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Maclaurin coefficient estimates of te-univalent functions connected with the (p,q)-derivative

Ahmed M. Abd-Eltawab[a,*], Abbas Kareem Wanas[b]

[a]*Department of Mathematics, Faculty of Science, Fayoum University, Fayoum 63514, Egypt*

[b]*Department of Mathematics, College of Science, University of Al-Qadisiyah, Al Diwaniyah, Al-Qadisiyah, Iraq*

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Abstract

In this paper, we introduce a new subclass of analytic and te-univalent functions in the open unit disc associated with the operator $\mathcal{T}_{\zeta}^{\lambda,p,q}$, which is defined by using the (p,q)-derivative. We obtain the coefficient estimates and Fekete-Szegő inequalities for the functions belonging to this class.The various results presented in this paper would generalize and improve those in related works of several earlier authors.

Keywords: bi-univalent functions, coefficient bounds, Fekete-Szegő inequality, Hadamard product, (p,q)-derivative operator, te-univalent function
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1 Introduction

Let A denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Further, by S we shall denote the class of all functions in A which are univalent in U.

For the function f given by (1.1) and $\zeta \in A$ given by

$$\zeta(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{1.2}$$

the Hadamard product (or convolution) of f and ζ is defined by

$$(f * \zeta)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (\zeta * f)(z).$$

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*Corresponding author
Email addresses: ams03@fayoum.edu.eg (Ahmed M. Abd-Eltawab), abbas.kareem.w@qu.edu.iq (Abbas Kareem Wanas)

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For $b_n = 1$, $n \geq 2$, let $\zeta(z) = I(z)$, then $(f * I)(z) = f(z)$.

The theory of $q-$calculus plays an important role in many fields of mathematical, physical, and engineering sciences. The first application of the $q-$calculus was introduced by Jackson in [17, 18]. Recently, there is an extension of $q-$calculus, denoted by $(p, q)-$calculus which is obtained by substituting q by q/p in q-calculus. The $(p, q)-$integer was introduced by Chakrabarti and Jagannathan in [10]. For definitions and properties of the $(p, q)-$calculus, one may refer to [8, 27].

For $0 < q < p \leq 1$, the $(p; q)-$derivative operator for $f * \zeta$ is defined as in [2]:

$$D_{pq}(f*\zeta)(z) = \begin{cases} \frac{(f*\zeta)(pz)-(f*\zeta)(qz)}{(p-q)z}, & if \quad z \in U^* := U - \{0\} \\ f'(0), & if \quad z = 0 \end{cases}. \tag{1.3}$$

From (1.3) we deduce that

$$D_{pq}(f*\zeta)(z) = 1 + \sum_{n=2}^{\infty} [n, p, q]\, a_n b_n z^{n-1} \qquad (z \in U),$$

where the $(p, q)-$bracket number is given by

$$\begin{aligned} [n, p, q] &= \frac{p^n - q^n}{p - q} = \sum_{j=0}^{n-1} p^{n-(j+1)} q^j \\ &= p^{n-1} + p^{n-2}q + p^{n-3}q^2 + ... + q^{n-1} \qquad (0 < q < p \leq 1), \end{aligned} \tag{1.4}$$

which is a natural generalization of the $q-$number. Clearly, we note that $[n, 1, q] = [n]_q = \frac{1-q^n}{1-q}$, and $\lim_{q \longrightarrow 1^-} [n, 1, q] = n$.

By using (1.4) the $(p, q)-$shifted factorial is given by

$$[n, p, q]! = \begin{cases} 1, & if \quad n = 0 \\ \prod_{i=1}^{n} [i, p, q], & if \quad n \in \mathbb{N} := \{1, 2, 3, ...\} \end{cases},$$

and for any positive number δ, the $(p, q)-$generalized Pochhammer symbol is defined by

$$[\delta, p, q]_n = \begin{cases} 1, & if \quad n = 0 \\ \prod_{i=1}^{n} [\delta + i - 1, p, q], & if \quad n \in \mathbb{N} : \end{cases}.$$

For the functions f and ζ are given by (1.1) and (1.2), respectively, we define the linear operator $\mathcal{T}_\zeta^{\lambda,p,q} : A \to A$ by

$$\mathcal{T}_\zeta^{\lambda,p,q} f(z) * \mathcal{M}_{p,q,\lambda+1} = zD_{pq}(f*\zeta)(z) \qquad (\lambda > -1, 0 < q < p \leq 1, z \in U),$$

where the function $\mathcal{M}_{p,q,\lambda+1}$ is given by

$$\mathcal{M}_{p,q,\lambda+1} = z + \sum_{n=2}^{\infty} \frac{[\lambda+1, p, q]_{n-1}}{[n-1, p, q]!} z^n \qquad (\lambda > -1, 0 < q < p \leq 1, z \in U).$$

It is easy to find that

$$\mathcal{T}_\zeta^{\lambda,p,q} f(z) = z + \sum_{n=2}^{\infty} \Psi_{n-1} a_n z^n \qquad (\lambda > -1, 0 < q < p \leq 1, z \in U), \tag{1.5}$$

where

$$\Psi_{n-1} := \frac{[n,p,q]!}{[\lambda+1,p,q]_{n-1}} b_n, \qquad n \geq 2. \tag{1.6}$$

We note that $\mathcal{T}_\zeta^{0,1,q} f(z) \longrightarrow z(f*\zeta)'(z)$ as $\lambda = 0$, $p = 1$, and $q \longrightarrow 1^-$, where $(f*\zeta)'$ is the ordinary derivative of the function $f*\zeta$. Also, for $\lambda = b_n = 1$, we have $\mathcal{T}_I^{1,p,q} f(z) = f(z)$.

Remark 1.1. The linear operator $\mathcal{T}_\zeta^{\lambda,p,q}$ is a generalization of many other linear operators considered earlier, we obtain the next special cases:

(i) For $p = 1$, we obtain the operators

$$\mathcal{H}_\zeta^{\lambda,q} f(z) := z + \sum_{n=2}^{\infty} \Phi_{n-1} a_n z^n \qquad (\lambda > -1, 0 < q < 1, z \in U),$$

where

$$\Phi_{n-1} = \frac{[n,q]!}{[\lambda+1,q]_{n-1}} b_n,$$

and

$$\mathcal{T}_\zeta^{\lambda} f(z) := \lim_{q\longrightarrow 1^-} \mathcal{T}_\zeta^{\lambda,1,q} f(z) = z + \sum_{n=2}^{\infty} \frac{n!}{(\lambda+1)_{n-1}} a_n b_n z^n \qquad (\lambda > -1, z \in U),$$

where the operators $\mathcal{H}_\zeta^{\lambda,q}$ and $\mathcal{T}_\zeta^{\lambda}$ were introduced and studied by El-Deeb et al. [15];

(ii) For $p = 1$ and $b_n = \frac{(-1)^{n-1}\Gamma(\nu+1)}{4^{n-1}(n-1)!\Gamma(n+\nu)}$, $\nu > 0, \lambda > -1$, we obtain the operator

$$\mathcal{N}_{v,q}^{\lambda} f(z) := z + \sum_{n=2}^{\infty} \frac{[n,q]!}{[\lambda+1,q]_{n-1}} \frac{(-1)^{n-1}\Gamma(\nu+1)}{4^{n-1}(n-1)!\Gamma(n+\nu)} a_n z^n \qquad (z \in U),$$

where the operator $\mathcal{N}_{v,q}^{\lambda}$ was studied by El-Deeb and Bulboacă [14]:

(iii) For $p = 1$ and $b_n = \left(\frac{k+1}{k+n}\right)^\alpha$, $\alpha > 0, k \geq 0$, we obtain the operator

$$\mathcal{M}_{k,q}^{\lambda,\alpha} f(z) := z + \sum_{n=2}^{\infty} \left(\frac{k+1}{k+n}\right)^\alpha \frac{[n,q]!}{[\lambda+1,q]_{n-1}} a_n z^n \ \ (z \in U),$$

where the operator $\mathcal{M}_{k,q}^{\lambda,\alpha}$ was studied by El-Deeb and Bulboacă [13];

(iv) For $p = 1$ and $b_n = 1$, we obtain the the operator

$$\mathcal{J}_q^{\lambda} f(z) := z + \sum_{n=2}^{\infty} \frac{[n,q]!}{[\lambda+1,q]_{n-1}} a_n z^n \ \ (z \in U),$$

where the operator $\mathcal{J}_q^{\lambda}$ was studied by Arif et al. [5];

(v) For $p = 1$ and $b_n = \frac{m^{n-1}}{(n-1)!} e^{-m}$, $m > 0$, we obtain the q-analogue of Poisson operator:

$$\mathcal{I}_q^{\lambda,m} f(z) := z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} . \frac{[n,q]!}{[\lambda+1,q]_{n-1}} a_n z^n \ \ (z \in U),$$

where the operator $\mathcal{I}_q^{\lambda,m}$ was studied by Porwal [25];

(vi) For $p = 1$ and $b_n = \left[\frac{1+\ell+\mu(k-1)}{1+\ell}\right]^m$, $m \in \mathbb{Z}, \ell > 0, \mu \geq 0$, we obtain the q-analogue of Prajapat operator [26], defined by:

$$\mathcal{J}_{q,\ell,\mu}^{\lambda,m} f(z) := z + \sum_{n=2}^{\infty} \left[\frac{1+\ell+\mu(n-1)}{1+\ell}\right]^m . \frac{[n,q]!}{[\lambda+1,q]_{n-1}} a_n z^n \ \ (z \in U).$$

According to the Koebe one-quarter theorem Duren [12], it ensures that the images of U under every univalent functions $f \in S$ contains a disc of radius $\frac{1}{4}$. Thus, every univalent function f on U has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in U),$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \tag{1.7}$$

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ denote the class of all bi-univalent functions in U given by (1.1). Some examples of functions in the class Σ are $\frac{z}{1-z}$, $-\log(1-z)$, and $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$.

Abd-Eltawab [1] introduced the concept of te-univalency associated with an operator, which is a generalization and extension of the concept of bi-univalency. Let $\mathcal{S}_\zeta^{\lambda,p,q}$ denote the class of all functions given by (1.5), which are univalent in U. It is well known that every function $\mathcal{T}_\zeta^{\lambda,p,q}f \in \mathcal{S}_\zeta^{\lambda,p,q}$ has an inverse $\left(\mathcal{T}_\zeta^{\lambda,p,q}f\right)^{-1}$, defined by

$$h(\mathcal{T}_\zeta^{\lambda,p,q}f(z)) = z \quad (z \in U)$$

and

$$\mathcal{T}_\zeta^{\lambda,p,q}f(h(w)) = w \quad \left(|w| < r_0(\mathcal{T}_\zeta^{\lambda,p,q}f); r_0(\mathcal{T}_\zeta^{\lambda,p,q}f) \geq \frac{1}{4}\right),$$

where

$$\begin{aligned} h(w) = \left(\mathcal{T}_\zeta^{\lambda,p,q}f\right)^{-1}(w) &= w - \Psi_1 a_2 w^2 + \left[2\Psi_1^2 a_2^2 - \Psi_2 a_3\right] w^3 \\ &- \left[5\Psi_1^3 a_2^3 - 5\Psi_1\Psi_2 a_2 a_3 + \Psi_3 a_4\right] w^4 + \cdots, \end{aligned} \tag{1.8}$$

and Ψ_{n-1} is given by (1.6). We note that $h(w) = g(w)$ as $\lambda = b_n = 1$, where g is given by (1.7)

A function f given by (1.1) is said to be te-univalent in U associated with the operator $\mathcal{T}_\zeta^{\lambda,p,q}$, if both $\mathcal{T}_\zeta^{\lambda,p,q}f$ and $\left(\mathcal{T}_\zeta^{\lambda,p,q}f\right)^{-1}$ are univalent in U. Let $\Sigma_\zeta^{\lambda,p,q}$ denote the class of all functions given by (1.1), which are te-univalent in U associated with $\mathcal{T}_\zeta^{\lambda,p,q}$.

For two functions f and ζ, which are analytic in U, we say that f is subordinate to ζ, written $f(z) \prec \zeta(z)$ if there exists a Schwarz function s, which (by definition) is analytic in U with $s(0) = 0$ and $|s(z)| < 1$ for all $z \in U$, such that $f(z) = \zeta(s(z))$, $z \in U$. Furthermore, if the function ζ is univalent in U, then we have the following equivalence, (cf., e.g., [9] , and [21]):

$$f(z) \prec \zeta(z) \Leftrightarrow f(0) = \zeta(0) \text{ and } f(U) \subset \zeta(U).$$

Ma and Minda [20] unified various subclasses of starlike and convex functions consist of functions $f \in A$ satisfying the subordination $\frac{zf'(z)}{f(z)} \prec \varphi(z)$ and $1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z)$ respectively. A function f is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f^{-1} are respectively Ma-Minda starlike or convex (see [3]). Many interesting examples of the functions of the class Σ, together with various other properties and characteristics associated with bi-univalent functions can be found in the earlier works (see [6, 19, 22] and others). Brannan and Taha [7] introduced certain subclasses of bi-univalent functions similar to the familiar subclasses of univalent functions consisting of starlike, convex and strongly starlike functions. They investigated the bound on the initial coefficients of the classes bi-starlike and bi-convex functions. Recently, many researchers (see [4, 11, 15, 23, 30, 32]) introduced and investigated some new subclasses of Σ and obtained bounds for the initial coefficients of the function given by (1.1). For a brief history and interesting examples in the class Σ (see [29]).

Earlier in 1933, Fekete and Szegö [16] made use of Lowner's parametric method in order to prove that, if $f \in S$ and is given by (1.1),

$$\left|a_3 - \mu a_2^2\right| \leq 1 + 2\exp\left(-\frac{2\xi}{1-\xi}\right) \quad (0 \leq \xi \leq 1, \mu \in \mathbb{C}).$$

For some history of Feketo-Szegő problem for class of starlike, convex and close-to-convex functions, refer to work produced by by Srivastava et al. [28]. Besides that, some authors [1, 15, 31] have studied the Feketo-Szegő inequalities for certain subclasses of bi-univalent functions.

The object of the present paper is to introduce a new subclass of analytic and te-univalent functions in the open unit disc associated with the operator $\mathcal{T}_{\zeta}^{\lambda,p,q}$, and the bound for second and third coefficients of functions in this class are obtained. Also the Fekete-Szegő inequality is determined for this function class. The results presented in this paper would generalize and improve some recent works of [3, 7, 11, 15].

In order to derive our main results we need to use the following lemma:

Lemma 1.2 ([24]). If $p \in \mathcal{P}$ then $|c_n| \leq 2$ for each n, where $\mathcal{P}$ is the family of all functions p, analytic in U, for which

$$Re\{p(z)\} > 0 \qquad (z \in U),$$

where

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots \quad (z \in U).$$

2 Coefficient Estimates for the Function Class $\mathfrak{T}_{\Sigma}^{\lambda,p,q}(\eta,\zeta,\varphi)$

We begin this section by assuming that φ is an analytic function with positive real part in U, with $\varphi(0) = 1$, $\varphi'(0) > 0$ and $\varphi(U)$ maps the unit disc U onto a region starlike with respect to 1, and symmetric with respect to the real axis. Such a function has a series expansion of the form:

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots. \quad \text{with} \quad B_1 > 0. \tag{2.1}$$

Unless otherwise mentioned, we assume throughout this paper that, the function φ satisfies the above conditions, $\lambda > -1, 0 < q < p \leq 1, \eta \in \mathbb{C} - \{0\}$ and $z \in U$.

Definition 2.1. A function f given by (1.1) is said to be in the class $\mathfrak{T}_{\Sigma}^{\lambda,p,q}(\eta,\zeta,\varphi)$, if the following subordination conditions hold true:

$$f \in \Sigma_{\zeta}^{\lambda,p,q}, \text{with} \;\; 1 + \frac{1}{\eta}\left(\frac{zD_{pq}\left(\mathcal{T}_{\zeta}^{\lambda,p,q} f(z)\right)}{\mathcal{T}_{\zeta}^{\lambda,p,q} f(z)} - 1\right) \prec \varphi(z), \tag{2.2}$$

and

$$1 + \frac{1}{\eta}\left(\frac{zD_{pq}(h(w))}{h(w)} - 1\right) \prec \varphi(z), \tag{2.3}$$

where the functions ζ and h are given by (1.2) and (1.8), respectively.

It is interesting to note that the special values of parameters λ, p, q, η, φ and b_n, $n \geq 2$, the class $\mathfrak{T}_{\Sigma}^{\lambda,p,q}(\eta,\zeta,\varphi)$ unifies the following known and new classes:

(i) $\mathfrak{T}_{\Sigma}^{\lambda,1,q}[\eta,\zeta,\varphi] = \mathfrak{T}_{\Sigma}^{\lambda,q}[\eta,\zeta,\varphi]$ improves the class $\mathcal{L}_{\Sigma}^{\lambda,q}[\eta,\zeta,\varphi]$, which was introduced and studied by El-Deeb et al.[15];

(ii) $\lim_{q\longrightarrow 1^-} \mathfrak{T}_{\Sigma}^{\lambda,1,q}[\eta,\zeta,\varphi] = \mathfrak{T}_{\Sigma}^{\lambda}[\eta,\zeta,\varphi]$ improves the class $\mathcal{G}_{\Sigma}^{\lambda}[\eta,\zeta,\varphi]$, which was introduced and studied by El-Deeb et al.[15];

(iii) $\mathfrak{T}_{\Sigma}^{\lambda,p,q}\left(\eta,\zeta,\left(\frac{1+z}{1-z}\right)^{\alpha}\right) = \mathcal{S}_{\Sigma}^{*\lambda,p,q}(\eta,\zeta,\alpha) \;\; (0 < \alpha \leq 1)$;

(iv) $\mathfrak{T}_{\Sigma}^{\lambda,p,q}\left(\eta,\zeta,\frac{1+(1-2\beta)z}{1-z}\right) = \mathcal{S}_{\Sigma}^{*\lambda,p,q}(\eta,\zeta,\beta) \;\; (0 \leq \beta < 1)$;

(v) $\lim_{q\longrightarrow 1^-} \mathfrak{T}_{\Sigma}^{1,1,q}(\eta,I,\varphi) = \mathcal{S}_{\Sigma}^{*}(\eta,\varphi)$, where the class $\mathcal{S}_{\Sigma}^{*}(\eta,\varphi)$ was introduced and studied by Deniz [11];

(vi) $\lim_{q\longrightarrow 1^-} \mathfrak{T}_{\Sigma}^{1,1,q}(1,I,\varphi) = \mathcal{S}_{\Sigma}^{*}(\varphi)$, where the class $\mathcal{S}_{\Sigma}^{*}(\varphi)$ was introduced and studied by Ali et al. [3];

(vii) $\lim_{q\longrightarrow 1^-} \mathfrak{T}_{\Sigma}^{1,1,q}\left(1, I, \left(\frac{1+z}{1-z}\right)^{\alpha}\right) = \mathcal{S}_{\Sigma}^{*}(\alpha)$ $(0 < \alpha \leq 1)$, where the class $\mathcal{S}_{\Sigma}^{*}(\alpha)$ was introduced and studied by Brannan and Taha [7];

(viii) $\lim_{q\longrightarrow 1^-} \mathfrak{T}_{\Sigma}^{1,1,q}\left(1, I, \frac{1+(1-2\beta)z}{1-z}\right) = \mathcal{S}_{\Sigma}^{*}(\beta)$ $(0 \leq \beta < 1)$, where the class $\mathcal{S}_{\Sigma}^{*}(\beta)$ was introduced and studied by Brannan and Taha [7].

Theorem 2.2. If the function f given by (1.1) belongs to the class $\mathfrak{T}_{\Sigma}^{\lambda,p,q}(\eta,\zeta,\varphi)$, then

$$|a_2| \leq \frac{|\eta|\, B_1\sqrt{B_1}}{\Psi_1\sqrt{\left|\eta\left[(q-1)(p+q)+p^2\right]B_1^2+(p+q-1)^2(B_1-B_2)\right|}}, \tag{2.4}$$

and

$$|a_3| \leq \frac{|\eta|}{\Psi_2}\left[\frac{B_1+|B_2-B_1|}{|(q-1)(p+q)+p^2|}\right], \tag{2.5}$$

where $\Psi_{n-1}, n\in\{2,3\}$ is given by (1.6).

Proof . If $f \in \mathfrak{T}_{\Sigma}^{\lambda,p,q}(\eta,\zeta,\varphi)$, from (2.2), (2.3), and the definition of subordination it follows that there exist two analytic functions $u, v : U \longrightarrow U$ with $u(0) = v(0) = 0$, such that

$$\frac{zD_{pq}\left(\mathcal{T}_{\zeta}^{\lambda,p,q}f(z)\right)}{\mathcal{T}_{\zeta}^{\lambda,p,q}f(z)} - 1 = \eta\left[\varphi(u(z))-1\right], \tag{2.6}$$

and

$$\frac{zD_{pq}(h(w))}{h(w)} - 1 = \eta\left[\varphi(v(w))-1\right]. \tag{2.7}$$

We define the functions r and s in $\mathcal{P}$ given by

$$r(z) = \frac{1+u(z)}{1-u(z)} = 1+u_1z+u_2z^2+u_3z^3+..., \tag{2.8}$$

and

$$s(z) = \frac{1+v(z)}{1-v(z)} = 1+v_1z+v_2z^2+v_3z^3+.... \tag{2.9}$$

It follows from (2.8) and (2.9) that

$$u(z) = \frac{r(z)-1}{r(z)+1} = \frac{u_1}{2}z+\frac{1}{2}\left(u_2-\frac{u_1^2}{2}\right)z^2+..., \tag{2.10}$$

and

$$v(z) = \frac{s(z)-1}{s(z)+1} = \frac{v_1}{2}z+\frac{1}{2}\left(v_2-\frac{v_1^2}{2}\right)z^2+.... \tag{2.11}$$

Using (2.10) and (2.11) with (2.1) lead us to

$$\eta\left[\varphi(u(z))-1\right] = \frac{\eta B_1 u_1}{2}z+\eta\left[\frac{1}{2}\left(u_2-\frac{u_1^2}{2}\right)B_1+\frac{1}{4}u_1^2B_2\right]z^2+...,$$

and

$$\eta\left[\varphi(v(z))-1\right] = \frac{\eta B_1 v_1}{2}z+\eta\left[\frac{1}{2}\left(v_2-\frac{v_1^2}{2}\right)B_1+\frac{1}{4}v_1^2B_2\right]z^2+....$$

On the other hand,

$$\begin{aligned}&\frac{zD_{pq}\left(\mathcal{T}_{\zeta}^{\lambda,p,q}f(z)\right)}{\mathcal{T}_{\zeta}^{\lambda,p,q}f(z)} - 1\\ =\ &(p+q-1)\Psi_1a_2z+\left[\left(q(p+q)+p^2-1\right)\Psi_2a_3-(p+q-1)\Psi_1^2a_2^2\right]z^2+...,\end{aligned}$$

and

$$\begin{aligned}\frac{zD_{pq}\left(h(w)\right)}{h(w)}-1 &= -\left(p+q-1\right)\Psi_1 a_2 w \\ &+ \left[\left(\left(2q-1\right)\left(p+q\right)+2p^2-1\right)\Psi_1^2 a_2^2-\left(q\left(p+q\right)+p^2-1\right)\Psi_2 a_3\right]w^2+....\end{aligned}$$

Now, equating the coefficients in (2.6) and (2.7), we get

$$\left(p+q-1\right)\Psi_1 a_2=\frac{\eta B_1 u_1}{2}, \tag{2.12}$$

$$\left(q\left(p+q\right)+p^2-1\right)\Psi_2 a_3-\left(p+q-1\right)\Psi_1^2 a_2^2=\eta\left[\frac{1}{2}\left(u_2-\frac{u_1^2}{2}\right)B_1+\frac{1}{4}u_1^2 B_2\right], \tag{2.13}$$

$$-\left(p+q-1\right)\Psi_1 a_2 w=\frac{\eta B_1 v_1}{2}, \tag{2.14}$$

and

$$\left(\left(2q-1\right)\left(p+q\right)+2p^2-1\right)\Psi_1^2 a_2^2-\left(q\left(p+q\right)+p^2-1\right)\Psi_2 a_3=\eta\left[\frac{1}{2}\left(v_2-\frac{v_1^2}{2}\right)B_1+\frac{1}{4}v_1^2 B_2\right]. \tag{2.15}$$

From (2.12) and (2.14), we get

$$u_1=-v_1 \tag{2.16}$$

and

$$2\left(p+q-1\right)^2\Psi_1^2 a_2^2=\frac{\eta^2 B_1^2}{4}\left(u_1^2+v_1^2\right). \tag{2.17}$$

Now from (2.13), (2.15) and (2.17), we obtain

$$\begin{aligned}2\left[\left(q-1\right)\left(p+q\right)+p^2\right]\Psi_1^2 a_2^2 &= \frac{\eta B_1}{2}\left(u_2+v_2\right)+\frac{\eta\left(B_2-B_1\right)}{4}\left(u_1^2+v_1^2\right) \\ &= \frac{\eta B_1}{2}\left(u_2+v_2\right)+\frac{2\left(B_2-B_1\right)\left(p+q-1\right)^2\Psi_1^2 a_2^2}{\eta B_1^2}.\end{aligned}$$

Therefore, we have

$$a_2^2=\frac{\eta^2 B_1^3\left(u_2+v_2\right)}{4\Psi_1^2\left[\eta\left[\left(q-1\right)\left(p+q\right)+p^2\right]B_1^2+\left(p+q-1\right)^2\left(B_1-B_2\right)\right]}. \tag{2.18}$$

Using the Lemma 1.2 that $|u_2|\leq 2$ and $|v_2|\leq 2$, we immediately have the bound for $|a_2|$ as asserted in (2.4).

Next, in order to find the bound on $|a_3|$, by subtracting (2.15) from (2.13) and using (2.16), we get

$$\begin{aligned}&2\left(q\left(p+q\right)+p^2-1\right)\Psi_2 a_3-\ 2\left(q\left(p+q\right)+p^2-1\right)\Psi_1^2 a_2^2 \\ &= \eta\left[\frac{1}{2}\left(u_2-\frac{u_1^2}{2}\right)B_1+\frac{1}{4}u_1^2 B_2\right]-\eta\left[\frac{1}{2}\left(v_2-\frac{v_1^2}{2}\right)B_1+\frac{1}{4}v_1^2 B_2\right] \\ &= \frac{\eta}{2}B_1\left(u_2-v_2\right).\end{aligned} \tag{2.19}$$

It follows from (2.15) and (2.19) that

$$\left(\left(q-1\right)\left(p+q\right)+p^2\right)\Psi_2 a_3=\frac{\eta\left[\left(2q-1\right)\left(p+q\right)+2p^2-1\right]B_1\left(u_2-v_2\right)}{4\left(q\left(p+q\right)+p^2-1\right)}+\frac{\eta}{2}B_1 v_2+\frac{\eta}{4}\left(B_2-B_1\right)v_1^2,$$

and then,

$$a_3=\frac{\eta}{\Psi_2}\left[\frac{\left[\left(\left(2q-1\right)\left(p+q\right)+2p^2-1\right)u_2+\left(p+q-1\right)v_2\right]B_1+v_1^2\left(q\left(p+q\right)+p^2-1\right)\left(B_2-B_1\right)}{4\left(q\left(p+q\right)+p^2-1\right)\left(\left(q-1\right)\left(p+q\right)+p^2\right)}\right]. \tag{2.20}$$

Taking the absolute value of (2.20), and applying Lemma 1.2 once again for the coefficients v_1, v_2 and u_2, we readily get the inequality (2.5). □

Taking $p=1$ in Theorem 2.2, we obtain the following corollary which improves the result of El-Deeb et al. [[15],Theorem 1].

Corollary 2.3. If the function f given by (1.1) belongs to the class $\mathfrak{T}_{\Sigma}^{\lambda,q}(\eta,\zeta,\varphi)$, then

$$|a_2| \leq \frac{|\eta| B_1\sqrt{B_1}}{q\Psi_1\sqrt{|\eta B_1^2 + B_1 - B_2|}},$$

and

$$|a_3| \leq \frac{|\eta|}{q^2\Psi_2}\left[B_1 + |B_2 - B_1|\right] ,$$

where $\Psi_{n-1}, n \in \{2,3\}$ is given by (1.6).

Taking $q \longrightarrow 1^-$ in Corollary 2.3, we obtain the following corollary which improves the result of El-Deeb et al. [[15], Corollary 1].

Corollary 2.4. If the function f given by (1.1) belongs to the class $\mathfrak{T}_{\Sigma}^{\lambda}(\eta,\zeta,\varphi)$, then

$$|a_2| \leq \frac{|\eta| B_1\sqrt{B_1}}{\Psi_1\sqrt{|\eta B_1^2 + B_1 - B_2|}},$$

and

$$|a_3| \leq \frac{|\eta|}{\Psi_2}\left[B_1 + |B_2 - B_1|\right] ,$$

where $\Psi_{n-1}, n \in \{2,3\}$ is given by (1.6).

Taking $\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + ...$ $(0 < \alpha \leq 1)$ in Theorem 2.2, we obtain the following corollary

Corollary 2.5. If the function f given by (1.1) belongs to the class $\mathcal{S}_{\Sigma}^{*\lambda,p,q}(\eta,\zeta,\alpha)$, then

$$|a_2| \leq \frac{2|\eta|\alpha}{\Psi_1\sqrt{\left|2\eta\left[(q-1)(p+q)+p^2\right]\alpha + (p+q-1)^2(1-\alpha)\right|}}, \tag{2.21}$$

and

$$|a_3| \leq \frac{|\eta|}{\Psi_2}\left[\frac{2\alpha(1+|\alpha-1|)}{|(q-1)(p+q)+p^2|}\right] , \tag{2.22}$$

where $\Psi_{n-1}, n \in \{2,3\}$ is given by (1.6).

Taking $\varphi(z) = \frac{1+(1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + ...$ $(0 \leq \beta < 1)$ in Theorem 2.2, we obtain the following corollary.

Corollary 2.6. If the function f given by (1.1) belongs to the class $\mathcal{S}_{\Sigma}^{*\lambda,p,q}(\eta,\zeta,\beta)$, then

$$|a_2| \leq \frac{1}{\Psi_1}\sqrt{\frac{2|\eta|(1-\beta)}{|(q-1)(p+q)+p^2|}}, \tag{2.23}$$

and

$$|a_3| \leq \frac{2|\eta|}{\Psi_2}\left[\frac{(1-\beta)}{|(q-1)(p+q)+p^2|}\right] , \tag{2.24}$$

where $\Psi_{n-1}, n \in \{2,3\}$ is given by (1.6).

Remark 2.7. (i) Taking $q \longrightarrow 1^-$ and $\lambda = b_n = 1$ in Corollary 2.3, we obtain the result obtained by Deniz [[11], Corollary 2.3];

(ii) Taking $q \longrightarrow 1^-$ and $\eta = \lambda = b_n = 1$ in Corollary 2.3, we obtain the result obtained by Ali et al. [[3], Corollary 2.1];

(iii) Taking $q \longrightarrow 1^-$ and $p = \eta = \lambda = b_n = 1$ in Corollary 2.5, the inequality in (2.21) reduces to the estimates obtained by Brannan and Taha [[7],Theorem 2.1];

(iv) Taking $q \longrightarrow 1^-$ and $p = \eta = \lambda = b_n = 1$ in Corollary 2.6, we obtain the result obtained by Brannan and Taha [[7],Theorem 3.1];

3 Fekete-Szegő Proplem for the Function Class $\mathfrak{T}_{\Sigma}^{\lambda,p,q}(\eta,\zeta,\varphi)$.

Theorem 3.1. If the function f given by (1.1) belongs to the class $\mathfrak{T}_{\Sigma}^{\lambda,p,q}(\eta,\zeta,\varphi)$, then

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|\eta|\, B_1}{2\Psi_2}\left(\left|L(\mu) + \frac{1}{q(p+q)+p^2-1}\right| + \left|L(\mu) - \frac{1}{q(p+q)+p^2-1}\right|\right) \tag{3.1}$$

with

$$L(\mu) = \frac{\eta B_1^2\left(1 - \frac{\Psi_2}{\Psi_1^2}\mu\right)}{\eta\left[(q-1)(p+q)+p^2\right]B_1^2 + (p+q-1)^2(B_1 - B_2)} \tag{3.2}$$

where $\mu \in \mathbb{C}$ and $\Psi_{n-1}, n \in \{2,3\}$ is given by (1.6).

Proof . If $f \in \mathfrak{T}_{\Sigma}^{\lambda,p,q}(\eta,\zeta,\varphi)$ like in the proof of Theorem 2.2, from (2.19) we have,

$$a_3 - \frac{\Psi_1^2}{\Psi_2}a_2^2 = \frac{\eta B_1(u_2 - v_2)}{4\Psi_2\left(q(p+q)+p^2-1\right)} \tag{3.3}$$

.

Multiplying (2.18) by $\left(\frac{\Psi_1^2}{\Psi_2} - \mu\right)$ we get:

$$\left(\frac{\Psi_1^2}{\Psi_2} - \mu\right)a_2^2 = \frac{\eta^2 B_1^3\left(\frac{\Psi_1^2}{\Psi_2} - \mu\right)(u_2 + v_2)}{4\Psi_1^2\left[\eta\left[(q-1)(p+q)+p^2\right]B_1^2 + (p+q-1)^2(B_1 - B_2)\right]} \tag{3.4}$$

Adding (3.3) and (3.4), it follows that

$$a_3 - \mu a_2^2 = \frac{\eta B_1}{4\Psi_2}\left[\left(L(\mu) + \frac{1}{q(p+q)+p^2-1}\right)u_2 + \left(L(\mu) - \frac{1}{q(p+q)+p^2-1}\right)v_2\right] \tag{3.5}$$

where $L(\mu)$ is given by (3.2).

Taking the absolute value of (3.5), and applying Lemma 1.2 for the coefficients v_2 and u_2 we obtain the inequality (3.1). □

Taking $p = 1$ in Theorem3.1, we obtain the following corollary which improves the result of El-Deeb et al. [[15],Theorem 2].

Corollary 3.2. If the function f given by (1.1) belongs to the class $\mathfrak{T}_{\Sigma}^{\lambda,q}(\eta,\zeta,\varphi)$, then

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|\eta|\, B_1}{2\Psi_2}\left(\left|L(\mu) + \frac{1}{q(q+1)}\right| + \left|L(\mu) - \frac{1}{q(q+1)}\right|\right)$$

with

$$L(\mu) = \frac{\eta B_1^2\left(1 - \frac{\Psi_2}{\Psi_1^2}\mu\right)}{q^2\left[\eta B_1^2 + B_1 - B_2\right]}$$

where $\mu \in \mathbb{C}$ and $\Psi_{n-1}, n \in \{2,3\}$ is given by (1.6).

Taking $q \longrightarrow 1^-$ in Corollary3.2, we obtain the following corollary which improves the result of El-Deeb et al. [[15], Corollary 5].

Corollary 3.3. If the function f given by (1.1) belongs to the class $\mathfrak{T}_{\Sigma}^{\lambda}(\eta,\zeta,\varphi)$, then

$$\left|a_3-\mu a_2^2\right| \leq \frac{|\eta|\, B_1}{2\Psi_2}\left(\left|L(\mu)+\frac{1}{2}\right|+\left|L(\mu)-\frac{1}{2}\right|\right)$$

with

$$L(\mu)=\frac{\eta B_1^2\left(1-\frac{\Psi_2}{\Psi_1^2}\mu\right)}{\eta B_1^2+B_1-B_2}$$

where $\mu\in\mathbb{C}$ and $\Psi_{n-1}, n\in\{2,3\}$ is given by (1.6).

Taking $\varphi(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}$ $(0<\alpha\leq 1)$ in Corollary3.2, we obtain the following corollary which improves the result of El-Deeb et al. [[15], Example 3].

Corollary 3.4. If the function f given by (1.1) belongs to the class $\mathfrak{T}_{\Sigma}^{\lambda,q}\left(\eta,\zeta,\left(\frac{1+z}{1-z}\right)^{\alpha}\right)$, then

$$\left|a_3-\mu a_2^2\right| \leq \frac{|\eta|\,\alpha}{\Psi_2}\left(\left|L(\mu)+\frac{1}{q(q+1)}\right|+\left|L(\mu)-\frac{1}{q(q+1)}\right|\right)$$

with

$$L(\mu)=\frac{2\eta\alpha\left(1-\frac{\Psi_2}{\Psi_1^2}\mu\right)}{q^2\left[(2\eta-1)\alpha+1\right]}$$

where $\mu\in\mathbb{C}$ and $\Psi_{n-1}, n\in\{2,3\}$ is given by (1.6).

Taking $\varphi(z)=\frac{1+(1-2\beta)z}{1-z}$ $(0\leq\beta<1)$ in Corollary3.2, we obtain the following corollary which improves the result of El-Deeb et al.[[15], Remark 6].

Corollary 3.5. If the function f given by (1.1) belongs to the class $\mathfrak{T}_{\Sigma}^{\lambda,q}\left(\eta,\zeta,\frac{1+(1-2\beta)z}{1-z}\right)$, then

$$\left|a_3-\mu a_2^2\right| \leq \frac{|\eta|\,(1-\beta)}{\Psi_2}\left(\left|L(\mu)+\frac{1}{q(q+1)}\right|+\left|L(\mu)-\frac{1}{q(q+1)}\right|\right)$$

with

$$L(\mu)=\frac{1}{q^2}\left(1-\frac{\Psi_2}{\Psi_1^2}\mu\right)$$

where $\mu\in\mathbb{C}$ and $\Psi_{n-1}, n\in\{2,3\}$ is given by (1.6).

Remark 3.6. We mention that all the above estimations for the first two Taylor-Maclaurin coefficients and Fekete-Szegő problem for the function class $\mathfrak{T}_{\Sigma}^{\lambda,p,q}(\eta,\zeta,\varphi)$ are not sharp. To find the sharp upper bounds for the above function class, it still is an interesting open problem, as well as for $|a_n|$, $n\geq 4$.

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