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On T_β-contractive mappings: Fixed point results with an application

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Abstract

We introduce the concept of T_β-contractive mappings in the framework of bounded metric spaces and prove the existence of a fixed point for such mappings without using neither the compactness nor the uniform convexity of the space. Furthermore, a fixed point theorem for T_β-weakly contractive maps has been given. We point out that, these results generalize and improve many previous works in the literature. Ultimately, one of our theoretical results has been implicated to study the existence of the solution to a class of functional equations arising in dynamic programming under new weak conditions.

Keywords: Fixed point, T_β-contractive mappings, T_β-weakly contractive maps, α-admissible, τ-distance
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1 Introduction

Let $(X, \| \,.\, \|)$ be a Banach space and $T : X \to X$ is a nonexpansive mapping whose Lipschitz's constant equal to 1, that is $\|Tx - Ty\| \leq \| x - y \|$ for all $x, y \in X$. Since 1965 considerable effort has been aimed to study the fixed point theory for nonexpansive mappings in the setting of Banach spaces and metric spaces. In 1965, F. E. Browder [5] and D. Göhde [9] independently proved that every nonexpansive mapping of a closed convex and bounded subset of the Banach space X has a fixed point, if the subset is supposed to be uniformly convex (for each $0 < \varepsilon \leq 2$, there exists $\delta > 0$ such that for all $\|x\| \leq 1, \|y\| \leq 1$ the condition $\|x - y\| \geq \varepsilon$ implies that $\|\frac{x+y}{2}\| \leq 1 - \delta$ see [6]).

In 2012, Samet et al [12] introduced the notion of α-admissible mappings. By using this concept, the authors defined α-ψ- contractive mappings and proved a remarkable fixed point for such mappings in the setting of metric spaces.

Very recently, the authors in [14](2019) established some fixed point theorems without using any additional condition on the space, in other words, they showed some results for a class of mappings $T : X \to X$ satisfying

$$\inf_{x \neq y \in X} \{d(x, y) - d(Tx, Ty)\} > 0, \tag{1.1}$$

where (X, d) is a complete metric space, not necessarily compact. In this direction, recent works can be found in [15, 16, 17, 18, 19, 20, 21].

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So, it is a very natural question if we can extend (1.1) to

$$\inf_{x\neq y\in X}\{d(x,y)-d(Tx,Ty)\}\geq 0 \tag{1.2}$$

and prove a fixed point theorem for this type. In this paper, using the notion of α-admissible mapping [12] and the concept of τ-distances defined in [1] we give an affirmative answer to the above question. That is to say, we introduce the concept of T_β-contractive mappings and establish a fixed point theorem for this type of contractions which is a class of nonexpansive mappings without using neither the compactness nor the uniform convexity.

Furthermore, based on our first result and motivated by the notion of E-weakly contractive maps defined in [14](see also, weakly contractive maps defined in [2] and comparison function in [11]), we have proved a fixed point for a new class of mappings called T_β-weakly contractive maps.

Moreover, all these results can be studied via the simulation function proposed by Khojasteh et al in [10] in 2015, which is an important tool in fixed point theory. This fact gives an added value to the theorems studied in this paper, for more detail we refer the reader to see [7, 8, 13].

On the other hand, the existence of solutions for functional equations arising in dynamic programming

$$f(x)=\sup_{y\in D}\{g(x,y)+G(x,y,f(\rho(x,y)))\}, \tag{1.3}$$

have been studied in the literature by using different fixed point theorems (see [3, 4]). In this work, as an application of our studies, we presented the existence of solutions for (1.3) under new and weak conditions.

2 Preliminaries

The aim of this section is to present some notions and results used in the paper. Let (X,τ) be a topological space and $p:X\times X\to[0,\infty)$ be a function. For any $\varepsilon>0$ and any $x\in X$, let $B_p(x,\varepsilon)=\{y\in X:p(x,y)<\varepsilon\}$.

Definition 2.1. (Definition 2.1 [1]) The function p is said to be $\tau-$distance if for each $x\in X$ and any neighborhood V of x, there exists $\varepsilon>0$ such that $B_p(x,\varepsilon)\subset V$.

Definition 2.2. A sequence $\{x_n\}$ in a topological space (X,τ) is a p-Cauchy if it satisfies the usual metric condition with respect to p, in other words, if $\lim_{n,m\to\infty}p(x_n,x_m)=0$.

Definition 2.3. (Definition 3.1 [1]) Let (X,τ) be a topological space with a τ-distance p.

1. X is S-complete if for every p-Cauchy sequence (x_n), there exists x in X with $\lim p(x,x_n)=0$.
2. X is p-Cauchy complete if for every p-Cauchy sequence (x_n), there exists x in X such that $\lim x_n=x$ with respect to τ.
3. X is said to be p-bounded if $\sup\{p(x,y):x,y\in X\}<\infty$.

Remark 2.4. The topology induced by p is finer than τ.

Lemma 2.5. (Lemma 3.1[1])
Let (x_n) be a sequence in a Hausdorff topological space (X,τ) with a τ-distance p and $x,y\in X$, then

1. If $(\alpha_n)\subset\mathbb{R}^+$ a sequence converging to 0 such that $p(x,x_n)\leq\alpha_n$ for all $n\in\mathbb{N}$, then (x_n) converges to x with respect to the topology τ.
2. $p(x,y)=0$ implies $x=y$.
3. If $\lim_{n\to\infty}p(x,x_n)=0$ and $\lim_{n\to\infty}p(y,x_n)=0$, then $x=y$.

Definition 2.6. ([1]) Ψ is the class of all functions $\psi:[0,+\infty)\longrightarrow[0,+\infty)$ satisfying:

i) ψ is nondecreasing,

ii) $\lim\psi^n(t)=0$, for all $t\in[0,\infty)$.

Theorem 2.7. (Theorem 4.1 [1])
Let (X,τ) be a Hausdorff topological space with a $\tau-$distance p. Suppose that X is $p-$bounded and $S-$complete. Let T be a selfmapping of X such that

$$p(Tx,Ty)\leq\psi(p(x,y)),$$

for all $x,y\in X$, where $\psi\in\Psi$. Then T has a unique fixed point.

On the other hand, in 2012, Samet et al. [12] introduced the concept of $\alpha-\psi$-contractive type mappings and established some fixed point theorems for these mappings in complete metric spaces.

Definition 2.8. (see [12]). Let (X, d) be a metric space, $T : X \to X$ and $\alpha : X \times X \to \mathbb{R}^+$ be two given mappings. Then, T is called an α-admissible mapping if

$$\alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1 \text{ for all } x, y \in X.$$

Theorem 2.9. (see [12]). Let be a complete metric space and $T : X \to X$ be an α-ψ-contractive mapping, that is,

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)),\ \forall x, y \in X$$

where $\psi \in \phi_1$ (see[12]). Assume that

i) T is α-admissible

ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$

iii) T is continuous

Then, T has a fixed point.

At the end of this section, we recall the proven results in [14]

Theorem 2.10. (Theorem 3 [14]) Let $T : X \to X$ be a mapping of a bounded complete metric space (X, d) such that

$$\inf_{x \neq y \in X} \{d(x, y) - d(Tx, Ty)\} > 0. \tag{2.1}$$

Then T has a unique fixed point.

Definition 2.11. (Definition 8 [14]) Let $T : X \to X$ be a mapping of a metric space (X, d). T will be said an E-weakly contractive map if for all $x, y \in X$

$$d(Tx, Ty) \leq d(x, y) - \phi[1 + d(x, y)],$$

where $\phi : [1, \infty) \to [0, \infty)$ is a function satisfying

i) $\phi(1) = 0$,

ii) $\inf\limits_{t>1} \phi(t) > 0$.

Theorem 2.12. (Theorem 9 [14]) Let $T : X \to X$ be an E-weakly contractive map of a bounded complete metric space (X, d). Then T has a unique fixed point.

3 Main results

In this section, we start with the following lemmas.

Lemma 3.1. Let (X, d) be a metric space and $p : X \times X \to \mathbb{R}^+$ be a function defined by

$$p(x, y) = e^{d(x,y)} - 1. \tag{3.1}$$

Then p is a τ_d-distance on X, where τ_d is the metric topology.

Proof . Let (X, τ_d) be the topological space with the metric topology τ_d, let $x \in X$ and V an arbitrary neighborhood of x, then there exists $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subset V$, where $B_d(x, \varepsilon) = \{y \in X, d(x, y) < \varepsilon\}$ is the open ball. It easy to see that $B_p(x, e^\varepsilon - 1) \subset B_d(x, \varepsilon)$, indeed:
Let $y \in B_p(x, e^\varepsilon - 1)$, then $p(x, y) < e^\varepsilon - 1$, which implies that $e^{d(x,y)} < e^\varepsilon$, and hence $d(x, y) < \varepsilon$. $\square$

Lemma 3.2. Let (X, d) be a bounded metric space. Then the function p defined in Lemma 3.1 is a bounded $\tau-$distance.

Lemma 3.3. Let (X,d) be a complete metric space. Then the function p defined in Lemma 3.1 is a S-complete $\tau-$distance.

Proof . Let (X,d) be a complete metric space and $\{x_n\} \subset X$ a p-Cauchy sequence, where p is the function defined in Lemma 3.1. Then $\lim\limits_{n,m\to\infty} p(x_n,x_m)=0$, and hence $\lim\limits_{n,m\to\infty} d(x_n,x_m)=0$. Since X is complete there exists $u\in X$ such that $\lim\limits_{n\to\infty} d(u,x_n)=0$. Finally, we deduce that there exists $u\in X$ such that $\lim\limits_{n\to\infty} p(u,x_n)=0$. $\square$ Before state another lemma we give the following definition

Definition 3.4. Let T be a selfmapping on a Hausdorff topological space (X,τ) with a τ-distance p. T is said to be p-continuous at $z\in X$ if for any $\{x_n\}\subset X$; $p(z,x_n)\to 0$ implies $p(Tz,Tx_n)\to 0$.

Lemma 3.5. Let (X,τ) be a Hausdorff topological space with a $\tau-$distance p. Suppose that X is $p-$bounded and $S-$complete. Let T be a selfmapping of X such that

$$\alpha(x,y)p(Tx,Ty)\leq \psi(p(x,y)),$$

for all $x,y\in X$, where $\psi\in\Psi$ and

i) T is α-admissible;

ii) there exists $x_0\in X$ such that $\alpha(x_0,T^nx_0)\geq 1$ for all $n\in\mathbb{N}$;

iii) T is p-continuous.

Then T has a fixed point.

Proof . Let $x_0\in X$ such that $\alpha(x_0,T^nx_0)\geq 1$ for all $n\in\mathbb{N}$. Define a sequence $\{x_n\}$ by $x_{n+1}=Tx_n$, for all $n\in\mathbb{N}$. Let $n,m\in\mathbb{N}$, since T is $\alpha-$admissible, by (ii) we obtain $\alpha(x_n,x_{n+m})\geq 1$. Then,

$$\begin{aligned} p(x_n,x_{n+m}) &\leq \alpha(x_n,x_{n+m})p(x_n,x_{n+m})\\ &\leq \psi(p(x_{n-1},x_{n+m-1}))\\ &\vdots\\ &\leq \psi^n(p(x_0,x_m))\\ &\leq \psi^n(M) \end{aligned}\tag{3.2}$$

where $M=\sup\{p(x,y): x,y\in X\}$. As $\lim\psi^n(M)=0$, so the sequence $\{x_n\}$ is a p-Cauchy sequence. Since X is S-complete, there exists $u\in X$ such that $\lim p(u,x_n)=0$. On the other hand, T is p-continuous, then $\lim p(Tu,Tx_n)=0$. Using Lemma 2.5 we obtain $Tu=u$. $\square$

Corollary 3.6. (Corollary 4.1 [1]) Let (X,τ) be a Hausdorff topological space with a τ-distance p. Suppose that X is p-bounded and S-complete. Let T be a selfmapping of X, if there exist $k\in[0,1)$ such that

$$p(Tx,Ty)\leq kp(x,y),\tag{3.3}$$

for all $x,y\in X$. Then T has a unique fixed point $u\in X$.

Now, we introduce the notion of T_β-contractive mapping.

Definition 3.7. Let T be a selfmapping of a bounded metric space (X,d), T is said to be T_β-contractive mapping if

$$\inf_{x\neq y\in X}\left\{d(x,y)-d(Tx,Ty)+\beta(x,y)\right\}>0,\tag{3.4}$$

where $\beta: X\times X\to\mathbb{R}$ is a function satisfying

$$\beta(x,y)\leq 0 \implies \beta(Tx,Ty)\leq 0.$$

Now, we are able to state our main results.

Theorem 3.8. Let T be a T_β-contractive mapping of a bounded complete metric space (X,d) such that

i) there exists $x_0 \in X$ such that $\beta(x_0, T^n x_0) \leq 0$ for all $n \in \mathbb{N}$;

ii) $\beta(a,b) \leq \inf\limits_{x \neq y \in X} \Big\{ d(x,y) - d(Tx,Ty) + \beta(x,y) \Big\}$ for all $a,b \in X$.

Then T has a fixed point $u \in X$.

Proof . Since T is a T_β-contractive mapping, so there exists a function $\beta : X \times X \to \mathbb{R}$ such that

$$\inf_{x \neq y \in X} \Big\{ d(x,y) - d(Tx,Ty) + \beta(x,y) \Big\} > 0.$$

We put

$$\gamma = \inf_{x \neq y \in X} \Big\{ d(x,y) - d(Tx,Ty) + \beta(x,y) \Big\},$$

which implies that for all $x \neq y \in X$, we have

$$d(Tx,Ty) - \beta(x,y) \leq d(x,y) - \gamma. \tag{3.5}$$

Thus

$$\alpha(x,y) e^{d(Tx,Ty)} \leq k e^{d(x,y)},$$

where $k = e^{-\gamma} < 1$ and $\alpha(x,y) = e^{-\beta(x,y)}$. Then, it follows from (ii) that

$$\alpha(x,y) p(Tx,Ty) \leq k p(x,y), \tag{3.6}$$

for all $x, y \in X$, with $p(x,y) = e^{d(x,y)} - 1$ is the $\tau-$distance defined in Lemma 3.1. Also, we get from (3.6) that T is p-continuous.
Finally, we deduce from Lemmas 3.1, 3.2, 3.3 and Lemma 3.5 that T has a fixed point $u \in X$. □

Corollary 3.9. (Theorem 3 [14]) Let $T : X \to X$ be a mapping of a bounded complete metric space (X,d) such that

$$\inf_{x \neq y \in X} \{ d(x,y) - d(Tx,Ty) \} > 0. \tag{3.7}$$

Then T has a fixed point.

Example 3.10. Let $X = [0,1] \times [0,1]$ endowed with the metric $d\big((x_1,y_1),(x_2,y_2)\big) = \|(x_1,y_1) - (x_2,y_2)\|_1 = |x_1 - x_2| + |y_1 - y_2|$. It is clear to see that (X,d) is not an uniform convex space, indeed:
For $\varepsilon = 1$, $x = (1,0)$ and $y = (0,1)$:
$\|x\|_1 = \|y\|_1 = 1$, $\|x - y\|_1 = 2 > 1 = \varepsilon$ and $\frac{1}{2}\|x + y\|_1 = 1 > 1 - \delta$ for each $\delta > 0$.
Consider the mapping $T : X \to X$ defined as

$$Tx = (1,1) - x, \text{ for all } x \in X.$$

Define a function $\beta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$\beta(x,y) = \begin{cases} \frac{1}{2} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}.$$

Let $x \neq y \in X$, then

$$\inf_{x \neq y \in X} \Big\{ d(x,y) - d(Tx,Ty) + \beta(x,y) \Big\} = \frac{1}{2} > 0,$$

and hence T is a T_β-contractive mapping. Also, we have :
$\beta\big((\frac{1}{2}, \frac{1}{2}), T^n(\frac{1}{2}, \frac{1}{2})\big) \leq 0$ for all $n \in \mathbb{N}$ and
$\beta(a,b) \leq \frac{1}{2} \leq \inf\limits_{x \neq y \in X} \Big\{ d(x,y) - d(Tx,Ty) + \beta(x,y) \Big\}$ for all $a, b \in X$.
Therefore, all conditions of Theorem 3.8 are satisfied and so T has the fixed point $(\frac{1}{2}, \frac{1}{2})$. On the other hand, since for all $x, y \in X$ we have $d(x,y) - d(Tx,Ty) = 0$, so Corollary 3.9 does not ensure the existence of the fixed point.

Remark 3.11. It is clear to see that the mapping defined in the above example is nonexampnsive, so our result ensures the existence for a class of nonexpansive contractions (i.e. $d(Tx,Ty) \leq d(x,y)$) without add any condition on the space (neither the compactness nor the uniform convexity).

In the following, $\beta : X \times X \to \mathbb{R}$ is the function defined (on a metric space X) in the Definition 3.7 and Theorem 3.8. In order to state the second result, we first give the following definition.

Definition 3.12. Φ_β is the class of all functions $\phi_\beta : [1, +\infty) \longrightarrow \mathbb{R}$ satisfying:

i) $\inf_{t>1} \phi_\beta(t) > 0$,

ii) $\phi_\beta(1) \leq \beta(x,x)$ for all $x \in X$.

Definition 3.13. Let $T : X \longrightarrow X$ be a mapping of a metric space (X,d). T will be said T_β-weakly contractive map if

$$d(Tx,Ty) - \beta(x,y) \leq d(x,y) - \phi_\beta(1 + d(x,y)),$$

for all $x, y \in X$ such that $\phi_\beta \in \Phi_\beta$.

Theorem 3.14. Let $T : X \longrightarrow X$ be a T_β-weakly contractive mapping of a bounded complete metric space (X,d). Then T has a fixed point.

Proof . Let $x \neq y \in X$, then from Definition 3.13, we have

$$\begin{aligned} 0 &< \inf_{t>1} \phi_\beta(t) \\ &\leq \phi_\beta(1 + d(x,y)) \\ &\leq d(x,y) - d(Tx,Ty) + \beta(x,y), \end{aligned}$$

thus

$$\inf_{x \neq y} \big\{ d(x,y) - d(Tx,Ty) + \beta(x,y) \big\} > 0.$$

According to Theorem 3.8, T has a fixed point in X. □

Corollary 3.15. (Theorem 9 [14]). Let $T : X \longrightarrow X$ be a E-weakly contractive mapping of a bounded complete metric space (X,d). Then T has a fixed point.

Example 3.16. Let $X = \{0,1,2\}$ with the usual metric $d(x,y) = |x-y|$ for all $x, y \in X$. Define a mapping $T : X \to X$ by

$$T0 = 0,\ T1 = 0 \ \text{ and } \ T2 = 1.$$

a function $\beta : X \times X \to \mathbb{R}$ defined by

$$\beta(x,y) = \begin{cases} -1 & \text{if } x = y \\ 2 & \text{if } x \neq y \end{cases}$$

and a function $\phi_\beta : [1,\infty) \to \mathbb{R}$ defined by

$$\phi_\beta(t) = \begin{cases} -2 & \text{if } t = 1 \\ 1 & \text{if } t > 1 \end{cases}.$$

It is clear to see that $\beta(0, T^n 0) \leq 0$ for all $n \in \mathbb{N}$, $\phi_\beta(1) \leq \beta(x,x)$ for all $x \in X$. Also, $\beta(a,b) \leq 2 \leq \inf_{x \neq y \in X} \Big\{ d(x,y) - d(Tx,Ty) + \beta(x,y) \Big\}$ for all $a, b \in X$.

So we have the following cases:

Case1: $d(T0,T1) - \beta(0,1) = -2 \leq 0 = d(0,1) - \phi_\beta(1 + d(0,1))$.

Case2: $d(T0,T2) - \beta(0,2) = -1 \leq 1 = d(0,2) - \phi_\beta(1 + d(0,2))$.

Case3: $d(T1,T2) - \beta(1,2) = -1 \leq 0 = d(1,2) - \phi_\beta(1 + d(1,2))$.

Therefore, T satisfies all assumptions in Theorem 3.14 and $T0 = 0$. But T does not satisfy Theorem 9 in [14], indeed:

$$d(T1,T2) = 1 > 0 = d(0,1) - \phi_\beta\big(1 + d(0,1)\big).$$

Remark 3.17. As a remark, the main results can be discussed via the simulation functions, for more detail we refer the reader to see [7, 8, 13].

4 Application

The existence of solutions for functional equations arising in dynamic programming have been studied by using different fixed point theorems (see [3, 4]).
Throughout this section we assume that X and Y are Banach spaces, $S \subset X$ is the state space and $D \subset Y$ is the decision space. Let $\rho : S \times D \to S$, $g : S \times D \to \mathbb{R}$ and $G : S \times D \times \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ is the field of real numbers. $B(S)$ denotes the set of all bounded real-valued functions on S. For $h, k \in B(S)$, let

$$d(h,k) = \sup\{|h(x) - k(x)| : x \in S\}.$$

It is easy to see that d is a metric on $B(S)$ and $(B(S), d)$ is a complete metric space. In this section, we study the existence of a solution of the following class of functional equations arising in dynamic programming.

$$f(x) = \sup_{y \in D}\{g(x,y) + G(x,y,f(\rho(x,y)))\}, \tag{4.1}$$

where g and G are bounded. We define $T : B(S) \to B(S)$ by

$$Tf(x) = \sup_{y \in D}\{g(x,y) + G(x,y,f(\rho(x,y)))\}. \tag{4.2}$$

Clearly, T is well-defined since g and G are bounded.

Now, we prove the existence and uniqueness of the solution for the functional equation (4.1).

Theorem 4.1. Let $T : B(S) \to B(S)$ be an operator defined by (4.2) and assume the following condition is satisfied: There exist $M > 0$ and a function $\eta : B(S) \times B(S) \to \mathbb{R}$ such that for all $h, k \in B(S)$ with $h \neq k$, we have:

$$\begin{aligned} \eta(h,k) \geq 0 &\implies |G(x,y,h(x)) - G(x,y,k(x))| \leq d(h,k) - M, \\ \eta(h,k) < 0 &\implies |G(x,y,h(x)) - G(x,y,k(x))| \leq d(h,k) \end{aligned} \tag{4.3}$$

for all $(x,y) \in S \times D$ and

- for all $h, k \in B(S)$, $\eta(h,k) \geq 0$ implies $\eta(Th, Tk) \geq 0$;
- there exists $h_0 \in B(S)$ such that $\eta(h_0, T^n h_0) \geq 0$ for all $n \in \mathbb{N}$.

Then the functional equation (4.1) has a bounded solution.

Proof . Let λ be an arbitrary positive number, let $x \in S$ and $h, k \in B(S)$ with $Th \neq Tk$, then there exist $y, z \in D$ such that

$$T(h(x)) < g(x,y) + G(x,y,h(\rho(x,y))) + \lambda, \tag{4.4}$$

$$T(k(x)) < g(x,z) + G(x,z,k(\rho(x,z))) + \lambda. \tag{4.5}$$

On the other hand, by the definition of T, we get

$$T(h(x)) \geq g(x,z) + G(x,z,h(\rho(x,z))) \tag{4.6}$$

$$T(k(x)) \geq g(x,y) + G(x,y,k(\rho(x,y))) \tag{4.7}$$

It follows from (4.4) and (4.7) that

$$\begin{aligned} T(h(x)) - T(k(x)) &< G(x,y,h(\rho(x,y))) - G(x,y,k(\rho(x,y))) + \lambda \\ &\leq |G(x,y,h(\rho(x,y))) - G(x,y,k(\rho(x,y)))| + \lambda. \end{aligned} \tag{4.8}$$

Similarly from (4.5) and (4.6)

$$T(k(x)) - T(h(x)) \leq |G(x,z,h(\rho(x,z))) - G(x,z,k(\rho(x,z)))| + \lambda. \tag{4.9}$$

In view of (4.8) and (4.9), we obtain

$$\begin{aligned} \eta(h,k) \geq 0 &\implies |T(h(x)) - T(k(x))| \leq d(h,k) - M + \lambda \\ \eta(h,k) < 0 &\implies |T(h(x)) - T(k(x))| \leq d(h,k) + \lambda \end{aligned}$$

or equivalently

$$\begin{aligned} \eta(h,k) \geq 0 &\implies d(T(h),T(k)) \leq d(h,k) - M + \lambda \\ \eta(h,k) < 0 &\implies d(T(h),T(k)) \leq d(h,k) + \lambda. \end{aligned} \tag{4.10}$$

Since λ is taken arbitrary, then we obtain

$$\begin{aligned} \eta(h,k) \geq 0 &\implies d(T(h),T(k)) \leq d(h,k) - M \\ \eta(h,k) < 0 &\implies d(T(h),T(k)) \leq d(h,k). \end{aligned} \tag{4.11}$$

for all $h \neq k \in B(S)$.
Now, define $\beta : B(S) \times B(S) \to \mathbb{R}$ by

$$\beta(k,h) = \begin{cases} 0 & \text{if } \eta(h,k) \geq 0 \\ 1 & \text{otherwise.} \end{cases}$$

Then, we have

$$\inf_{h \neq k \in B(S)} \left\{ d(h,k) - d(Th,Tk) + \beta(h,k) \right\} > 0. \tag{4.12}$$

Also, if we have $\beta(h,k) \leq 0$, we obtain from the definition of β that $\beta(Th,Tk) \leq 0$, which implies that T is a T_β-contractive mapping.
Now, it remains to show (i) and (ii) of Theorem 3.8:
It is clear to see that $\beta(h_0, T^n h_0) \leq 0$ for all $n \in \mathbb{N}$. Moreover, we have for all $a, b \in B(S)$

$$\beta(a,b) \leq \inf_{h \neq k \in B(S)} \left\{ d(h,k) - d(Th,Tk) + \beta(h,k) \right\},$$

Finally, we conclude by Theorem 3.8 that the functional equation (4.1) has a bounded solution. □

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