t_1}^{t_2} (\log\frac{t_2}{s})^{\alpha-1}\frac{ds}{s}\Big| \\ &\leq \frac{M}{\Gamma(\alpha+1)}\left[\Big|(\log t_1)^\alpha + (\log\frac{t_2}{t_1})^\alpha - (\log t_2)^\alpha\Big| + \Big|(\log\frac{t_2}{t_1})^\alpha\Big|\right] \\ &\leq \frac{M}{\Gamma(\alpha+1)}\left[\Big|(\log t_1)^\alpha - (\log t_2)^\alpha\Big|\right]. \end{aligned}$$

The right hand side of the above inequality tends to zero as $t_2 \longrightarrow t_1$, which implies that $\mathcal{F}$ is equicontinuous. As a consequence of Steps 1 to 3, by the Ascoli-Arzela theorem, we can conclude that the operator $\mathcal{F} : C([1-r_1, e+$

$r_2], \mathbb{R}) \longrightarrow C([1-r_1, e+r_2], \mathbb{R})$ is continuous and completely continuous.
Step 4: Now it remains to show that the set

$$\xi = \Big\{ x \in C([1-r_1, e+r_2], \mathbb{R}) : \; x = \lambda \mathcal{F} x, \text{ for some } \lambda \in (0,1) \Big\},$$

is bounded. Let $x \in \xi$, then $x = \lambda \mathcal{F} x$ for some $0 < \lambda < 1$. Thus for each $t \in J$, we have

$$x(t) = \lambda \Bigg(\frac{1}{\Gamma(\alpha)} \int_1^t (\log \frac{t}{s})^{\alpha-1} \sigma_x(s) \frac{ds}{s} + \frac{a \log t}{\Lambda \Gamma(\alpha+\delta)} \int_1^\eta (\log \frac{\eta}{s})^{\alpha+\delta-1} \sigma_x(s) \frac{ds}{s}$$
$$+ \frac{b \log t}{\Lambda \Gamma(\alpha)} \int_1^{e^-} (\log \frac{e^-}{s})^{\alpha-1} \sigma_x(s) \frac{ds}{s} + \frac{c (\log t)^{\gamma-1}}{\Lambda} \Bigg),$$

It follows from (3.19) that for each $t \in J$,

$$|x(t)| := l_3 < \infty,$$

it $t \in [1-r_1, 1]$ and by (3.17), then

$$|x(t)| := \|\phi\|_{[1-r_1,1]} + M_1 := l_1 < \infty,$$

it $t \in [e, e+r_2]$ and by (3.18), then

$$|x(t)| := \|\phi\|_{[e,e+r_2]} + M_2 := l_2 < \infty.$$

From which if follows that for each $t \in [1-r_1, e+r_2]$, we have $\|x\|_{[1-r_1,e+r_2]} \le k < \infty$, such that $k > 0$ is constant, this implies that ξ is bounded. As a consequence of the Schaefer's fixed point theorem, we deduce that $\mathcal{F}$ has a fixed point x which is a solution to problem (1.1)-(1.4). □

4 Example

Consider the following nonlinear problem

$$\begin{cases} {}^H_H D^{\frac{3}{2},\frac{1}{2}} x(t) = \dfrac{1}{e^t+9} \left(\dfrac{x_t}{x_t+1} - \dfrac{|{}^H_H D^{\frac{3}{2},\frac{1}{2}} x(t)|}{|{}^H_H D^{\frac{3}{2},\frac{1}{2}} x(t)| + 1} \right), & (4.1) \\ x(1^+) = 0, \; \dfrac{1}{2} {}_H I^{\frac{1}{3}} x(2) + 2 {}^H_H D^{1,1} x(e^-) = \dfrac{3}{4}, \; t \in (1,e), & (4.2) \\ x(t) + \dfrac{\sin x(t)}{e^t(1+x(t))} = \phi(t), \quad t \in [0,1], & (4.3) \\ x(t) + \dfrac{|x(t)|}{30 e^t (1+x(t))} = \psi(t), \quad t \in [e, e+2]. & (4.4) \end{cases}$$

We see that $\alpha = \frac{3}{2}$, $\beta = \frac{1}{2}$, $\eta = 2$, $\delta = \frac{1}{3}$, $a = \frac{1}{2}$, $b = 2$, $c = \frac{3}{4}$
and

$$f(t,u,v) = \frac{1}{e^t+9} \Big(\frac{u}{u+1} - \frac{v}{v+1} \Big),$$

for $t \in (1,e)$, $u \in C([-1,2], \mathbb{R})$, $v \in \mathbb{R}$. Clearly, the function f is continuous, and for u, $\overline{u} \in [-1,2]$, v, $\overline{v} \in \mathbb{R}$ and $t \in (1,e)$. We have

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \le \frac{1}{e^t+9} \big(\|u - \overline{u}\|_{[-1,2]} + |v - \overline{v}| \big)$$
$$\le \frac{1}{10} \|u - \overline{u}\|_{[-1,2]} + \frac{1}{10} |v - \overline{v}|.$$

Hence, condition (H2) is satisfied with $L_1 = L_2 = \dfrac{1}{10}$.

Thus condition (H3) is satised with $M = \dfrac{1}{10}$.

$$g_1(t) = \frac{\sin x(t)}{e^t(1+x(t))}, \; t \in [0,1], \; x \in C([0, e+2], \mathbb{R}),$$

$$g_2(t) = \frac{|x(t)|}{30e^t(1+x(t))}, \ t \in [e, e+2], \ x \in C([0, e+2], \mathbb{R}),$$

and let $x_1, \ x_2 \in C([0, e+2], \mathbb{R})$, we have, if $t \in [0, 1]$

$$|g_1(x_1)(t) - g_1(x_2)(t)| \leq \frac{1}{e^t} \|x_1 - x_2\|_{[0.e+2]},$$

then

$$\|g_1(x_1) - g_1(x_2)\|_{[0,1]} \leq \frac{1}{e} \|x_1 - x_2\|_{[0.e+2]},$$

and if

$$|g_2(x_1)(t) - g_2(x_2)(t)| \leq \frac{1}{30e^t} \|x_1 - x_2\|_{[0.e+2]},$$

then

$$\|g_2(x_1) - g_2(x_2)\|_{[e,e+2]} \leq \frac{1}{30e} \|x_1 - x_2\|_{[0.e+2]}.$$

Hence, condition (H4) is satisfied with $L_3 = \frac{1}{e}$, $L_4 = \frac{1}{30e}$.

$$\Omega := \frac{L_1}{(1 - L_2)} \left(\frac{1}{\Gamma(\alpha + 1)} + \frac{|a|(\log \eta)^{\alpha+\delta}}{|\Lambda|\Gamma(\alpha + \delta + 1)} + \frac{|b|}{|\Lambda|\Gamma(\alpha + 1)} \right) = 0.28528,$$

$$\rho = \max(\Omega, L_3, L_4) = 0.36788 < 1.$$

Since all the conditions of Theorem 3.1 are satisfied, it follows that the problem (4.1)-(4.4) has a unique solution $x \in C^2([1 - r_1, e + r_2], \mathbb{R})$.

5 Conclusion

In this work, we consider the implicit Hilfer-Hadamard fractional differential equations involving both retarded and advanced arguments and nonlocal mixed boundary conditions. We prove two theorems and we consider example to illustrate our results. In the first theorem, we prove the existence and uniqueness of the solution and in the second, we deal with the existence of at least one solution. The methods used are the Banach's fixed point Theorem and Schaefer's fixed point Theorem. Here, we remark that the Hilfer-Hadamard fractional derivative reduces to the Hadamard and Caputo fractional derivatives whenever $\beta = 0$ and $\beta = 1$, respectively.

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References

[1] S. Abbas, M. Benchohra and J. Henderson, *Existence and attractivity results for Hilfer fractional differential equations*, J. Math. Sci. **243** (2019), 347–357.

[2] S. Abbas, M. Benchohra and J.R. Graef, *Weak solutions to implicit differential equations involving the Hilfer fractional derivative*, Nonlinear Dyn. Syst. Theory **18** (2018), no. 1, 1-–11.

[3] S. Abbas, M. Benchohra, J.-E. Lagreg, A. Alsaedi and Y. Zhou, *Existence and Ulam stability for fractional differential equations of Hilfer-Hadamard type*, Adv. Difference. Equ. **180** (2017), 1–14.

[4] S. Abbas, M. Benchohra, J.E. Lagreg and Y. Zhou, *A survey on Hadamard and Hilfer fractional differential equations, Analysis and stability*, Chaos Solitons Fractals **102** (2017), 47–71.

[5] B. Ahmadn and S.K. Ntouyas, *Initial value problem of fractional order Hadamard-type functional differential equations*, Electron. J. Differ. Equ. **77** (2015), 1–9.

[6] B. Ahmad and S.K. Ntouyas, *Hilfer–Hadamard fractional boundary value problems with nonlocal mixed boundary conditions*, Fractal Fractional **5** (2021), 195.

[7] R.P. Agarwal, S.K. Ntouyas, B. Ahmad and A.K. Alzahrani, *Hadamard-type fractional functional differential equations and inclusions with retarded and advanced arguments*, Adv. Difference. Equ. **2016** (2016), 92.

[8] Y. Arioua and N. Benhamidouche, *Boundary value problem for Caputo-Hadamard fractional differential equations*, Surv. Math. Appl. **12** (2017), 103-–115.

[9] S.P. Bhairat, *Existence and continuation of solution of Hilfer fractional differential equations*, J. Math. Model. **7** (2019), no. 1, 1–20.

[10] M. Benchohra S. Bouriah and J. Henderson, *Nonlinear implicit Hadamard's fractional differential equations with retarded and advanced arguments*, Azer. J. Math. **8** (2018), no. 2.

[11] M. Benchohra, S. Bouriah and J.R. Graef, *Boundary value problems for nonlinear implicit Caputo–Hadamard-Type fractional differential equations with impulses*, Mediterr. J. Math. **14** (2017), no. 206, 1–21.

[12] M. Benchohra, J. R. Graef, N. Guerraiche and S. Hamani, *Nonlinear boundary value problems for fractional differential inclusions with Caputo-Hadamard derivatives on the half line*, AIMS Math. **6** (2021), no. 6, 6278—6292.

[13] M. Benchohra, J. Lazreg and G. N'Gúerékata, *Nonlinear implicit Hadamard's fractional differential equations on Banach space retarded and advanced arguments*, Int. J. Evol. Equ. **10** (2015), no. 3-4, 283—295.

[14] P.L. Butzer, A.A. Kilbas and J.J. Trujillo, *Fractional calculus in the Mellin setting and Hadamard-type fractional integrals*, J. Math. Anal. Appl. **269** (2019), 1—27.

[15] A. Boutiara, K. Guerbati and M. Benbachir, *Caputo-Hadamard fractional differential equation with three-point boundary conditions in Banach spaces*, AIMS Math. **5** (2019), no. 1, 259—272.

[16] Y.Y. Gambo, F. Jarad, D. Baleanu and T. Abdeljawad, *On Caputo modification of the Hadamard fractional derivatives*, Adv. Differ. Equ. **2014** (2014), no. 1, 1–12.

[17] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer, New York 2003.

[18] M. Haoues, A. Ardjouni and A. Djoudi, *Existence and uniqueness of solutions for the nonlinear retarded and advanced implicit Hadamard fractional differential equations with nonlocal conditions*, nonlinear stud. **27** (2020), no. 2, 433–445.

[19] R. Hilfer, *Threefold introduction to fractional derivatives*, Anomalous Transport: Foundations and Applications, 2008, pp. 17—73.

[20] R. Hilfer, *Applications of fractional calculus in physics*, Singapore: World Sc. Publ. Co., 2000.

[21] F. Jarad, T. Abdeljawad and D. Baleanu, *Caputo-type modification of the Hadamard fractional derivatives*, Adv. Differ. Equ. **2012** (2012), no. 1, Article 142, 8 pages.

[22] M.D. Kassim and N.E. Tatar, *Well-posedness and stability for a differential problem with Hilfer-Hadamard fractional derivative*, Abstr. Appl. Anal. **2013** (2013), 1–12.

[23] A.A. Kilbas, *Hadamard-type fractional calculus*, J. Korean Math. Soc. **38** (2001), no. 6, 1191–1204.

[24] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of the Fractional Differential Equations*, North-Holland Mathematics Studies, Elsevier: Amsterdam, The Netherlands, 2006.

[25] K. Oldham and J. Spanier, *The Fractional Calculus*, Acad. Press, New York, 1974.

[26] D.S. Oliveira and E.C. Oliveira, *Hilfer-Katugampola fractional derivative*, eprint arXiv:1705.07733v1 [math.CA]. 2017.

[27] I. Podlubny, *Fractional Differential Equations, in: Mathematics in Science and Engineering*, 198, Acad. Press, New York, 1999.

[28] M.D. Qassim, K.M. Furati and N.E. Tatar, *On a differential equation involving Hilfer-Hadamard fractional derivative*, Abst. Appl. Anal. **2012** (2012).

[29] D. Qian, Z. Gong and C. Li, *A generalized Grönwall inequality and its application to fractional differential equations with Hadamard derivatives*, Proc. the 3rd. Conf. on Nonlinear Sci. Complexity, Cankaya University, 2010.

[30] A.Y.A. Salamooni and D.D. Pawar, *Existence and uniqueness of nonlocal boundary conditions for Hilfer–Hadamard-type fractional differential*, Adv. Differ. Equ. **2021** (2021).

[31] A.Y.A. Salamooni and D.D. Pawar *Existence and stability results for Hilfer–Katugampola-type fractional implicit differential equations with nonlocal conditions*, J. Nonlinear Sci. Appl. **14** (2021), no. 3, 124-–138.

[32] A.Y.A. Salamooni and D.D. Pawar, *Existence and uniqueness of boundary value problems for Hilfer-Hadamard-type fractional differential equations*, Ganita **70** (2020), no. 2, 1-–16.

[33] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon Breach, Tokyo-Paris-Berlin, 1993.

[34] D. Vivek, K. Shah and K. Kanagarajan, *Dynamical analysis of Hilfer–Hadamard type fractional pantograph equations via successive approximation*, J. Taibah Univ. Sci. **13** (2019), no. 1, 225—230.

[35] D. Viveka, K. Kanagarajana and E.M. Elsayedb, *Nonlocal initial value problems for implicit differential equations with Hilfer–Hadamard fractional derivative*, Nonlinear Anal. Model. Control. **23** (2018), no. 3, 341—360.

[36] D. Vivek, K. Kanagrajan and E.M. Elsayed, *Some existence and stability results for Hilfer fractional implicit differential equations with nonlocal conditions*, Mediterr. J. Math. **15** (2018).

[37] J.R. Wang, Ch. Zhu and M. Feckan, *Solvability of fully nonlinear functional equations involving Erdélyi-Kober fractional integrals on the unbounded interval*, Optim. **63** (2014), 1235–1248.

[38] J.R. Wang and Y. Zhang, *Nonlocal initial value problems for differential equations with Hilfer fractional derivative*, Appl. Math. Comput. **266** (2015), 850–859.

[39] Y. Zhou. *Basic theory of fractional differential equations*. World Scientific, Singapore, 2014.