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Study on fractional order atmospheric internal waves model by Aboodh transform homotopy perturbation method

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Abstract

The study of atmospheric internal waves which occur inside a fluid rather than on the surface is represented by a mathematical model named the atmospheric internal waves model. Under the shallow-fluid assumption, internal atmospheric waves are modeled by a nonlinear partial differential equation system. The proposed shallow flow model is based on the assumption that waves spread out horizontally before rising vertically. The Aboodh transform homotopy perturbation method (ATHPM) has been applied to obtain an approximate solution for a given model. This model helps to understand global atmospheric modeling, which has applications in climate and weather predictions. The ATHPM solution is compared with the EADM solution, HAM solution, FRDTM solution, and q-HAShTM solution to examine the accuracy and efficiency of the suggested method.

Keywords: Atmospheric internal waves, Fractional order nonlinear partial differential equation, Homotopy perturbation method, Aboodh transform, climate and weather predictions
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1 Introduction

ATHPM is used to find the analytical solution to nonlinear fractional differential equations. K.S. Aboodh introduced the Aboodh transform to facilitate the process of solving differential equations in the time domain [1]. The homotopy perturbation method (HPM) was proposed by J. H. He [15]. HPM is a semi-analytical approach for solving linear and nonlinear differential equations. Aboodh transform is extension of Laplace transform and the converse duality relation existing between the Aboodh transform and the Laplace transform [2].

Internal waves are also known as gravity waves which are only seen inside of fluids, never on the exterior. They are mostly found below the ocean's surface layer and atmosphere. The temperature change of a three-dimensional fluid is essential for identifying the propagation of internal waves and the three-dimensional wavelength. Internal waves gain velocity and energy when external waves, such as winds and other waves, collide with them [23]. Wave clouds are particularly interested in atmospheric waves. Morning Glory clouds in Australia show internal waves. For the interaction and propagation of gravity waves, researchers have developed many mathematical models and simulations [10, 26]. Internal waves are disturbances that flow along the pycnocline as they pass through fluid interfaces and diseases. Internal waves in the water force the free surface of the water to move vertically, making them invisible to

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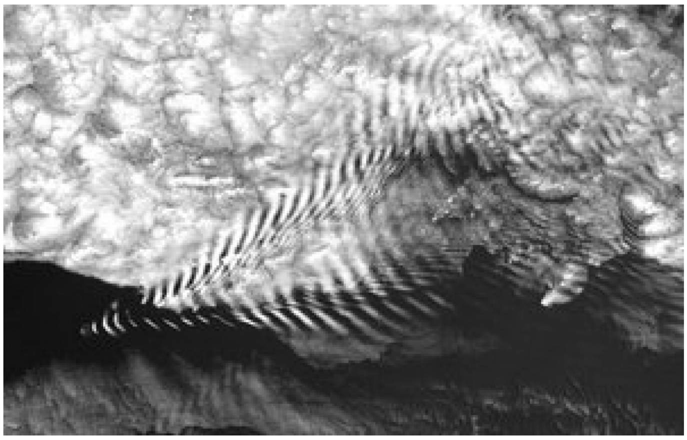


Figure 1: Over the New Amsterdam, a wave cloud pattern formed.

floating structures. Although they have been spotted in very turbulent conditions, these waves are difficult to detect. The pycnocline and thermocline are two deepwater density and temperature shifts roughly 50 m below the surface. Pycnocline refers to the gradual contact of two fluids of varying thicknesses. There is a chance that these waves will be pretty big. Internal waves are only detectable via their surface signature and direct pycnocline or thermocline measurements because they are not pparently visible. When the density interface is shallow enough to allow the internal wave crests to contact the sea surface, the waves can be identified by the increased roughness of the ocean surface. Internal waves have an essential role in mixing different layers of water in the seas, which impacts climate. The rate of such mixing is one of the significant areas of uncertainty in present climate models [8].

Internal waves can be represented mathematically using the shallow fluid assumption. The term "shallow fluid" denotes that the perturbation's horizontal scale is more essential compared to the depth of fluid layer [10]. The assumption of shallow fluid helped several models, including Kelvin waves models [9, 14], climate model [27], tsunami model [24], and Rossby waves model [11]. To solve these models, many scholars have employed analytical techniques [26, 27, 13, 21] but it is difficult to figure out the exact solution to that problems. As a result, various numerical and analytical methodologies are offered. However, numerical methods offer discrete points in the solution. Analytical methods were highly dominant. The partial differential equations of internal atmospheric waves were solved with VIM [7], EADM [26], FRDTM[20] and q-HAShTM [22].

The fluid must be layered for the phenomena to occur and each layer must have the same temperature and density, which should only fluctuate with height. The waves propagate horizontally like surface waves when the density varies over a small vertical height [16]. Internal waves in the atmosphere arise when a homogenous layer of air passes over a major barrier, such as a mountain range. Horizontal stripes of homogeneous air are disturbed when the air meets the barrier to form a wave pattern. They can cause the formation of wave clouds, which are generated when steady air flows over a barrier like a mountain. As the mass of air moves through the wave, it ascends and descends repeatedly. Clouds will form at the cooled peaks of these waves if the atmosphere is humid enough. Due to adiabatic heating, such clouds will vaporize in the descendant section of the wave, resulting in the classic clouded and transparent stripes. As indicated in Figure-1, this phenomenon has been observed in several places worldwide. Morning Glory Clouds arise in the southern section of the Gulf of Carpentaria due to the internal waves phenomenon in Australia. Figure-2 shows the morning glory clouds above the gulf of carpentaria.

Figure-3 illustrates the internal wave phenomenon in various areas around the world. Water moves near the surface, making it impossible to see or measure directly but, it leaves a unique path on the surface that satellites can detect. NASA's shuttle captured internal waves in the South China Sea in June, 1983 and on the Peruvian coast in 1984. Internal waves have an influence on chemistry binding, ambient temperature, and turbulence according to research undertaken during the previous decade. Internal airwaves are typically simulated using shallow-fluid equations, sometimes known as shallow water equations.

The major purpose of this research is to generalize the use of the Aboodh transform homotopy perturbation method (ATHPM) to shallow fluid equation analysis [17], which are a system of non-linear partial differential equations and apply it to the phenomenon of internal atmospheric waves propagating through a fluid medium. The advantage of this method is that it is to implement with minimum computational effort.

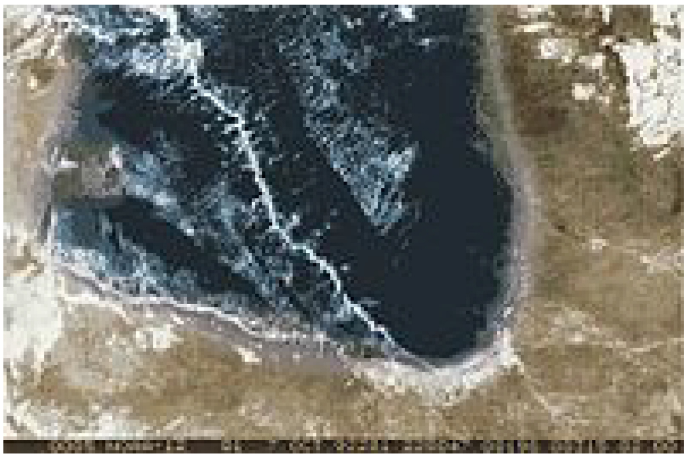
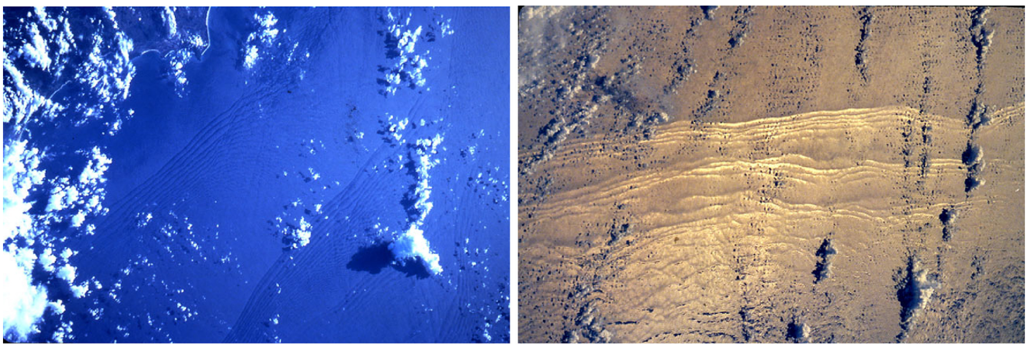


Figure 2: Clouds of Morning Glory above the Gulf of Carpentaria



(a) (b)

Figure 3: Internal waves in (a) the South China Sea (STS-7, June 1983, photo #7-05-0245) and (b) the Eastern Pacific Sea (STS-41C, April 1984, photo #13-31-980)

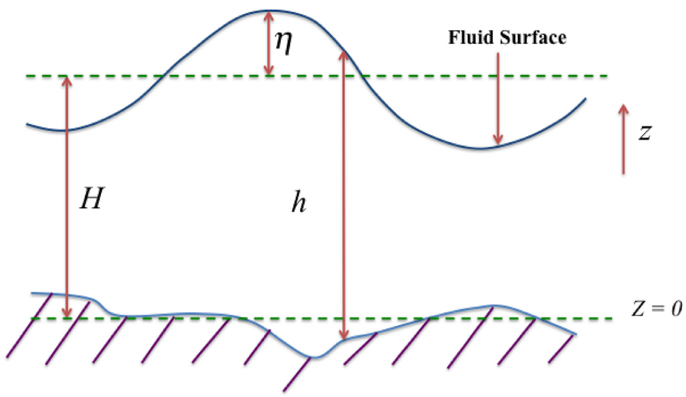


Figure 4: Schematic representation of internal waves

2 Preliminaries

2.1 Aboodh transform

Consider piecewise continuous and of exponential order functions in the set A given by

$$A = \{f(t) : \exists M, k_1, k_2 > 0, |f(t)| < Me^{-vt}\}. \tag{2.1}$$

The constant M must be a finite number, k_1, k_2 may be finite or infinite. The Aboodh transform is defined by [1]

$$A[f(t)](v) = K(v) = \frac{1}{v}\int_0^\infty f(t)e^{-vt}dt, t \geq 0, k_1 \leq v \leq k_2. \tag{2.2}$$

2.2 Properies of Aboodh transform

Table 1: Aboodh transform of some functions [1]

$f(t)$	$A[f(t)]$
1	$\frac{1}{v^2}$
t	$\frac{1}{v^3}$
t^n	$\frac{n!}{v^{n+2}}$
$\cos at$	$\frac{1}{v^2+a^2}$
$\sin at$	$\frac{a}{v(v^2+a^2)}$
e^{at}	$\frac{1}{v^2-av}$

Definition 2.1. Inverse Aboodh transform of a function $f(t), t \in (0, \infty)$ if $A[f(t)] = K(v)$, then $f(t) = A^{-1}[K(v)]$.

Definition 2.2. Aboodh transform of a caputo fractional derivative for $\alpha > 0$, $m \in \mathbb{N}$ and $m - 1 < \alpha \leq m$ is

$$A[D^\alpha f(x)] = v^\alpha K(v) - \prod_{k=0}^{\alpha-1} \frac{f^k(0)}{v^{2-\alpha+k}}.$$

3 Mathematical model

Internal atmospheric waves are described using a non-linear partial differential equations system based on the shallow water concept. Mass and momentum conservation are used to derive the fundamental equations of fluid motion in differential form. A fluid layer with a shallow depth compared to its height is referred to as "shallow fluid".

In such kind of atmosphere where the density determined by pressure, incompressible, hydrostatic, and inviscid fluid and it does not vary with space in homogeneous flow then the fractional order partial differential equations can be written as [20, 22],

$$\begin{aligned} \frac{\partial^\alpha m}{\partial t^\alpha} + m\frac{\partial m}{\partial x} - fp + g\frac{\partial \eta}{\partial x} = 0 \\ \frac{\partial^\alpha p}{\partial t^\alpha} + m\frac{\partial p}{\partial x} + fm + g\bar{H} = 0 \\ \frac{\partial^\alpha \eta}{\partial t^\alpha} + m\frac{\partial \eta}{\partial x} + p\bar{H} + \eta\frac{\partial m}{\partial x} = 0 \end{aligned} \tag{3.1}$$

which is fractional order atmospheric internal waves model [20, 22]. Where constant of pressure gradient of desired magnitude $\bar{H} = -\frac{f}{g}\bar{U}$, $\bar{U} = 2.5m/s$, gravitational constant $g = 9.8m/s^2$, at $\alpha = \frac{\pi}{3}$ and $\omega = 7.29 \times 10^{-5} rad/s$, x is a space coordinate, t is time, m and p are cartesian velocities, η is fluid depth, f is the Coriolis parameter, g is gravity acceleration, $\bar{H}$ represents mean fluid depth, and $\bar{U}$ is the defined, constant mean geostrophic speed [26].

4 Implementation of ATHPM

Consider a nonlinear and nonhomogeneous fractional order partial differential equation in the Caputo sense.

$$\frac{\partial^\alpha \nu}{\partial t^\alpha} + R\nu(\varkappa, t) + N\nu(\varkappa, t) = f(\varkappa, t) \tag{4.1}$$

where $f(x, y)$ is the source term, R is the linear differential operator and N is the nonlinear differential equation. Apply Aboodh transform on it,

$$A[\frac{\partial^\alpha \nu}{\partial t^\alpha}] + A[R\nu(\varkappa, t)] + A[N\nu(\varkappa, t)] = A[f(\varkappa, t)],$$

$$K(s) = \frac{1}{s^2}\nu(\varkappa, 0) - \frac{1}{s^\alpha}A[R\nu(\varkappa, t) + N\nu(\varkappa, t) - f(\varkappa, t)].$$

Implement Aboodh inverse transform,

$$\nu(\varkappa, t) = \nu(\varkappa, 0) - A^{-1}[\frac{1}{s^\alpha}A(R\nu(\varkappa, t) + N\nu(\varkappa, t) - f(\varkappa, t))]. \tag{4.2}$$

For nonlinear term, use HPM [23]

$$\sum_{n=0}^{\infty} p^n \nu_n(\varkappa, t) = \nu(\varkappa, 0) - pA^{-1}[\frac{1}{s^\alpha}A(R\sum_{n=0}^{\infty} p^n H_n(\varkappa, t) - N\sum_{n=0}^{\infty} p^n H_n(\varkappa, t))].$$

We combine coefficients such as powers of p on both sides, we get

$$p^0 : \nu_0(\varkappa, t) = \nu(\varkappa, 0)$$

$$p^i : \nu_i(\varkappa, t) = -A^{-1}[\frac{1}{s^\alpha}A(R\nu_{i-1}(\varkappa, t) - N\nu_{i-1}(\varkappa, t))]; i > 0.$$

Finally, the approximate analytical solution is as follows:

$$\nu(\varkappa, t) = \nu_0 + \nu_1 + \nu_2 + \nu_3 + \cdots . \tag{4.3}$$

5 Uniqueness and Convergence analysis

Theorem 5.1. When $0 < \sigma < 1$, the solution derived by ATHPM of a nonlinear partial differential equation has a unique solution.

Proof . Suppose $\nu(\varkappa,t)$ and $u(\varkappa,t)$ are two distinct solutions of equation (4.1), and both having the same initial condition then

$$|\nu-u| = |(A^{-1}[\frac{1}{s^{\alpha}}A(R\nu(\varkappa,t)+N\nu(\varkappa,t)-f(\varkappa,t))]-(A^{-1}[\frac{1}{s^{\alpha}}A(Ru(\varkappa,t)+Nu(\varkappa,t)-f(\varkappa,t))])|$$

$$|\nu-u| \le |-A^{-1}[\frac{1}{s^{\alpha}}A(R(\nu-u)+N(\nu-u))]|.$$

Now, Applying convolution theorem [18],

$$|\nu-u| \le \int_0^t (|R(\nu)-R(u)|+|L(\nu)-L(u)|)|\frac{(t-\tau)^n}{n!}|\,d\tau,$$

R and N are bounded operators as satisfies the Lipschitz conditions so that $|R(\nu)-R(u)| \le K|\nu-u|$ and $|N(\nu)-N(u)| \le M|\nu-u|$ for $K, M > 0$. Therefore

$$|\nu-u| \le \int_0^t ((K+M)|\nu-u|)|\frac{(t-\tau)^n}{n!}|\,d\tau.$$

Using mean value theorem for integrals, therefore

$$|\nu-u| \le ((K+M)|\nu-u|)LT,$$

where, $L = \max(t-\tau)^n\, and\, t \in [0,T]$
Hence $|\nu-u| \le |\nu-u|\sigma$, where $\sigma = (K+M)LT$. Therefore $(1-\sigma)|\nu-u| \le 0$. That implies $\nu = u$ whenever, $0 < \sigma < 1$. Hence, the solution is unique. □

Theorem 5.2. Let ν is an exact solution and $N : H \to H$ be a nonlinear operator in Hilbert space H , Approximate analytical solution $\nu(x,t)$ is converges to ν , if $\exists\, \gamma; 0 < \gamma < 1$, such that $||\nu_{i+1}|| \le \gamma||\nu_i||, \forall i \in \mathbb{N} \cup 0$ [26].

Definition 5.3. $\forall i \in N \cup \{0\}; \gamma_i$ can be obtain as

$$\gamma_i = \begin{cases} \frac{\|\nu_{i+1}\|}{\|\nu_i\|} & \text{if } \|\nu_i\| \ne 0 \\ 0 & \text{if } \|\nu_i\| = 0 \end{cases}$$

Corollary 5.4. $\sum_{i=0}^{\infty} \nu_i$ converges to exact solution ν, whem $0 < \gamma_i < 1, i = 1,2,3,....$

6 Solution of the problem by ATHPM

Consider fractional order partial differential equations in equation (3.3),

$$\begin{aligned} \frac{\partial^{\alpha} m}{\partial t^{\alpha}} + m\frac{\partial m}{\partial x} - fp + g\frac{\partial \eta}{\partial x} &= 0 \\ \frac{\partial^{\alpha} p}{\partial t^{\alpha}} + m\frac{\partial p}{\partial x} + fm + g\bar{H} &= 0 \\ \frac{\partial^{\alpha} \eta}{\partial t^{\alpha}} + m\frac{\partial \eta}{\partial x} + p\bar{H} + \eta\frac{\partial m}{\partial x} &= 0 \end{aligned} \tag{6.1}$$

with the following initial conditions [26]:

$$\begin{aligned} m(x,0) &= e^{x}\ \text{sech}^2(x), \\ p(x,0) &= 2x\ \text{sech}^2(2x), \\ \eta(x,0) &= x^2\ \text{sech}^2(2x). \end{aligned} \tag{6.2}$$

Apply Aboodh transform on it,

$$\begin{aligned} A[\frac{\partial^\alpha m}{\partial t^\alpha}] &= A[-m\frac{\partial m}{\partial x} + fp - g\frac{\partial \eta}{\partial x}] \\ A[\frac{\partial^\alpha p}{\partial t^\alpha}] &= A[-m\frac{\partial p}{\partial x} - fm - g\bar{H}] \\ A[\frac{\partial^\alpha \eta}{\partial t^\alpha}] &= A[-m\frac{\partial \eta}{\partial x} - p\bar{H} - \eta\frac{\partial m}{\partial x}] \end{aligned} \tag{6.3}$$

$$\begin{aligned} v^\alpha\{K(v) - \frac{1}{v^2}m(x,0)\} &= A[-m\frac{\partial m}{\partial x} + fp - g\frac{\partial \eta}{\partial x}] \\ v^\alpha\{K(v) - \frac{1}{v^2}p(x,0)\} &= A[-m\frac{\partial p}{\partial x} - fm - g\bar{H}] \\ v^\alpha\{K(v) - \frac{1}{v^2}\eta(x,0)\} &= A[-m\frac{\partial \eta}{\partial x} - p\bar{H} - \eta\frac{\partial m}{\partial x}]. \end{aligned} \tag{6.4}$$

Apply inverse Aboodh transform

$$\begin{aligned} m(x,t) &= m(x,0) + A^{-1}[\frac{1}{v^\alpha}A(-m\frac{\partial m}{\partial x} + fp - g\frac{\partial \eta}{\partial x})] \\ p(x,t) &= p(x,0) + A^{-1}[\frac{1}{v^\alpha}A[-m\frac{\partial p}{\partial x} - fm - g\bar{H}] \\ \eta(x,t) &= \eta(x,0) + A^{-1}[\frac{1}{v^\alpha}A[-m\frac{\partial \eta}{\partial x} - p\bar{H} - \eta\frac{\partial m}{\partial x}]. \end{aligned} \tag{6.5}$$

By HPM
Comparing coefficients of p^0 :

$$\begin{aligned} m_0 &= m(x,0) = e^x \ \text{sech}^2(x) \\ p_0 &= p(x,0) = 2x \ \text{sech}^2(2x) \\ \eta_0 &= \eta(x,0) = x^2 \ \text{sech}^2(2x). \end{aligned} \tag{6.6}$$

Comparing coefficients of p^1 :

$$\begin{aligned} m_1 &= A^{-1}[\frac{1}{v^\alpha}A(-m_0\frac{\partial m_0}{\partial x} + fp_0 - g\frac{\partial \eta_0}{\partial x})] \\ p_1 &= A^{-1}[\frac{1}{v^\alpha}A[-m_0\frac{\partial p_0}{\partial x} - fm_0 - g\bar{H}] \\ \eta_1 &= A^{-1}[\frac{1}{v^\alpha}A[-m_0\frac{\partial \eta_0}{\partial x} - p_0\bar{H} - \eta_0\frac{\partial m_0}{\partial x}]. \end{aligned} \tag{6.7}$$

So,

$$\begin{aligned} m_1 = (-e^x \,\text{sech}^2(x)(e^x \,\text{sech}^2(x) - 2e^x \,\text{sech}^2(x)\tanh(x)) - 19.5997474\, x \ \text{sech}^2(2x) \\ + 39.2\, x^2 \,\text{sech}^2(2x)\tanh(2x))\frac{t^\alpha}{(\alpha)!} \end{aligned}$$

$$\begin{aligned} p_1 = (-e^x \,\text{sech}^2(x)(2\,\text{sech}^2(2x) - 8\,x\,\text{sech}^2(2x)\tanh(2x)) - 0.0001263\, e^x \,\text{sech}^2(x) \\ + 0.0003438166667)\frac{t^\alpha}{(\alpha)!} \end{aligned}$$

$$\begin{aligned} \eta_1 = (-e^x \,\text{sech}^2(x)(2x\,\text{sech}^2(x) - 4x^2 \,\text{sech}^2(2x)\tanh(2x)) + 0.0000701667\, x \ \text{sech}^2(2x) \\ - x^2 \,\text{sech}^2(2x)(e^x \,\text{sech}^2(x) - 2e^x \,\text{sech}^2(x)\tanh(x)))\frac{t^\alpha}{(\alpha)!}. \end{aligned}$$

Comparing coefficients of p^2:

$$\begin{aligned} m_2 &= A^{-1}[\frac{1}{v^\alpha}A(-m_1\frac{\partial m_0}{\partial x}-m_0\frac{\partial m_1}{\partial x}+fp_1-g\frac{\partial \eta_1}{\partial x})] \\ p_2 &= A^{-1}[\frac{1}{v^\alpha}A[-m_0\frac{\partial p_1}{\partial x}-m_1\frac{\partial p_0}{\partial x}-fm_1] \\ \eta_2 &= A^{-1}[\frac{1}{v^\alpha}A[-m_0\frac{\partial \eta_1}{\partial x}-m_1\frac{\partial \eta_0}{\partial x}-p_1\bar{H}-\eta_0\frac{\partial m_1}{\partial x}-\eta_1\frac{\partial m_0}{\partial x}] \end{aligned} \tag{6.8}$$

so,

$$\begin{aligned} m_2 =&((470.4e^x \operatorname{sech}^2(x)x^2\tanh^2(2x)+((-117.6x^2-305.5989896x+235.2\tanh(x)x^2)e^x \\ &\operatorname{sech}^2(x)+0.002751x)\tanh(2x)-0.00068763+(58.8\tanh^2(x)x^2+(-117.5995x \\ &-39.2x^2)\tanh(x)+37.1997474+58.7997474x-166.6x^2)e^x\operatorname{sech}^2(x))+0.000343817 \\ &+(14\tanh^2(x)-12\tanh(x)+1)e^{3x}\operatorname{sech}^6(x)-0.0001263e^x\operatorname{sech}^2(x))\frac{t^{2\alpha}}{(2\alpha)!} \end{aligned}$$

$$\begin{aligned} p_2 =&((-235.198\tanh(2x)x^2+39.133495x+313.6\tanh^2(2x)x^3)\operatorname{sech}^4(2x)+(48e^{2x} \\ &\operatorname{sech}^4(x)\tanh^2(2x)x+((-16x+32x\tanh(x)-16)(e^{2x})\operatorname{sech}^4(x)-0.00495096x^2)\tanh(2x) \\ &+(4-8\tanh(x)-16x)e^{2x}\operatorname{sech}^4(x)+0.00248x)\operatorname{sech}^2(2x)+(0.00025-0.0005\tanh(x))e^{2x} \\ &\operatorname{sech}^4(x))\frac{t^{2\alpha}}{(2\alpha)!} \end{aligned}$$

$$\begin{aligned} \eta_2 =&((-313.59798\tanh(2x)x^3+392\tanh^2(2x)x^4+58.79924x^2-78.4x^4)\operatorname{sech}^4(2x) \\ &+(24e^{2x}\operatorname{sech}^4(x)\tanh^2(2x)x^2+((-16x-16x^2+32\tanh(x)x^2)e^{2x}\operatorname{sech}^4(x) \\ &+0.000561e^x\operatorname{sech}^2(x)x)\tanh(2x)+(20\tanh^2(x)x^2-16(x-x^2)\tanh(x)+2-8x^2 \\ &+8x)e^{2x}\operatorname{sech}^4(x)+(-0.00014033+0.00014033\tanh(x)x-0.00007017x)e^x\operatorname{sech}^2(x)) \\ &\operatorname{sech}^2(2x)+(1.206223\times 10^{-8})-(4.431025\times 10^{-9})e^x\operatorname{sech}^2(x))\frac{t^{2\alpha}}{(2\alpha)!} \end{aligned}$$

$$\vdots$$

By equation (4.20), solution can be obtained as

$$\begin{aligned} m(x,t) =&e^x\ \operatorname{sech}^2(x)+[-e^x\operatorname{sech}^2(x)(e^x\operatorname{sech}^2(x)-2e^x\operatorname{sech}^2(x)\tanh(x))-19.5997474\,x\ \operatorname{sech}^2(2x) \\ &+39.2\,x^2\operatorname{sech}^2(2x)\tanh(2x)]\frac{t^\alpha}{(\alpha)!}+[(470.4e^x\operatorname{sech}^2(x)x^2\tanh^2(2x)+ \\ &((-117.6x^2-305.59899x+235.2\tanh(x)x^2)e^x\operatorname{sech}^2(x)+0.002751x)\tanh(2x) \\ &-0.00068763+(58.8\tanh^2(x)x^2+(-117.5995x-39.2x^2)\tanh(x)+37.1997474 \\ &+58.7997474x-166.6x^2)e^x\operatorname{sech}^2(x))+0.000343817+(14\tanh^2(x)-12\tanh(x) \\ &+1)e^{3x}\operatorname{sech}^6(x)-0.0001263e^x\operatorname{sech}^2(x)]\frac{t^{2\alpha}}{(2\alpha)!}+\cdots \end{aligned}$$

$$\begin{aligned} p(x,t) =&2x\ \operatorname{sech}^2(2x)+[-e^x\operatorname{sech}^2(x)(2\operatorname{sech}^2(2x)-8\,x\operatorname{sech}^2(2x)\tanh(2x))-0.0001263\,e^x \\ &\operatorname{sech}^2(x)+0.0003438166667]\frac{t^\alpha}{(\alpha)!}+[(-235.198\tanh(2x)x^2+39.133495x \\ &+313.6\tanh^2(2x)x^3)\operatorname{sech}^4(2x)+(48e^{2x}\operatorname{sech}^4(x)\tanh^2(2x)x+((-16x+32x\tanh(x) \\ &-16)(e^{2x})\operatorname{sech}^4(x)-0.004951x^2)\tanh(2x)+(4-8\tanh(x)-16x)e^{2x}\operatorname{sech}^4(x) \\ &+0.00248x)\operatorname{sech}^2(2x)+(0.00025-0.0005\tanh(x))e^{2x}\operatorname{sech}^4(x)]\frac{t^{2\alpha}}{(2\alpha)!}+\cdots \end{aligned}$$

$$\begin{aligned}\eta(x,t) =&x^2\ \operatorname{sech}^2(2x) + [-e^x \operatorname{sech}^2(x)(2x \operatorname{sech}^2(x) - 4x^2 \operatorname{sech}^2(2x)\tanh(2x)) + 0.0000701667\, x\ \operatorname{sech}^2(2x)\\
&- x^2 \operatorname{sech}^2(2x)(e^x \operatorname{sech}^2(x) - 2e^x \operatorname{sech}^2(x)\tanh(x))]\frac{t^\alpha}{(\alpha)!} + [(-313.59798\tanh(2x)x^3\\
&+ 392\tanh^2(2x)x^4 + 58.79924x^2 - 78.4x^4)\operatorname{sech}^4(2x) + (24e^{2x}\operatorname{sech}^4(x)\tanh^2(2x)x^2\\
&+ ((-16x - 16x^2 + 32\tanh(x)x^2)e^{2x}\operatorname{sech}^4(x) + 0.000561e^x \operatorname{sech}^2(x)x)\tanh(2x)\\
&+ (20\tanh^2(x)x^2 - 16(x - x^2)\tanh(x) + 2 - 8x^2 + 8x)e^{2x}\operatorname{sech}^4(x) + (-117.5995x\\
&- 39.2x^2)\tanh(x) + 37.1997474 + 58.7997474x - 166.6x^2)e^x \operatorname{sech}^2(x)) + 0.000343817\\
&+ (14\tanh^2(x) - 12\tanh(x) + 1)e^{3x}\operatorname{sech}^6(x) - 0.0001263e^x \operatorname{sech}^2(x)]\frac{t^{2\alpha}}{(2\alpha)!} + \cdots .\end{aligned}$$

Applying convergence analysis, we have

$$\gamma_{0,m} = \frac{\|m_1\|}{\|m_0\|} = 0.06267143884 < 1$$

$$\gamma_{1,m} = \frac{\|m_2\|}{\|m_1\|} = 0.03965327031 < 1$$

$$\gamma_{0,p} = \frac{\|p_1\|}{\|p_0\|} = 0.08169127510 < 1$$

$$\gamma_{1,p} = \frac{\|p_2\|}{\|p_1\|} = 0.02811701493 < 1$$

$$\gamma_{0,\eta} = \frac{\|\eta_1\|}{\|\eta_0\|} = 0.2132868240 < 1$$

$$\gamma_{1,\eta} = \frac{\|\eta_2\|}{\|\eta_1\|} = 0.07027548980 < 1$$

$$\vdots$$

Hence, we can say that all series solutions are convergent. Therefore the obtained ATHPM solution is convergent. The case $\alpha = 1$ is convert fractional order partial differential equation into classical partial differential equation, and it has approximate analytic solution

$$\begin{aligned}m(x,t) =&e^x\ \operatorname{sech}^2(x) + [-e^x \operatorname{sech}^2(x)(e^x \operatorname{sech}^2(x) - 2e^x \operatorname{sech}^2(x)\tanh(x)) - 19.5997474\, x\ \operatorname{sech}^2(2x)\\
&+ 39.2\, x^2 \operatorname{sech}^2(2x)\tanh(2x)]t + [(470.4e^x \operatorname{sech}^2(x)x^2\tanh^2(2x) + ((-117.6x^2\\
&- 305.59899x + 235.2\tanh(x)x^2)e^x \operatorname{sech}^2(x) + 0.002751x)\tanh(2x) - 0.00068763\\
&+ (58.8\tanh^2(x)x^2 + (-117.5995x - 39.2x^2)\tanh(x) + 37.1997474 + 58.7997474x\\
&- 166.6x^2)e^x \operatorname{sech}^2(x)) + 0.000343817 + (14\tanh^2(x) - 12\tanh(x) + 1)e^{3x}\operatorname{sech}^6(x)\\
&- 0.0001263e^x \operatorname{sech}^2(x)]\frac{t^2}{(2)!} + \cdots\end{aligned}$$

$$\begin{aligned}p(x,t) =&2x\ \operatorname{sech}^2(2x) + [-e^x \operatorname{sech}^2(x)(2\operatorname{sech}^2(2x) - 8\, x\operatorname{sech}^2(2x)\tanh(2x))\\
&- 0.0001263\, e^x \operatorname{sech}^2(x) + 0.0003438166667]t + [(-235.198\tanh(2x)x^2 + 39.133495x\\
&+ 313.6\tanh^2(2x)x^3)\operatorname{sech}^4(2x) + (48e^{2x}\operatorname{sech}^4(x)\tanh^2(2x)x + ((-16x + 32x\tanh(x)\\
&- 16)(e^{2x})\operatorname{sech}^4(x) - 0.004951x^2)\tanh(2x) + (4 - 8\tanh(x) - 16x)e^{2x}\operatorname{sech}^4(x)\\
&+ 0.00248x)\operatorname{sech}^2(2x) + (0.00025 - 0.0005\tanh(x))e^{2x}\operatorname{sech}^4(x)]\frac{t^2}{(2)!} + \cdots\end{aligned}$$

$$\begin{aligned}\eta(x,t) =& x^2 \operatorname{sech}^2(2x) + [-e^x \operatorname{sech}^2(x)(2x \operatorname{sech}^2(x) - 4x^2 \operatorname{sech}^2(2x)\tanh(2x)) + 0.0000701667\, x \operatorname{sech}^2(2x) \\ & - x^2 \operatorname{sech}^2(2x)(e^x \operatorname{sech}^2(x) - 2e^x \operatorname{sech}^2(x)\tanh(x))]t + [(-313.59798 \tanh(2x)x^3 \\ & + 392 \tanh^2(2x)x^4 + 58.79924x^2 - 78.4x^4)\operatorname{sech}^4(2x) + (24e^{2x} \operatorname{sech}^4(x)\tanh^2(2x)x^2 \\ & + ((-16x - 16x^2 + 32\tanh(x)x^2)e^{2x} \operatorname{sech}^4(x) + 0.000561e^x \operatorname{sech}^2(x)x)\tanh(2x) \\ & + (20\tanh^2(x)x^2 - 16(x - x^2)\tanh(x) + 2 - 8x^2 + 8x)e^{2x} \operatorname{sech}^4(x) + (-117.5995x \\ & - 39.2x^2)\tanh(x) + 37.1997474 + 58.7997474x - 166.6x^2)e^x \operatorname{sech}^2(x)) + 0.000343817 \\ & + (14\tanh^2(x) - 12\tanh(x) + 1)e^{3x} \operatorname{sech}^6(x) - 0.0001263e^x \operatorname{sech}^2(x)]\frac{t^2}{(2)!} + \cdots\end{aligned}$$

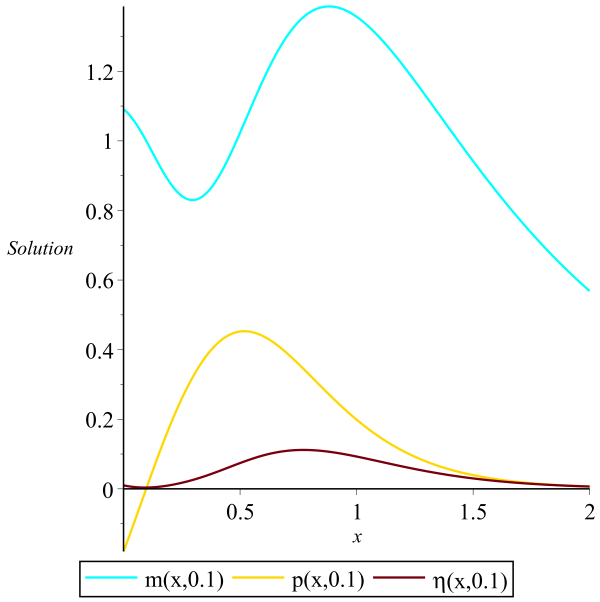


Figure 5: Solution behaviour for $\alpha = 1$ and time $t = 0.1$

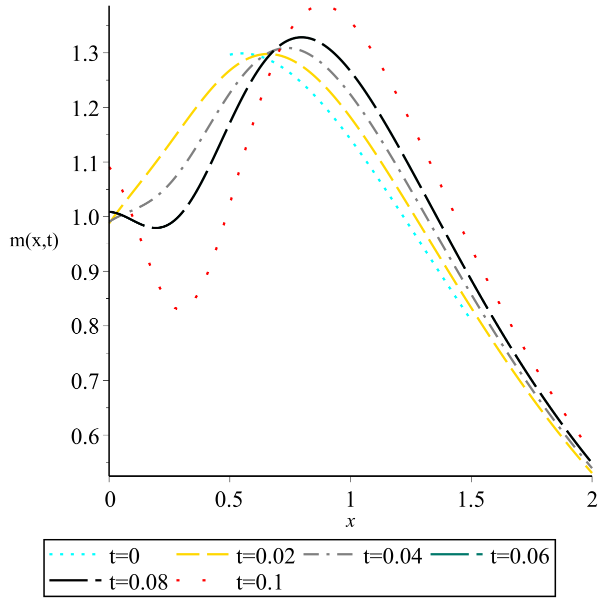


Figure 6: In east-west component of wind solution behaviour of $m(x,t)$ for different times t in 2D

7 Results and Discussion

The graphical and numerical results obtained with ATHPM are discussed in this section.. Maple software is used to generate numerical solutions and graphs.

The graphical solutions of $\eta(x,t), p(x,t)$ and $m(x,t)$ with $\alpha = 1$, $0 \leq x \leq 2$ and fixed $t = 0.1$ are shown in Figure-5. For various time t and $\alpha = 1$ figure-6 represented the solution behaviour of the east-west wind component $m(x,t)$ versus distance $0 \leq x \leq 2$. Tables 2–4 demonstrates a numerical values of $\eta(x,t), p(x,t)$ and $m(x,t)$ for $t = 0 - 0.1$ and $x = 0.2 - 2$. On the other hand tables 5-8 shows that the comparison of FRDTM, EADM , HAM, and q-HAShTM with ATHPM for $\alpha = 1$. $m(x,t), p(x,t)$ and $\eta(x,t)$ for different values of α are shown in table-9. The obtained findings show that the ATHPM can quickly solve nonlinear time-fractional differential equations without finding Adomian polynomials and without using shape parameters, discretization, transformation, restricted assumptions, or linearization. ATHPM can also be applied to equations that emerge in real-life situations.

8 Conclusion

To study fractional order systems involving partial differential equations of atmospheric internal waves models, the analytical approach known as ATHPM is applied which results in an accurate view of the phenomenon. To further understand fractional behavior, a graphical and numerical simulation is shown. The EADM, HAM, FRDTM, and q -HAShTM are compared to the ATHPM. The Aboodh transform provides a more quick fractional-order solution, hence the ATHPM delivers a superior solution with faster convergence. All findings show the significance of applying

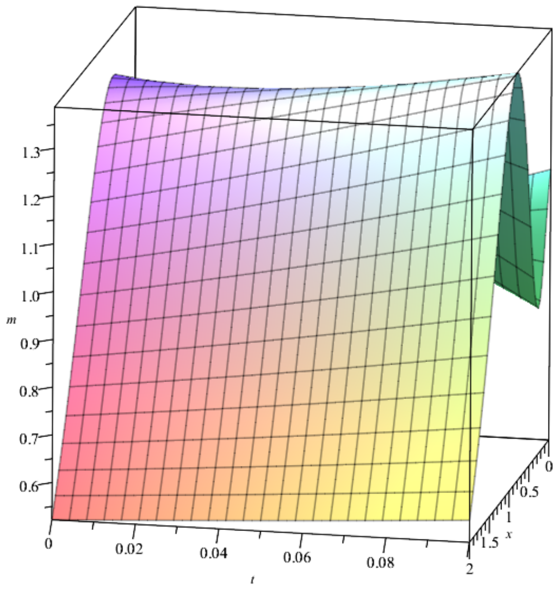


Figure 7: Solution for $\alpha = 1$ of $m(x, t)$

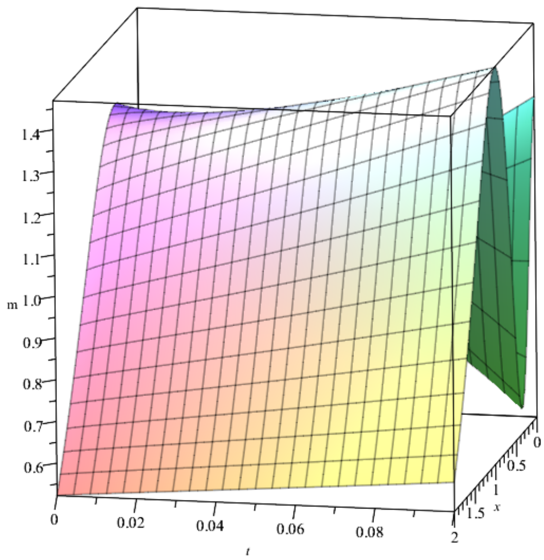


Figure 8: Solution for $\alpha = 0.87$ of $m(x, t)$

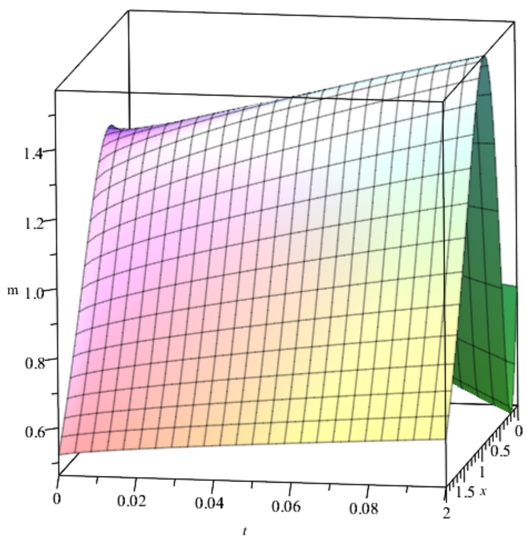


Figure 9: Solution for $\alpha = 0.75$ of $m(x, t)$

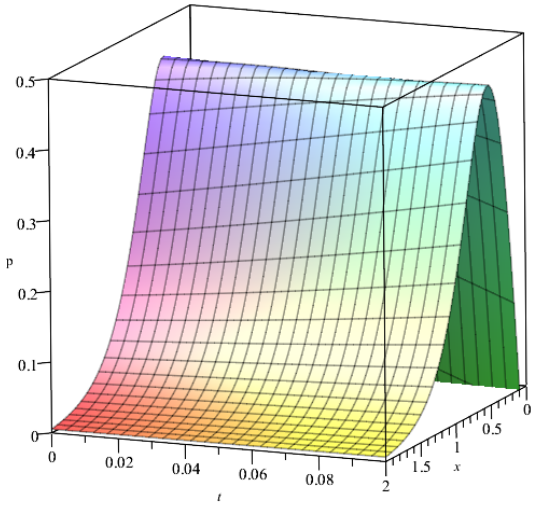


Figure 10: Solution for $\alpha = 1$ of $p(x,t)$

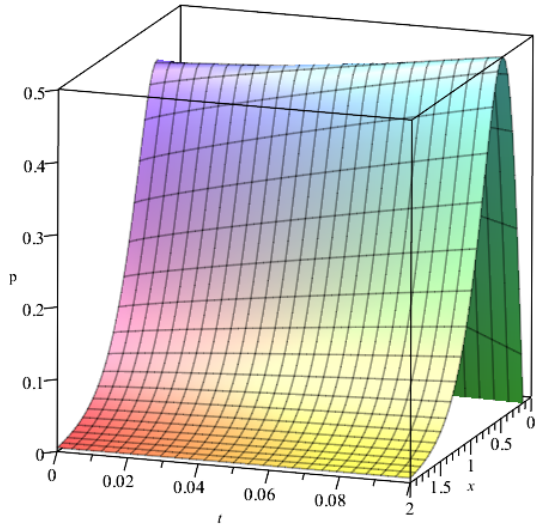


Figure 11: Solution for $\alpha = 0.87$ of $p(x,t)$

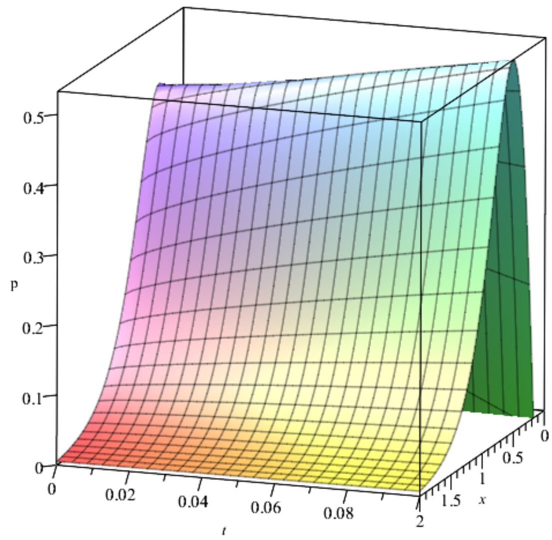


Figure 12: Solution for $\alpha = 0.75$ of $p(x,t)$

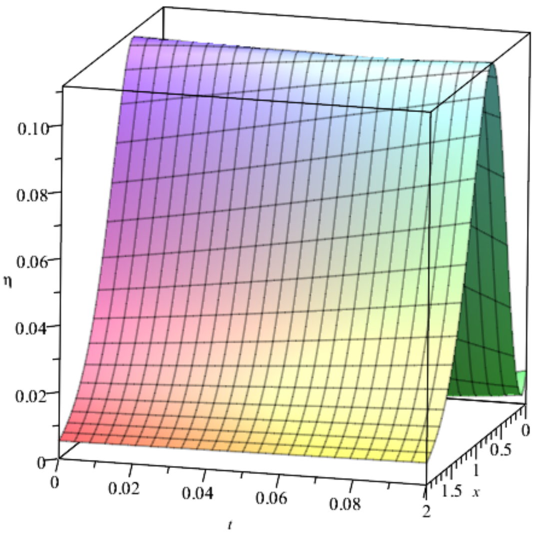


Figure 13: Solution for $\alpha = 1$ of $\eta(x,t)$

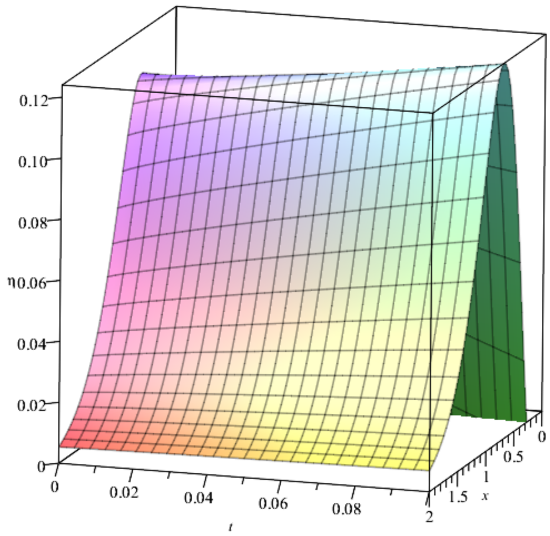


Figure 14: Solution for $\alpha = 0.87$ of $\eta(x,t)$

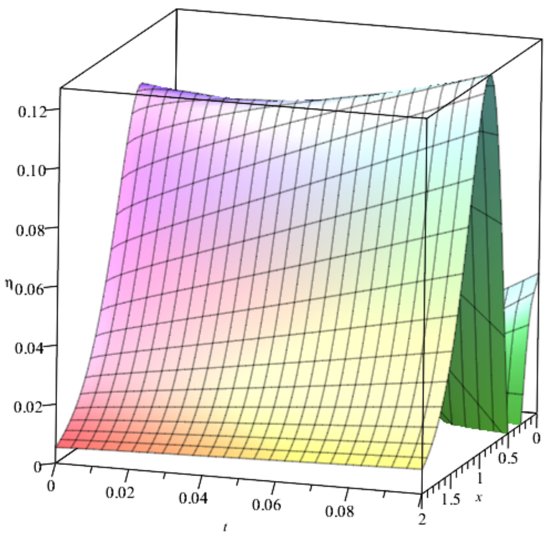


Figure 15: Solution for $\alpha = 0.75$ of $\eta(x,t)$

Table 2: $m(x,t)$ for different times and spaces

x	t=0	t=0.02	t=0.04	t=0.06	t=0.08	t=0.1
0.2	1.173820550	1.103172621	1.038358879	0.9793793252	0.9262339585	0.8789227794
0.4	1.276463074	1.221795678	1.155624440	1.077949358	0.9887704326	0.8880876644
0.6	1.296579219	1.293786401	1.280368008	1.256324040	1.221654496	1.176359377
0.8	1.244200157	1.275269388	1.303347263	1.328433782	1.350528944	1.369632750
1	1.141608621	1.181684582	1.223232313	1.266251819	1.310743094	1.356706143
1.2	1.012702051	1.048371315	1.086544320	1.127221067	1.170401555	1.216085785
1.4	0.8765414446	0.9037527431	0.9330246567	0.9643571852	0.9977503289	1.033204088
1.6	0.7455654871	0.7647116275	0.7852061297	0.8070489936	0.8302402192	0.8547798067
1.8	0.6264916546	0.6393585415	0.6530114265	0.6674503098	0.6826751912	0.6986860709
2	0.5220429082	0.5304667269	0.5393190947	0.5486000112	0.5583094769	0.5684474914

Table 3: $p(x,t)$ for different times and spaces

x	t=0	t=0.02	t=0.04	t=0.06	t=0.08	t=0.1
0.2	0.3422555144	0.3135100932	0.2831924086	0.2513024605	0.2178402488	0.1828057736
0.4	0.4472441342	0.4474319116	0.4444222236	0.4382150702	0.4288104515	0.4162083673
0.6	0.3660239954	0.3815899132	0.3966174619	0.4111066415	0.4250574519	0.4384698934
0.8	0.2408433212	0.2559374897	0.2720058396	0.2890483713	0.3070650843	0.3260559789
1	0.1413016497	0.1509766225	0.1615646546	0.1730657458	0.1854798961	0.1988071058
1.2	0.0777210584	0.0828614652	0.0885099221	0.0946664291	0.1013309861	0.1085035932
1.4	0.0411114623	0.0435768285	0.0462699408	0.0491907990	0.0523394032	0.0557157534
1.6	0.0211974330	0.0223104026	0.0235141643	0.0248087181	0.0261940639	0.0276702018
1.8	0.0107348007	0.0112194393	0.0117377811	0.0122898262	0.0128755746	0.0134950262
2	0.00536380273	0.00557119144	0.0057905238	0.00602179995	0.00626501974	0.00652018322

Table 4: $\eta(x,t)$ for different times and spaces

x	t=0	t=0.02	t=0.04	t=0.06	t=0.08	t=0.1
0.2	0.03422555144	0.02743869345	0.02167783781	0.01694298455	0.01323413363	0.01055128506
0.4	0.08944882683	0.08309263639	0.07582468284	0.06764496616	0.05855348637	0.04855024346
0.6	0.1098071986	0.1094337009	0.1078833848	0.1051562503	0.1012522974	0.09617152618
0.8	0.09633732851	0.09984669912	0.1031119894	0.1061331995	0.1089103292	0.1114433787
1	0.07065082484	0.07460828512	0.07880431829	0.08323892424	0.08791210301	0.09282385470
1.2	0.04663263506	0.04953865976	0.05271176788	0.05615195940	0.05985923436	0.06383359274
1.4	0.02877802360	0.03053330834	0.03245661574	0.03454794577	0.03680729845	0.03923467378
1.6	0.01695794646	0.01790538959	0.01893714073	0.02005319986	0.02125356701	0.02253824216
1.8	0.009661320681	0.01013715998	0.01065032848	0.01120082618	0.01178865308	0.01241380919
2	0.005363802732	0.005591277308	0.005834030580	0.006092062549	0.006365373213	0.006653962571

Table 5: Comparision of ATHPM with EADM for $t=0$ and $t=0.02$

	$t=0$						$t=0.02$					
x	ATHPM			EADM			ATHPM			EADM		
	$m(x,t)$	$p(x,t)$	$\eta(x,t)$	$m(x,t)$	$p(x,t)$	$\eta(x,t)$	$m(x,t)$	$p(x,t)$	$\eta(x,t)$	$m(x,t)$	$p(x,t)$	$\eta(x,t)$
0.2	1.173820550	0.3422555144	0.03422555144	1.17382	0.342256	0.0342256	1.103172621	0.3135100932	0.02743869345	1.10345	0.31351	0.0274387
0.4	1.276463074	0.4472441342	0.08944882683	1.27646	0.447244	0.0894488	1.221795678	0.4474319116	0.08309263639	1.22178	0.447431	0.0830926
0.6	1.296579219	0.3660239954	0.1098071986	1.29658	0.366024	0.109807	1.293786401	0.3815899132	0.1094337009	1.29363	0.381589	0.109434
0.8	1.244200157	0.2408433212	0.09633732851	1.2442	0.240843	0.0963373	1.275269388	0.2559374897	0.09984669912	1.27512	0.255937	0.0998467
1	1.141608621	0.1413016497	0.07065082484	1.14161	0.141302	0.0706508	1.181684582	0.1509766225	0.07460828512	1.18159	0.150976	0.0746083
1.2	1.012702051	0.0777210584	0.04663263506	1.0127	0.0777211	0.0466326	1.048371315	0.0828614652	0.04953865976	1.04832	0.0828609	0.0495387
1.4	0.8765414446	0.0411114623	0.02877802360	0.876541	0.0411115	0.028778	0.9037527431	0.0435768285	0.03053330834	0.903729	0.0435763	0.0305333
1.6	0.7455654871	0.0211974330	0.01695794646	0.745565	0.0211974	0.0169579	0.76471162756	0.0223104026	0.01790538959	0.764701	0.0223098	0.0179054
1.8	0.6264916546	0.0107348007	0.009661320681	0.626492	0.0107348	0.00966132	0.6393585415	0.0112194393	0.01013715998	0.639354	0.0112189	0.0101372
2	0.5220429082	0.00536380273	0.005363802732	0.522043	0.0053638	0.0053638	0.5304667269	0.00557119144	0.005591277308	0.530465	0.00557063	0.00559128

Table 6: Comparision of ATHPM with EADM for $t=0.04$ and $t=0.06$

	$t=0.04$						$t=0.06$					
x	ATHPM			EADM			ATHPM			EADM		
	$m(x,t)$	$p(x,t)$	$\eta(x,t)$	$m(x,t)$	$p(x,t)$	$\eta(x,t)$	$m(x,t)$	$p(x,t)$	$\eta(x,t)$	$m(x,t)$	$p(x,t)$	$\eta(x,t)$
0.2	1.038358879	0.2831924086	0.02167783781	1.03948	0.283191	0.0057894	0.9793793252	0.2513024605	0.01694298455	0.981895	0.251301	0.0169429
0.4	1.155624440	0.4444222236	0.07582468284	1.15555	0.444421	0.0758246	1.077949358	0.4382150702	0.06764496616	1.07779	0.438213	0.0676449
0.6	1.280368008	0.3966174619	0.1078833848	1.27973	0.396616	0.107883	1.256324040	0.4111066415	0.1051562503	1.2549	0.411105	0.105156
0.8	1.303347263	0.2720058396	0.1031119894	1.30276	0.272005	0.103112	1.328433782	0.2890483713	0.1061331995	1.32712	0.289047	0.106133
1	1.223232313	0.1615646546	0.07880431829	1.22286	0.161564	0.0788043	1.266251819	0.1730657458	0.08323892424	1.26542	0.173064	0.0832389
1.2	1.086544320	0.0885099221	0.05271176788	1.08635	0.0885088	0.0527118	1.127221067	0.0946664291	0.05615195940	1.12678	0.0946647	0.0561519
1.4	0.9330246567	0.0462699408	0.03245661574	0.932931	0.0462688	0.0324566	0.9643571852	0.0491907990	0.03454794577	0.964146	0.0946647	0.0345479
1.6	0.7852061297	0.0235141643	0.01893714073	0.785163	0.023513	0.0189371	0.8070489936	0.0248087181	0.02005319986	0.806953	0.024807	0.0200532
1.8	0.6530114265	0.0117377811	0.01065032848	0.652993	0.0117367	0.0106503	0.6674503098	0.0122898262	0.01120082618	0.667408	0.0122881	0.0112008
2	0.5393190947	0.0057905238	0.005834030580	0.539311	0.0057894	0.00583403	0.5486000112	0.00602179995	0.006092062549	0.548582	0.00602011	0.00609206

Table 7: Comparision of ATHPM with EADM for $t = 0.08$ and $t = 0.1$

	$t=0.08$						$t=0.1$					
x	ATHPM			EADM			ATHPM			EADM		
	$m(x,t)$	$p(x,t)$	$\eta(x,t)$	$m(x,t)$	$p(x,t)$	$\eta(x,t)$	$m(x,t)$	$p(x,t)$	$\eta(x,t)$	$m(x,t)$	$p(x,t)$	$\eta(x,t)$
0.2	0.9262339585	0.2178402488	0.01323413363	0.930707	0.217838	0.0132341	0.8789227794	0.1828057736	0.01055128506	0.885912	0.182803	0.0105512
0.4	0.9887704326	0.4288104515	0.05855348637	0.988485	0.428808	0.0585534	0.8880876644	0.4162083673	0.04855024346	0.887641	0.416206	0.0485501
0.6	1.221654496	0.4250574519	0.1012522974	1.21912	0.425055	0.101252	1.176359377	0.4384698934	0.09617152618	1.1724	0.438467	0.0961714
0.8	1.350528944	0.3070650843	0.1089103292	1.34819	0.307063	0.10891	1.369632750	0.3260559789	0.1114433787	1.36598	0.326053	0.111443
1	1.310743094	0.1854798961	0.08791210301	1.30927	0.185478	0.0879121	1.356706143	0.1988071058	0.09282385470	1.3544	0.198804	0.0928238
1.2	1.170401555	0.1013309861	0.05985923436	1.16962	0.101329	0.0598592	1.216085785	0.1085035932	0.06383359274	1.21486	0.108501	0.0638336
1.4	0.9977503289	0.0523394032	0.03680729845	0.997374	0.0523372	0.0368073	1.033204088	0.0557157534	0.03923467378	1.03262	0.0557129	0.0392347
1.6	0.8302402192	0.0261940639	0.02125356701	0.830069	0.0261918	0.0212536	0.8547798067	0.0276702018	0.02253824216	0.854513	0.0276674	0.0225382
1.8	0.6826751912	0.0128755746	0.01178865308	0.6826	0.0128733	0.0117887	0.6986860709	0.0134950262	0.01241380919	0.698569	0.0134922	0.0124138
2	0.5583094769	0.00626501974	0.006365373213	0.558277	0.00626277	0.00636537	0.5684474914	0.00652018322	0.006653962571	0.568397	0.00651737	0.00665396

Table 8: Comparision of ATHPM with q-HAShTM and FRDTM

		$m(x,t)$			$p(x,t)$			$\eta(x,t)$		
t	x	ATHPM	q-HAShTM	FRDTM	ATHPM	q-HAShTM	FRDTM	ATHPM	q-HAShTM	FRDTM
	0.2	1.103172621	1.10345	1.10403	0.3135100932	0.31350	0.31364	0.02743869345	0.02743	0.02745
	0.4	1.221795678	1.12217	1.22231	0.4474319116	0.44743	0.44744	0.08309263639	0.083092	0.08317
0.02	0.6	1.293786401	1.29362	1.29335	0.3815899132	0.38158	0.38151	0.1094337009	0.10943	0.10940
	0.8	1.275269388	1.27512	1.27483	0.2559374897	0.25593	0.25593	0.09984669912	0.099846	0.09981
	1	1.181684582	1.18159	1.18152	0.1509766225	0.15097	0.15099	0.07460828512	0.07460	0.07460
	0.2	1.038358879	1.03947	1.04412	0.2831924086	0.28319	0.28422	0.02167783781	0.02167	0.02180
	0.4	1.155624440	1.15555	1.15984	0.4444222236	0.44442	0.44446	0.07582468284	0.07582	0.07642
0.04	0.6	1.280368008	1.27934	1.27748	0.3966174619	0.39661	0.39599	0.1078833848	0.10788	0.10764
	0.8	1.303347263	1.30276	1.30042	0.2720058396	0.27200	0.27191	0.1031119894	0.10311	0.10285
	1	1.223232313	1.22286	1.22226	0.1615646546	0.16156	0.16169	0.07880431829	0.07880	0.07877

Table 9: $m(x,t), p(x,t)$ and $\eta(x,t)$ for different values of α

x	t	$\alpha=0.7$			$\alpha=0.8$			$\alpha=0.9$		
		m	p	η	m	p	η	$m(x,t)$	$p(x,t)$	$\eta(x,t)$
	0.02	0.9611309699	0.2295197932	0.01688377572	1.020616909	0.2713531978	0.02051851477	1.068320863	0.2972160908	0.02433995623
	0.04	0.8781292798	0.1456886615	0.01481581576	0.9322570604	0.2120217399	0.01478829825	0.9892496458	0.2548946451	0.01794119246
0.2	0.06	0.8375803351	0.06594963945	0.01838115186	0.8710500172	0.1536723316	0.01285406039	0.9247746668	0.2118912000	0.01372572932
	0.08	0.8249549153	-0.01250269780	0.02581756331	0.8295641884	0.09493503456	0.01380676792	0.8722248341	0.1676931900	0.01136921875
	0.1	0.8335752126	-0.0906958639	0.03628421525	0.8041797316	0.03530762350	0.01718416892	0.8302331396	0.1221067403	0.01069759163
	0.02	1.294000000	0.1817939521	0.08631826809	1.238894884	0.1659988930	0.08046014816	1.204015431	0.1566663603	0.07686336903
	0.04	1.401738083	0.2148791161	0.09814007397	1.317345807	0.1882426846	0.08875949601	1.261166342	0.1718979432	0.08274675390
1	0.06	1.501365291	0.2478692445	0.1094723791	1.392590382	0.2111351588	0.09698005489	1.318059640	0.1880031554	0.08876113734
	0.08	1.597359803	0.2814063943	0.1206840057	1.466840388	0.2349765153	0.1053011593	1.375514740	0.2050986502	0.09497386721
	0.1	1.691462518	0.3156720389	0.1319069450	1.540973169	0.2598413684	0.1137865124	1.433868065	0.2232144324	0.1014096041
	0.02	0.5548662379	0.006181138977	0.006274759222	0.5427508611	0.005876642750	0.005931040609	0.5352244229	0.005689068813	0.005721737855
	0.04	0.5790243733	0.006793568518	0.006974136009	0.5599471060	0.006308256373	0.006417361970	0.5475438659	0.005995948013	0.006063940797
2	0.06	0.6018757573	0.007378480125	0.007650605727	0.5767734209	0.006734316195	0.006903148041	0.5600091392	0.006308754875	0.006416300267
	0.08	0.6242678649	0.007955641450	0.008324135889	0.5936447607	0.007164454257	0.007398050824	0.5727751353	0.006631100028	0.006782459605
	0.1	0.6465155957	0.008532208619	0.009001643064	0.6107162224	0.007602146292	0.007905346888	0.5859016265	0.006964327090	0.007163692548

ATHPM to better understand the behavior of internal atmospheric waves. Finally, because internal waves have such a large impact on climate change, the fractional solution offers a practical look into their physical behavior which can help in the refinement of climate and weather models.

References

[1] K.S. Aboodh, *The new integral transform'Aboodh transform*, Glob. J. Pure Appl. Math. **9** (2013), no. 1, 35–43.

[2] K.S. Aboodh, A. Idris and R.I. Nuruddeen, *On the Aboodh transform connections with some famous integral transforms*, Int. J. Eng. Inf. Syst. **1** (2017), 143–151.

[3] A.N. ALbukhuttar and Y.A. AL-Rikabi, *General formulas for solution some important partial differential equations using El-Zaki transform*, Int. J. Nonlinear Anal. Appl. **13** (2022), no. 1, 3181–3190.

[4] M.G.S. AL-Safi, A.A. Yousif and M.S. Abbas, *Numerical investigation for solving non-linear partial differential equation using Sumudu-Elzaki transform decomposition method*, Int. J. Nonlinear Anal. Appl. **13** (2022), no. 1, 963–973.

[5] A.A. Alshikh and M.M. Abdelrahim Mahgob, *A comparative study between Laplace transform and two new integrals "Elzaki" transform and "Aboodh" transform*, Pure Appl. Math. J. **5** (2016), no. 5, 145–150.

[6] B. Bagharzadehtvasani, A.H. Refahi Sheikhani and H. Aminikhah, *A numerical scheme for solving variable order Caputo-Prabhakar fractional integro-differential equation*, Int. J. Nonlinear Anal. Appl. **13** (2022), no. 1, 467–484.

[7] Z. Busrah, and T. Bakhtiar, *Application of homotopy and variational iteration methods to the atmospheric internal waves model*, ISOR J. Math. **10** (2014), no. 5, 46–53.

[8] S. Chakrabarti, *Handbook of Offshore Engineering (2-volume set)*, Elsevier, 2005.

[9] C. Clancy and P. Lynch, *Laplace transform integration of the shallow-water equations, Part I: Eulerian formulation and Kelvin waves*, Quart. J. Royal Meteor. Soc. **137** (2011), no. 656, 792–799.

[10] J. Coiffier, *Fundamentals of numerical weather prediction*, Cambridge University Press, 2011.

[11] C.T. Duba and J.F. McKenzie, *Propagation properties of Rossby waves for latitudinal β-plane variations of f and zonal variations of the shallow water speed*, Ann. Geophys. **30** (2012), no. 5, 849–855.

[12] F. Ferdous and M.d. Golam Hafez, *Oblique closed form solutions of some important fractional evolution equations via the modified Kudryashov method arising in physical problems*, J. Ocean Engin. Sci. **3** (2018), no. 3, 244–252.

[13] A. Goswami, J. Singh, D. Kumar and S. Gupta, *An efficient analytical technique for fractional partial differential equations occurring in ion acoustic waves in plasma*, J. Ocean Engin. Sci. **4** (2019), no. 2, 85–99.

[14] V.S. Gulakaram, N.K. Vissa and P.K. Bhaskaran, *Role of mesoscale eddies on atmospheric convection during summer monsoon season over the Bay of Bengal: A case study*, J. Ocean Engin. Sci. **3** (2018), no. 4, 343–354.

[15] J.-H. He, *A coupling method of a homotopy technique and a perturbation technique for non-linear problems*, Int. J. Nonlinear Mech. **35** (2000), no. 1, 37–43.

[16] A.A. Imani, D.D. Ganji and Houman B. Rokni, *Approximate traveling wave solution for shallow water wave equation*, Appl. Math. Modell. **36** (2012), no. 4, 1550–1557.

[17] H.P. Jani and T.R. Singh, *Study of concentration arising in longitudinal dispersion phenomenon by Aboodh transform homotopy perturbation method*, Int. J. Appl. Comput. Math. **8** (2022), no. 4, 1–10.

[18] A.C. Loyinmi and T.K. Akinfe, *Exact solutions to the family of Fisher's reaction-diffusion equation using Elzaki homotopy transformation perturbation method*, Engin. Rep. **2** (2020), no. 2, e12084.

[19] V.J. Prajapati and R. Meher, *A robust analytical approach to the generalized Burgers-Fisher equation with fractional derivatives including singular and non-singular kernels*, J. Ocean Engin. Sci. In Press, (2022). https://doi.org/10.1016/j.joes.2022.06.035

[20] T. Patel, H. Patel and R. Meher, *Analytical study of atmospheric internal waves model with fractional approach*, J. Ocean Engin. Sci. In Press, (2022). https://doi.org/10.1016/j.joes.2022.02.004

[21] S. Rashid, R. Ashraf and Z. Hammouch, *New generalized fuzzy transform computations for solving*

fractional partial differential equations arising in oceanography, J. Ocean Engin. Sci. In Press, (2021). https://doi.org/10.1016/j.joes.2021.11.004

[22] P.P. Sartanpara and R. Meher, *A robust fuzzy-fractional approach for the atmospheric internal wave model*, J. Ocean Engin. Sci. In Press, (2022). https://doi.org/10.1016/j.joes.2022.02.001

[23] R. Seager and D.S. Battisti, *Challenges to our understanding of the general circulation: Abrupt climate change*, Glob. Circul. Atmosphere **332** (2007), 372.

[24] P. Tandel, H. Patel and T. Patel, *Tsunami wave propagation model: A fractional approach*, J. Ocean Engin. Sci. In Press, (2021). https://doi.org/10.1016/j.joes.2021.10.004

[25] M.Y. Turki, S.Y. Alku and M.S. Mechee, *The general implicit-block method with two-points and extra derivatives for solving fifth-order ordinary differential equations*, Int. J. Nonlinear Anal. Appl. **13** (2022), no. , 1081–1097.

[26] A.C. Varsoliwala and T.R. Singh, *Mathematical modelling of atmospheric internal waves phenomenon and its solution by Elzaki Adomian decomposition method*, J. Ocean Engin. Sci. **7** (2022), no. 3, 203–212.

[27] D.L. Williamson, J.B. Drake, J.J. Hack, R. Jakob and P.N. Swarztrauber, *A standard test set for numerical approximations to the shallow water equations in spherical geometry*, J. Comput. Phys. **102** (1992), no 1, 211–224.