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Hyers–Ulam–Rassias stability of Jensen's functional equation on fuzzy normed linear spaces

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Abstract

In this paper, the Hyers–Ulam–Rassias stability of Jensen's functional equation is discussed in detail on fuzzy normed linear spaces. It is shown through the analysis that every fuzzy approximate Jensen-type mapping can be approximated by an additive mapping. The obtained results are the fuzzy versions of some classical theorems.

Keywords: Fuzzy norm, Fuzzy normed linear space, Functional equation
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1 Introduction

In 1992, Felbin [12] proposed an alternative concept of fuzzy norm on a linear space with a Kaleva and Seikkala-type related metric [19]. He proved the existence of a completion for every finite-dimensional normed linear space. This notion of fuzzy norm was then updated by Xiao and Zhu [38], who investigated the topological features of fuzzy normed linear spaces. Bag and Samanta [4] defined yet another fuzzy norm. Weak fuzzy boundedness, strong fuzzy boundedness, fuzzy continuity, strong fuzzy continuity, weak fuzzy continuity, sequential fuzzy continuity, and fuzzy norm of linear operators with respect to an associated fuzzy norm were defined by Bag and Samanta [5].

The first stability problem was proposed by Ulam [37] in 1940. In the next year, Hyers [15] solved the problem posed by Ulam. Hyers' conclusion is known as the direct technique, in which an approximate solution close to the exact solution is directly produced from the provided function, and it is the most influential process for achieving the stability of many types of functional equations. Th. M. Rassias [27] used this direct technique to extend Hyers' conclusion and develop a new result validating the stability of the Cauchy additive functional equation using the sum of powers of norms as an upper bound. Aoki [1] established a special example of Th. M. Rassias' theorem on the Ulam–Hyers stability of additive mappings. The results obtained by Rassias have been extended and called the Hyers–Ulam–Rassias stability theory for functional equations. J. M. Rassias ([25, 26, 28]) proposed a further generalization of Hyers' conclusion and proved a theorem based on weaker assumptions constrained by a product of powers of norms between 1982 and 1989. By considering a broad control function as a function of variables, Găvruta [14] extended Hyers' theorem. Ravi et al. [29] studied the stability of a new quadratic functional equation in 2008.

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Jung [18] established the Hyers–Ulam stability of the Jensen equation on a constrained domain, and used the finding to analyze the asymptotic characteristic of an intriguing additive mapping. In [20], the stability of functional inequalities was investigated related to Cauchy–Jensen additive functional equalities in non-Archimedean Banach spaces. Several mathematicians have pointed out intriguing applications of the Hyers–Ulam–Rassias stability theory to a variety of issues.

Other techniques for investigating the stability of functional equations include fixed point methods. The fixed point alternative can provide an approximate answer that is close to the exact solution. In 1996, Isac and Th. M. Rassias [16] determined the way the stability theorem of functional equations could be used to prove novel fixed point theorems with applications. Further details on the fixed point approach to the stability of functional equations can be found in [6, 9, 10, 24, 33].

An important extension of the additive Cauchy equation is Jensen's functional equation:

$$2f((x+y)/2) = f(x) + f(y).$$

Every solution of Jensen's functional equation is called a *Jensen function*. Kominek [21] was the first to study the stability of this equation. Also, the fuzzy version of the Hyers–Ulam–Rassias stability of Jensen's functional equation has been studied. We refer the reader to [23] for more details.

The generalized Hyers–Ulam stability of the reciprocal functional equation

$$g(x+y) = (g(x)g(y))/(g(x) + g(y))$$

was studied by Ravi and Senthil Kumar [31]. The "reciprocal formula" of every electric circuit with a couple of parallel resistors matches this functional equation [30]. Jung [17] investigated the Hyers–Ulam stability of the reciprocal functional equation using the fixed point approach. Some other examples of reciprocal-type functional equations can be found in [8, 32]. Furthermore, in [34], the new rational functional equation

$$j((x+y)/2) = (2j(x)j(y))/(j(x) + j(y))$$

and its solution were studied, and various stabilities were investigated for the equation by using a fixed point approach.

Moreover, some mathematicians have introduced appropriate versions of fuzzy approximately additive functions on fuzzy normed spaces and intuitionistic fuzzy normed spaces [2, 3, 7, 11, 22, 35, 36].

In this paper, we use the definition of fuzzy normed spaces given in [5], and present the fuzzy version of Hyers–Ulam–Rassias stability of Jensen's functional equations proved by Jung [18] on classical normed linear spaces. Also, we show that every approximate Jensen-type mapping can be approximated by an additive mapping on fuzzy normed linear spaces.

2 Preliminaries

In this section, we present some preliminaries that will be used in the next section of this paper.

Definition 2.1. ([19]) A mapping $\tilde{\eta} : \mathbb{R} \longrightarrow [0,1]$ is called a *fuzzy real number* with α-level set $[\tilde{\eta}]_\alpha = \{t : \tilde{\eta}(t) \geq \alpha\}$ if it satisfies the following conditions.

(N1) There exists $t_0 \in \mathbb{R}$ such that $\tilde{\eta}(t_0) = 1$.

(N2) For each $\alpha \in (0,1]$, there exist real numbers $\eta_\alpha^- \leq \eta_\alpha^+$ such that the $\alpha-$level set $[\tilde{\eta}]_\alpha$ is equal to the closed interval $[\eta_\alpha^-, \eta_\alpha^+]$.

The set of all fuzzy real numbers is denoted by $F(\mathbb{R})$. Since each $r \in \mathbb{R}$ can be considered as the fuzzy real number $\tilde{r} \in F(\mathbb{R})$ defined by

$$\tilde{r}(t) = \begin{cases} 1, & t = r \\ 0, & \text{otherwise,} \end{cases}$$

it follows that $\mathbb{R}$ can be embedded in $F(\mathbb{R})$.

Definition 2.2. ([19]) On $F(\mathbb{R}) \times F(\mathbb{R})$, the arithmetic operations $+$, $-$, $\times$ and $/$ are defined by

$$\begin{aligned}
(\tilde{\eta}+\tilde{\gamma})(t) &= \sup_{t=x+y}(\min(\tilde{\eta}(x),\tilde{\gamma}(y))),\\
(\tilde{\eta}-\tilde{\gamma})(t) &= \sup_{t=x-y}(\min(\tilde{\eta}(x),\tilde{\gamma}(y))),\\
(\tilde{\eta}\times\tilde{\gamma})(t) &= \sup_{t=xy}(\min(\tilde{\eta}(x),\tilde{\gamma}(y))),\\
\text{and}\quad & \\
(\tilde{\eta}/\tilde{\gamma})(t) &= \sup_{t=x/y}(\min(\tilde{\eta}(x),\tilde{\gamma}(y))),
\end{aligned}$$

which are special cases of Zadeh's extension principle.

Definition 2.3. ([19]) Let $\tilde{\eta} \in F(\mathbb{R})$. If $\tilde{\eta}(t) = 0$ for all $t < 0$, then $\tilde{\eta}$ is called a *positive fuzzy real number*. The set of all positive fuzzy real numbers is denoted by $F^+(\mathbb{R})$.

Lemma 2.4. ([19]) Let $\tilde{\eta}, \tilde{\gamma} \in F(\mathbb{R})$ and $[\tilde{\eta}]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$, $[\tilde{\gamma}]_\alpha = [\gamma_\alpha^-, \gamma_\alpha^+]$. Then,

i) $[\tilde{\eta}+\tilde{\gamma}]_\alpha = [\eta_\alpha^- + \gamma_\alpha^-, \eta_\alpha^+ + \gamma_\alpha^+]$,
ii) $[\tilde{\eta}-\tilde{\gamma}]_\alpha = [\eta_\alpha^- - \gamma_\alpha^+, \eta_\alpha^+ - \gamma_\alpha^-]$,
iii) $[\tilde{\eta}\times\tilde{\gamma}]_\alpha = [\eta_\alpha^-\gamma_\alpha^-, \eta_\alpha^+\gamma_\alpha^+]$ for $\tilde{\eta}, \tilde{\gamma} \in F^+(R)$, and
iv) $[1/\tilde{\eta}]_\alpha = [\dfrac{1}{\eta_\alpha^+}, \dfrac{1}{\eta_\alpha^-}]$ if $\eta_\alpha^- > 0$.

Definition 2.5. ([19]) Let $\tilde{\eta}, \tilde{\gamma} \in F(\mathbb{R})$, and $[\tilde{\eta}]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$, $[\tilde{\gamma}]_\alpha = [\gamma_\alpha^-, \gamma_\alpha^+]$ for all $\alpha \in (0,1]$. Define a partial ordering by $\tilde{\eta} \le \tilde{\gamma}$ if and only if $\eta_\alpha^- \le \gamma_\alpha^-$ and $\eta_\alpha^+ \le \gamma_\alpha^+$ for all $\alpha \in (0,1]$. In $F(\mathbb{R})$, strict inequality is defined by $\tilde{\eta} < \tilde{\gamma}$ if and only if $\eta_\alpha^- < \gamma_\alpha^-$ and $\eta_\alpha^+ < \gamma_\alpha^+$ for all $\alpha \in (0,1]$.

Lemma 2.6. Let $\tilde{\eta} \in F(\mathbb{R})$. Then, $\tilde{\eta} \in F^+(\mathbb{R})$ if and only if $\tilde{0} \le \tilde{\eta}$.

Definition 2.7. ([5]) Let X be a linear space over $\mathbb{R}$ (the field of real numbers). Let N be a fuzzy subset of $X \times \mathbb{R}$ such that for all $x, u \in X$ and $c \in \mathbb{R}$, the following statements are true.

$(N1)$ $N(x,t) = 0$ for all $t \le 0$.
$(N2)$ $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$.
$(N3)$ If $c \ne 0$, then $N(cx,t) = N(x, t/|c|)$ for all $t \in \mathbb{R}$.
$(N4)$ $N(x+u, s+t) \ge \min\{N(x,s), N(u,t)\}$ for all $s,t \in \mathbb{R}$.
$(N5)$ $N(x,.)$ is a nondecreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.

Then, N is called a *fuzzy norm* on X.

We consider the following extra assumptions.

$(N6)$ $N(x,t) > 0$ for all $t > 0$ implies $x = 0$.
$(N7)$ For each $x \ne 0$, $N(x,.)$ is a continuous function on $\mathbb{R}$ which is strictly increasing on the subset $\{t : 0 < N(x,t) < 1\}$ of $\mathbb{R}$.

Definition 2.8. ([5]) Let (X,N) be a fuzzy normed linear space.

i) A sequence $\{x_n\} \subseteq X$ is said to *converge* to $x \in X$ ($\lim_{n\to\infty} x_n = x$) if $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$.
ii) A sequence $\{x_n\} \subseteq X$ is called *Cauchy* if $\lim_{n,m\to\infty} N(x_n - x_m, t) = 1$ for all $t > 0$.

3 Jensen's Functional Equation

Jung [18] studied the Hyers–Ulam–Rassias stability of Jensen's functional equation.

Theorem 3.1. ([18]) Let X be a real normed space, and Y be a real Banach space. Assume that $\delta, \theta \geq 0$ are fixed, and let $p > 0$ be given with $p \neq 1$. Suppose that a function $f : X \longrightarrow Y$ satisfies the functional inequality

$$\|2f((x+y)/2) - f(x) - f(y)\| \leq \delta + \theta(\|x\|^p + \|y\|^p), \tag{3.1}$$

for all $x, y \in X$. Furthermore, assume that $f(0) = 0$ and $\delta = 0$ in (3.1) for the case $p > 1$. Then, there exists a unique additive function $A : X \longrightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \begin{cases} \delta + \|f(0)\| + (2^{1-p} - 1)^{-1}\theta\|x\|^p & , \quad p < 1 \\ 2^{p-1}(2^{p-1} - 1)^{-1}\theta\|x\|^p & , \quad p > 1, \end{cases}$$

for all $x \in X$.

First, we state and prove the fuzzy version of the Hyers–Ulam–Rassias stability of Jensen's functional equation presented in Theorem 3.1 for $p > 1$.

Theorem 3.2. Let (X, N_1) and (Y, N_2) be fuzzy Banach spaces such that N_2 satisfies (N7), $p > 1$ and $\{\theta_\alpha\}_{\alpha \in (0,1)} \subseteq \mathbb{R}^+$. Moreover, let $f : X \longrightarrow Y$ be a function such that $f(0) = 0$ and for every $\alpha \in (0, 1)$, $x, y \in X$ and $s, t \geq 0$, $N_2(2f((x + y)/2) - f(x) - f(y), \theta_\alpha(t^p + s^p)) \geq \alpha$, where $N_1(x, t) \geq \alpha$ and $N_1(y, s) \geq \alpha$. Then, there exists a unique additive function $A : X \longrightarrow Y$ such that for every $\alpha \in (0, 1)$, $x \in X$ and $t \geq 0$, $N_2(f(x)-A(x), 2^p(2^{p-1}-1)^{-1}\theta_\alpha t^p) \geq \alpha$, where $N_1(x, t) \geq \alpha$.

Proof . Let $x \in X$, $t > 0$, $\alpha \in (0, 1)$ and $N_1(x, t) \geq \alpha$. If we put $y = 0$, then $N_1(y, s) \geq \alpha$ for all $s > 0$. Hence,

$$N_2(2f(x/2) - f(x), \theta_\alpha(t^p + s^p)) \geq \alpha, \text{ for all } s > 0.$$

If $t = s$, then $N_2(2f(x/2) - f(x), 2\theta_\alpha t^p) \geq \alpha$. By induction on n,

$$N_2(f(x) - 2^n f(2^{-n}x), 2\theta_\alpha t^p \sum_{k=0}^{n-1} 2^{(1-p)k}) \geq \alpha, \text{ for all } n > 0.$$

Thus, for every $n > m$,

$$\begin{aligned} N_2(2^n f(2^{-n}x) - 2^m f(2^{-m}x), s') &\geq N_2(2^n f(2^{-n}x) - 2^m f(2^{-m}x), s) \\ &= N_2(2^m 2^{(n-m)} f(2^{-(n-m)} 2^{-m}x) - 2^m f(2^{-m}x), s) \\ &\geq \alpha, \end{aligned}$$

where $s = 2\theta_\alpha t^p 2^{(1-p)m} \sum_{k=0}^{n-m-1} 2^{(1-p)k}$ and $s' = 2\theta_\alpha t^p 2^{(1-p)m}/(1 - 2^{1-p})$.

Now, let $r > 0$. Then, $\lim_{m \longrightarrow \infty}(r/(2\theta_\alpha t^p 2^{(1-p)m}/(1 - 2^{1-p})))^{1/p} = \infty$. Therefore,

$$\lim_{m \longrightarrow \infty} N_1(x, (r/(2\theta_\alpha t^p 2^{(1-p)m}/(1 - 2^{1-p})))^{1/p}) = 1.$$

Hence, there exists $N > 0$ such that $N_1(x, (r/(2\theta_\alpha t^p 2^{(1-p)m}/(1 - 2^{1-p})))^{1/p}) \geq \alpha$ for all $m > N$. Thus,

$$N_2(2^n f(2^{-n}x) - 2^m f(2^{-m}x), r) \geq \alpha, \text{ for all } n > m > N.$$

So, $\{2^n f(2^{-n}x)\}$ is a Cauchy sequence for each $x \in X$. Hence, the limit $A(x) = \lim_{n \to \infty} 2^n f(2^{-n}x)$ exists for each $x \in X$.

Now, we show that the function $A : X \longrightarrow Y$ is additive. Let $x, y \in X$, $t > 0$ and $\alpha \in (0, 1)$. Then,

$$\begin{aligned} N_2(A(x+y) - A(x) - A(y), t) \geq \min\{&N_2(A(x+y) - 2^{n+1} f(2^{-n-1}(x+y)), t/4), \\ &N_2(2^n f(2^{-n}x) - A(x), t/4), \\ &N_2(2^n f(2^{-n}y) - A(y), t/4), \\ &N_2(2^{n+1} f(2^{-n-1}(x+y)) - 2^n f(2^{-n}x) - 2^n f(2^{-n}y), t/4)\}. \end{aligned}$$

Since $\lim_{n\longrightarrow\infty} 2^{n(1-(1/p))}(t/8\theta_\alpha)^{1/p} = \infty$, it follows that

$$\begin{aligned}\lim_{n\longrightarrow\infty} N_1(x, 2^{n(1-(1/p))}(t/8\theta_\alpha)^{1/p}) &= \lim_{n\longrightarrow\infty} N_1(2^{-n}x, (2^{-n}t/8\theta_\alpha)^{1/p})\\ &= \lim_{n\longrightarrow\infty} N_1(y, 2^{n(1-(1/p))}(t/8\theta_\alpha)^{1/p})\\ &= \lim_{n\longrightarrow\infty} N_1(2^{-n}y, (2^{-n}t/8\theta_\alpha)^{1/p})\\ &= 1.\end{aligned}$$

So, there exists $N' > 0$ such that $N_1(2^{-n}x, (2^{-n}t/8\theta_\alpha)^{1/p}) \geq \alpha$ and $N_1(2^{-n}y, (2^{-n}t/8\theta_\alpha)^{1/p}) \geq \alpha$ for all $n \geq N'$. Thus,

$$\begin{aligned}&N_2(2^{n+1}f(2^{-n-1}(x+y)) - 2^n f(2^{-n}x) - 2^n f(2^{-n}y), t/4) =\\ &\quad N_2(2f((2^{-n}(x+y))/2) - f(2^{-n}x) - f(2^{-n}y), 2^{-n}t/4) \geq \alpha, \text{ for all } n \geq N'.\end{aligned}$$

Since $A(x) = \lim_{n\to\infty} 2^n f(2^{-n}x)$ for each $x \in X$, there exists $N'' > 0$ such that

$$\begin{aligned}&\min\{N_2(A(x+y) - 2^{n+1}f(2^{-n-1}(x+y)), t/4),\\ &\quad N_2(2^n f(2^{-n}x) - A(x), t/4), N_2(2^n f(2^{-n}y) - A(y), t/4)\} \geq \alpha,\end{aligned}$$

for all $n \geq N''$.

If we let $N = \max\{N', N''\}$, then

$$\begin{aligned}N_2(A(x+y) - A(x) - A(y), t) \geq &\min\{N_2(A(x+y) - 2^{n+1}f(2^{-n-1}(x+y)), t/4),\\ &N_2(2^n f(2^{-n}x) - A(x), t/4), N_2(2^n f(2^{-n}y) - A(y), t/4),\\ &N_2(2^{n+1}f(2^{-n-1}(x+y)) - 2^n f(2^{-n}x) - 2^n f(2^{-n}y), t/4)\}\\ \geq &\alpha, \text{ for all } n \geq N.\end{aligned}$$

This implies that $N_2(A(x+y) - A(x) - A(y), t) \geq \alpha$ for all $\alpha \in (0,1)$. Therefore,

$$N_2(A(x+y) - A(x) - A(y), t) = 1 \text{ for all } t > 0.$$

So, $A(x+y) = A(x) + A(y)$ for all $x, y \in X$.

In what follows, we show that for every $\alpha \in (0,1)$, $x \in X$ and $t \geq 0$,

$$N_2(f(x) - A(x), 2^p(2^{p-1}-1)^{-1}\theta_\alpha t^p) \geq \alpha, \text{ where } N_1(x,t) \geq \alpha.$$

Let $x \in X$, $t > 0$, $\alpha \in (0,1)$ and $N_1(x,t) \geq \alpha$. Then,

$$\lim_{n\to\infty} 2\theta_\alpha t^p \sum_{k=0}^{n-1} 2^{(1-p)k} = (1-2^{1-p})^{-1} 2\theta_\alpha t^p.$$

Assume that $\epsilon > 0$ is given. Since $\{2^n f(2^{-n}x)\}$ is a Cauchy sequence, there exists $N > 0$ such that

$$N_2(2^n f(2^{-n}x) - 2^m f(2^{-m}x), \epsilon) \geq \alpha, \text{ for all } m, n \geq N.$$

Since $A(x) = \lim_{n\to\infty} 2^n f(2^{-n}x)$, there exists $N' > 0$ such that

$$N_2(A(x) - 2^n f(2^{-n}x), (1-2^{p-1})^{-1}2\theta_\alpha t^p - 2\theta_\alpha t^p \sum_{k=0}^{N_4-1} 2^{(1-p)k}) \geq \alpha, \text{ for all } n \geq N'.$$

If $n \geq \max\{N, N'\}$, then

$$\begin{aligned}N_2(A(x) - f(x), s+\epsilon) \geq &\min\{, N_2(2^n f(2^{-n}x) - 2^{N_4} f(2^{-N_4}x), \epsilon)\\ &N_2(A(x) - 2^n f(2^{-n}x), s - 2\theta_\alpha t^p \sum_{k=0}^{N_4-1} 2^{(1-p)k}),\\ &N_2(f(x) - 2^{N_4} f(2^{-N_4}x), 2\theta_\alpha t^p \sum_{k=0}^{N_4-1} 2^{(1-p)k})\}\\ \geq &\alpha,\end{aligned}$$

where $s = (1 - 2^{1-p})^{-1} 2\theta_\alpha t^p$. Hence, $N_2(A(x) - f(x), (1 - 2^{1-p})^{-1} 2\theta_\alpha t^p + \epsilon) \geq \alpha$. By (N7) we obtain

$$N_2(A(x) - f(x), (1 - 2^{1-p})^{-1} 2\theta_\alpha t^p) = \lim_{\epsilon \to 0} N_2(A(x) - f(x), (1 - 2^{1-p})^{-1} 2\theta_\alpha t^p + \epsilon) \geq \alpha.$$

Finally, we show that the function $A : X \longrightarrow Y$ is unique. Let $A' : X \longrightarrow Y$ be another additive function which satisfies $N_2(f(x) - A'(x), 2^p(2^{p-1} - 1)^{-1}\theta_\alpha t^p) \geq \alpha$ for every $\alpha \in (0, 1)$, $x \in X$ and $t \geq 0$, where $N_1(x, t) \geq \alpha$. Let $t > 0$, $x \in X$ and $\alpha \in (0, 1)$. Since

$$\lim_{n \to \infty} 2^n([(2^{p-1} - 1)/(2^p\theta_\alpha)]2^{-n-1}t)^{1/p} = \infty,$$

there exists $N > 0$ such that

$$N_1(x, 2^n([(2^{p-1} - 1)/(2^p\theta_\alpha)]2^{-n-1}t)^{1/p}) = N_1(2^{-n}x, ([(2^{p-1} - 1)/(2^p\theta_\alpha)]2^{-n-1}t)^{1/p}) \geq \alpha,$$

for all $n \geq N$. Since A and A' are linear,

$$\begin{aligned} N_2(A(x) - A'(x), t) &= N_2(2^n(A(2^{-n}x) - A'(2^{-n}x)), t) \\ &= N_2(A(2^{-n}x) - A'(2^{-n}x), 2^{-n}t) \\ &\geq \min\{N_2(A(2^{-n}x) - f(2^{-n}x), 2^{-n-1}t), N_2(A'(2^{-n}x) - f(2^{-n}x), 2^{-n-1}t)\} \\ &\geq \alpha, \text{ for all } \alpha \in (0, 1). \end{aligned}$$

So, $N_2(A(x) - A'(x), t) = 1$ for all $t > 0$. Hence, $A(x) = A'(x)$ for all $x \in X$. □

In the next example, we show that if a function f satisfies the conditions of Theorem 3.1, then f satisfies the conditions of Theorem 3.2 for some fuzzy normed linear space. Therefore, Theorem 3.2 is a generalized fuzzy version of Theorem 3.1 for $p > 1$.

Example 3.3. Let $(X, \|.\|)$ be a Banach space, $\theta \in \mathbb{R}^+$, $p > 1$ and $f : X \longrightarrow X$ be a function such that $f(0) = 0$ and $\|2f((x + y)/2) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$.

We define a fuzzy norm N on X by

$$N(x, t) = \begin{cases} t/\|x\|, & 0 < t \leq \|x\|, \\ 1, & \|x\| < t, \\ 0, & t \leq 0, \end{cases}$$

where $x \in X$ and $t \in \mathbb{R}$.

Let $\{\alpha^{1-p}\theta\}_{\alpha \in (0,1)} \subseteq \mathbb{R}^+$. Suppose that $x, y \in X$, $s, t \in \mathbb{R}$, $\alpha \in (0, 1)$, $N(x, t) \geq \alpha$ and $N(y, s) \geq \alpha$. Now, we show that $N(2f((x + y)/2) - f(x) - f(y), \alpha^{1-p}\theta(t^p + s^p)) \geq \alpha$.

Case 1: If $\|2f((x + y)/2) - f(x) - f(y)\| < \alpha^{1-p}\theta(t^p + s^p)$, then

$$N(2f((x + y)/2) - f(x) - f(y), \alpha^{1-p}\theta(t^p + s^p)) = 1 \geq \alpha.$$

Case 2: $\alpha^{1-p}\theta(t^p + s^p) \leq \|2f((x + y)/2) - f(x) - f(y)\|$. Since $N(x, t) \geq \alpha$ and $N(y, s) \geq \alpha$, it follows that $t/\|x\| \geq \alpha$ and $s/\|y\| \geq \alpha$. Hence, $\alpha\|x\| \leq t$ and $\alpha\|y\| \leq s$. So,

$$\alpha^p\|f(x + y) - f(x) - f(y)\| \leq \alpha^p\theta(\|x\|^p + \|y\|^p) \leq \theta(t^p + s^p).$$

Thus,

$$N(f(x + y) - f(x) - f(y), \alpha^{1-p}\theta(t + s)) = \alpha^{1-p}\theta(t + s)/\|f(x + y) - f(x) - f(y)\| \geq \alpha.$$

Then, there exists a unique additive function $A : X \longrightarrow Y$ such that for every $\alpha \in (0, 1)$, $x \in X$ and $t \geq 0$, $N(f(x) - A(x), 2^p(2^{p-1} - 1)^{-1}\theta_\alpha t^p) \geq \alpha$, where $N(x, t) \geq \alpha$.

Corollary 3.4. Let (X, N_1) and (Y, N_2) be fuzzy Banach spaces such that N_2 satisfies (N7), $p > 1$ and $\theta \in F^+(\mathbb{R})$. Moreover, let $f : X \longrightarrow Y$ be a function such that $f(0) = 0$ and for every $\alpha \in (0, 1)$, $x, y \in X$ and $s, t \geq 0$, $N_2(2f((x + y)/2) - f(x) - f(y), \theta_\alpha^-(t^p + s^p)) \geq \alpha$, where $N_1(x, t) \geq \alpha$ and $N_1(y, s) \geq \alpha$. Then, there exists a unique additive function $A : X \longrightarrow Y$ such that for every $\alpha \in (0, 1)$, $x \in X$ and $t \geq 0$, $N_2(f(x) - A(x), 2^p(2^{p-1} - 1)^{-1}\theta_\alpha^- t^p) \geq \alpha$, where $N_1(x, t) \geq \alpha$.

Corollary 3.5. Let (X, N_1) and (Y, N_2) be fuzzy Banach spaces such that N_2 satisfies (N7), $p > 1$ and $\theta \in F^+(\mathbb{R})$. Moreover, let $f : X \longrightarrow Y$ be a function such that $f(0) = 0$ and for every $\alpha \in (0, 1)$, $x, y \in X$ and $s, t \geq 0$, $N_2(2f((x + y)/2) - f(x) - f(y), \theta^+_\alpha(t^p + s^p)) \geq \alpha$, where $N_1(x, t) \geq \alpha$ and $N_1(y, s) \geq \alpha$. Then, there exists a unique additive function $A : X \longrightarrow Y$ such that for every $\alpha \in (0, 1)$, $x \in X$ and $t \geq 0$, $N_2(f(x)-A(x), 2^p(2^{p-1}-1)^{-1}\theta^+_\alpha t^p) \geq \alpha$, where $N_1(x, t) \geq \alpha$.

Corollary 3.6. Let (X, N_1) and (Y, N_2) be fuzzy Banach spaces such that N_2 satisfies (N7), $p > 1$ and $\{\theta_\alpha\}_{\alpha\in[0,1]} \subseteq \mathbb{R}^+$. Moreover, let $f : X \longrightarrow Y$ be a function such that $f(0) = 0$ and

$$N_2(2f((x + y)/2) - f(x) - f(y), \theta_{\min\{N_1(x,t),N_1(y,s)\}}(t^p + s^p)) \geq \min\{N_1(x, t), N_1(y, s)\}, \text{ for all } x, y \in X.$$

Then, there exists a unique additive function $A : X \longrightarrow Y$ such that

$$N_2(f(x) - A(x), 2^p(2^{p-1} - 1)^{-1}\theta_{N_1(x,t)}t^p) \geq N_1(x, t), \text{ for all } x \in X.$$

In the following theorem, we state and prove the fuzzy version of the Hyers–Ulam–Rassias stability of Jensen's functional equation given in Theorem 3.1 for $0 < p < 1$.

Theorem 3.7. Let (X, N_1) and (Y, N_2) be fuzzy Banach spaces such that N_2 satisfies (N7), $0 < p < 1$ and $\{\theta_\alpha\}_{\alpha\in(0,1)}, \{\delta_\alpha\}_{\alpha\in(0,1)} \subseteq \mathbb{R}^+$. Moreover, let $f : X \longrightarrow Y$ be a function such that for every $\alpha \in (0, 1)$, $x, y \in X$, and $s, t \geq 0$, $N_2(2f((x+y)/2)-f(x)-f(y), \delta_\alpha + \theta_\alpha(t^p + s^p)) \geq \alpha$, where $N_1(x, t) \geq \alpha$ and $N_1(y, s) \geq \alpha$. Then, there exists a unique additive function $A : X \longrightarrow Y$ such that for every $\alpha \in (0, 1)$, $x \in X$ and $t, s \geq 0$,

$$N_2(f(x) - A(x), \delta_\alpha + s + (2^{1-p} - 1)^{-1}2\theta_\alpha t^p) \geq \alpha,$$

where $N_1(x, t) \geq \alpha$ and $N_2(f(0), s) \geq \alpha$.

Proof . Let $x \in X$, $s, t > 0$, $\alpha \in (0, 1)$, $N_1(x, t) \geq \alpha$ and $N_2(f(0), s) \geq \alpha$. If $y = 0$, then $N_1(y, r) \geq \alpha$ for all $r > 0$. Hence, $N_2(2f(x/2) - f(x) - f(0), \delta_\alpha + \theta_\alpha(t^p + s^p)) \geq \alpha$, for all $r > 0$. We put $t = r$ and then, we obtain $N_2(2f(x/2) - f(x) - f(0), \delta_\alpha + 2\theta_\alpha t^p) \geq \alpha$. We have $N_1(2x, 2t) \geq \alpha$. So, $N_2(2f(x) - f(2x) - f(0), \delta_\alpha + 2\theta_\alpha 2^pt^p) \geq \alpha$. Hence,

$$N_2(2f(x) - f(2x), s + \delta_\alpha + 2\theta_\alpha 2^pt^p) \geq \min\{N_2(2f(x) - f(2x) - f(0), \delta_\alpha + 2\theta_\alpha 2^pt^p), N_2(f(0), s)\} \geq \alpha.$$

Thus, $N_2(f(x) - 2^{-1}f(2x), 2^{-1}(s + \delta_\alpha) + 2\theta_\alpha 2^{p-1}t^p) \geq \alpha$. By induction on n, we obtain

$$N_2(f(x) - 2^{-n}f(2^nx), (\sum_{k=1}^{n} 2^{-k})(s + \delta_\alpha) + 2\theta_\alpha t^p \sum_{k=1}^{n} 2^{(p-1)k}) \geq \alpha, \text{ for all } n > 0.$$

Hence, for every $n > m$,

$$\begin{aligned} N_2(2^{-n}f(2^nx) - 2^{-m}f(2^mx), r) \geq& N_2(2^{-n}f(2^nx) - 2^{-m}f(2^mx), u) \\ =& N_2(2^{-m}2^{-(n-m)}f(2^{n-m}2^mx) - 2^{-m}f(2^mx), u) \\ \geq& \alpha, \end{aligned}$$

where $u = 2^{-m}[(\sum_{k=1}^{n-m} 2^{-k})(s + \delta_\alpha) + 2\theta_\alpha 2^{mp}t^p \sum_{k=1}^{n-m} 2^{(p-1)k}]$ and
$v = 2^{-m}[(s + \delta_\alpha) + 2\theta_\alpha 2^{mp}t^p/(2^{1-p} - 1)]$.

Now, let $r > 0$. Then, $\lim_{m\longrightarrow\infty}(r/[8\theta_\alpha 2^{m(p-1)}/(2^{1-p} - 1))])^{1/p} = \lim m \longrightarrow \infty 2^{m-2}r = \infty$. So,

$$\lim_{m\longrightarrow\infty} N_1(x, (r/[8\theta_\alpha 2^{m(p-1)}/(2^{1-p} - 1))])^{1/p}) = N_2(f(0), 2^{m-2}r) = 1.$$

Thus, there exists $N > 0$ such that $N_1(x, (r/[8\theta_\alpha 2^{m(p-1)}/(2^{1-p} - 1))])^{1/p}) \geq \alpha,\ N_2(f(0), 2^{m-2}r) \geq \alpha,$ and $2^{-m}\delta_\alpha \leq r/2$, for all $m \geq N$. Therefore,

$$N_2(2^{-n}f(2^nx) - 2^{-m}f(2^mx), r) \geq N_2(2^{-n}f(2^nx) - 2^{-m}f(2^mx), r/2 + 2^{-m}\delta_\alpha) \geq \alpha,$$

for all $n > m > N$. So, $\{2^{-n}f(2^nx)\}$ is a Cauchy sequence for each $x \in X$. Hence, the limit $A(x) = \lim_{n\to\infty} 2^{-n}f(2^nx)$ exists for each $x \in X$.

Now, we show that the function $A : X \longrightarrow Y$ is additive. Let $x, y \in X$, $t > 0$ and $\alpha \in (0, 1)$. Then,

$$\begin{aligned} N_2(A(x+y) - A(x) - A(y), t) \geq \min\{&N_2(A(x+y) - 2^{-n}f(2^n(x+y)), t/4), \\ &N_2(2^{-n-1}f(2^{n+1}x) - A(x), t/4), \\ &N_2(2^{-n-1}f(2^{n+1}y) - A(y), t/4), \\ &N_2(2^{-n}f(2^n(x+y)) - 2^{-n-1}f(2^{n+1}x) - 2^{-n-1}f(2^{n+1}y), t/4)\}. \end{aligned}$$

Since $\lim_{n\longrightarrow\infty}(2^{n+1}t/16\theta_\alpha)^{1/p} = \infty$, it follows that

$$\lim_{n\longrightarrow\infty} N_1(x, (2^{n+1}t/16\theta_\alpha)^{1/p}) = 1 = \lim_{n\longrightarrow\infty} N_1(y, (2^{n+1}t/16\theta_\alpha)^{1/p}).$$

So, there exists $N' > 0$ such that $N_1(x, (2^{n+1}t/16\theta_\alpha)^{1/p}) \geq \alpha,\ N_1(y, (2^{n+1}t/16\theta_\alpha)^{1/p}) \geq \alpha$ and $2^{-n-1}\delta_\alpha \leq t/8$ for all $n \geq N'$. Thus,

$$\begin{aligned} &N_2(2^{-n}f(2^n(x+y)) - 2^{-n-1}f(2^{n+1}x) - 2^{-n-1}f(2^{n+1}y), t/4) \geq \\ &N_2(2^{-n}f(2^n(x+y)) - 2^{-n-1}f(2^{n+1}x) - 2^{-n-1}f(2^{n+1}y), 2^{-n-1}\delta_\alpha + t/8) = \\ &N_2(2f((2^{n+1}(x+y))/2) - f(2^{n+1}x) - f(2^{n+1}y), \delta_\alpha + 2^{n+1}t/8) \geq \alpha, \end{aligned}$$

for all $n \geq N'$.

Since $A(x) = \lim_{n\to\infty} 2^{-n}f(2^n x)$ for each $x \in X$, there exists $N'' > 0$ such that

$$\begin{aligned} \min\{&N_2(A(x+y) - 2^{-n}f(2^n(x+y)), t/4), \\ &N_2(2^{-n-1}f(2^{n+1}x) - A(x), t/4), \\ &N_2(2^{-n-1}f(2^{n+1}y) - A(y), t/4)\} \geq \alpha, \end{aligned}$$

for all $n \geq N''$.

Letting $N = \max\{N', N''\}$ we obtain

$$\begin{aligned} N_2(A(x+y) - A(x) - A(y), t) \geq \min\{&N_2(A(x+y) - 2^{-n}f(2^n(x+y)), t/4), \\ &N_2(2^{-n-1}f(2^{n+1}x) - A(x), t/4), \\ &N_2(2^{-n-1}f(2^{n+1}y) - A(y), t/4), \\ &N_2(2^{-n}f(2^n(x+y)) - 2^{-n-1}f(2^{n+1}x) - 2^{-n-1}f(2^{n+1}y), t/4)\} \\ \geq &\alpha, \ \text{for all } n \geq N. \end{aligned}$$

Hence, $N_2(A(x+y) - A(x) - A(y), t) \geq \alpha$ for all $\alpha \in (0, 1)$. Thus, $N_2(A(x+y) - A(x) - A(y), t) = 1$ for all $t > 0$. So, $A(x+y) = A(x) + A(y)$.

In what follows, we show that for every $\alpha \in (0, 1)$, $x \in X$ and $t, s \geq 0$,

$$N_2(f(x) - A(x), \delta_\alpha + s + (2^{1-p} - 1)^{-1}2\theta_\alpha t^p) \geq \alpha,$$

where $N_1(x, t) \geq \alpha$ and $N_2(f(0), s) \geq \alpha$. Let $x \in X$, $t, s > 0$, $\alpha \in (0, 1)$, $N_1(x, t) \geq \alpha$ and $N_2(f(0), s) \geq \alpha$. Then,

$$\lim_{n\to\infty} 2\theta_\alpha t^p \sum_{k=1}^{n} 2^{(p-1)k} = 2^p(1 - 2^{p-1})^{-1}\theta_\alpha t^p.$$

Assume that $\epsilon > 0$ is given. Since $\{2^{-n}f(2^n x)\}$ is a Cauchy sequence, there exists $N' > 0$ such that

$$N_2(2^{-n}f(2^n x) - 2^{-m}f(2^m x), \epsilon) \geq \alpha, \ \text{for all } n \geq N'.$$

Since $A(x) = \lim_{n\to\infty} 2^{-n}f(2^n x)$, there exists $N'' > 0$ such that

$$N_2(A(x) - 2^{-n}f(2^n x), 2^p(1 - 2^{p-1})^{-1}\theta_\alpha t^p - 2\theta_\alpha t^p \sum_{k=1}^{N_4} 2^{(p-1)k}) \geq \alpha, \ \text{for all } n \geq N''.$$

If $n \geq \max\{N', N''\}$, then

$$\begin{aligned}N_2(A(x)-f(x),\epsilon+s+\delta_\alpha+u) \geq &\min\{N_2(2^{-n}f(2^nx)-2^{-N'}f(2^{N'}x),\epsilon),\\ &N_2(A(x)-2^{-n}f(2^nx),u-2\theta_\alpha t^p\sum_{k=1}^{N'}2^{(p-1)k}),\\ &N_2(f(x)-2^{-N'}f(2^{N'}y),s+\delta_\alpha+2\theta_\alpha t^p\sum_{k=1}^{N'}2^{(p-1)k})\}\\ \geq &\min\{N_2(2^{-n}f(2^nx)-2^{-N'}f(2^{N'}x),\epsilon),\\ &N_2(A(x)-2^{-n}f(2^nx),u-2\theta_\alpha t^p\sum_{k=1}^{N'}2^{(p-1)k}),\\ &N_2(f(x)-2^{-N'}f(2^{N'}y),(\sum_{k=1}^{N'}2^{-k})(s+\delta_\alpha)+2\theta_\alpha t^p\sum_{k=1}^{N'}2^{(p-1)k})\}\\ \geq&\alpha,\end{aligned}$$

where $u = 2^p(1-2^{p-1})^{-1}\theta_\alpha t^p$. Hence, $N_2(A(x)-f(x),\epsilon+s+\delta_\alpha+2^p(1-2^{p-1})^{-1}\theta_\alpha t^p) \geq \alpha$. By (N7) we obtain

$$N_2(A(x)-f(x),s+\delta_\alpha+2^p(1-2^{p-1})^{-1}\theta_\alpha t^p) = \lim_{\epsilon\to 0} N_2(A(x)-f(x),\epsilon+s+\delta_\alpha+2^p(1-2^{p-1})^{-1}\theta_\alpha t^p) \geq \alpha.$$

Finally, we show that the function $A : X \longrightarrow Y$ is unique. Let $A' : X \longrightarrow Y$ be another additive function which satisfies $N_2(f(x)-A'(x),s+\delta_\alpha+2^p(1-2^{p-1})^{-1}\theta_\alpha t^p) \geq \alpha$ for every $\alpha \in (0,1)$, $x \in X$ and $t,s \geq 0$, where $N_1(x,t) \geq \alpha$ and $N_2(f(0),s) \geq \alpha$. Let $t,s > 0$, $x \in X$, $\alpha \in (0,1)$ and $N_2(f(0),s) \geq \alpha$. Then,

$$\lim_{n\to\infty}([(1-2^{p-1})/(2^p\theta_\alpha)]2^{n(1-p)-1}t)^{1/p} = \infty.$$

Hence, there exists $N > 0$ such that

$$N_1(2^nx,([(1-2^{p-1})/(2^p\theta_\alpha)]2^{n-1}t)^{1/p}) = N_1(x,([(1-2^{p-1})/(2^p\theta_\alpha)]2^{n(1-p)-1}t)^{1/p}) \geq \alpha, \text{ for all } n \geq N.$$

Therefore,

$$\begin{aligned}N_2(A(x)-A'(x),2^{1-n}(s+\delta_\alpha)+t) =&N_2(A(2^nx)-A'(2^nx),2(s+\delta_\alpha)+2^nt)\\ \geq &\min\{N_2(A(2^nx)-f(2^nx),s+\delta_\alpha+2^{n-1}t),\\ &N_2(A'(2^nx)-f(2^nx),s+\delta_\alpha+2^{n-1}t)\}\\ \geq&\alpha, \text{ for all } n \geq N.\end{aligned}$$

By (N7) we obtain

$$N_2(A(x)-A'(x),t) = \lim_{n\to\infty} N_2(A(x)-A'(x),2^{1-n}(s+\delta_\alpha)+t) \geq \alpha.$$

So, $N_2(A(x)-A'(x),t) = 1$ for all $t > 0$. Hence, $A(x) = A'(x)$ for all $x \in X$. □

In the following example, we present a function f that satisfies the conditions of Theorem 3.1. Moreover, we present a fuzzy normed linear space such that the function f satisfies the conditions of Theorem 3.7. Therefore, Theorem 3.7 is a generalized fuzzy version of Theorem 3.1 for $0 < p < 1$.

Now, we show that Theorem 3.7 is the fuzzy version of Theorem 3.1 for $0 < p < 1$.

Example 3.8. Let $(X, \|.\|)$ be a Banach space, $\delta, \theta \in \mathbb{R}^+$, $0 < p < 1$ and $f : X \longrightarrow X$ be a function such that $\|2f((x+y)/2)-f(x)-f(y)\| \leq \delta+\theta(\|x\|^p+\|y\|^p)$, for all $x, y \in X$.

We define a fuzzy norm N on X by

$$N(x,t) = \begin{cases} t/\|x\|, & 0 < t \leq \|x\| \\ 1, & \|x\| < t, \\ 0, & t \leq 0, \end{cases}$$

where $x \in X$ and $t \in \mathbb{R}$.

Let $\{\alpha^{1-p}\theta\}_{\alpha\in(0,1)}, \{\alpha\delta\}_{\alpha\in(0,1)} \subseteq \mathbb{R}^+$. Suppose that $x, y \in X$, $s, t \in \mathbb{R}$, $\alpha \in (0,1)$, $N(x,t) \geq \alpha$ and $N(y,s) \geq \alpha$. Now, we show that $N(2f((x+y)/2)-f(x)-f(y),\alpha\delta+\alpha^{1-p}\theta(t^p+s^p)) \geq \alpha$.

Case 1: If $\|2f((x+y)/2) - f(x) - f(y)\| < \alpha\delta + \alpha^{1-p}\theta(t^p + s^p)$, then

$$N(2f((x+y)/2) - f(x) - f(y), \alpha\delta + \alpha^{1-p}\theta(t^p + s^p)) = 1 \geq \alpha.$$

Case 2: $\alpha\delta + \alpha^{1-p}\theta(t^p + s^p) \leq \|2f((x+y)/2) - f(x) - f(y)\|$. Since $N(x,t) \geq \alpha$ and $N(y,s) \geq \alpha$, it follows that $t/\|x\| \geq \alpha$ and $s/\|y\| \geq \alpha$. Hence, $\alpha\|x\| \leq t$ and $\alpha\|y\| \leq s$. So,

$$\alpha^p\|2f((x+y)/2) - f(x) - f(y)\| \leq \alpha^p\delta + \alpha^p\theta(\|x\|^p + \|y\|^p) \leq \alpha^p\delta + \theta(t^p + s^p).$$

Thus,

$$N(2f((x+y)/2) - f(x) - f(y), \alpha\delta + \alpha^{1-p}\theta(t+s)) =$$
$$[\alpha\delta + \alpha^{1-p}\theta(t+s)]/\|2f((x+y)/2) - f(x) - f(y)\| \geq \alpha.$$

Therefore, a unique additive function $A : X \longrightarrow Y$ exists such that for every $\alpha \in (0,1)$, $x \in X$ and $t,s \geq 0$, $N(f(x) - A(x), \delta_\alpha + s + (2^{1-p}-1)^{-1}2\theta_\alpha t^p) \geq \alpha$, where $N(x,t) \geq \alpha$ and $N(f(0), s) \geq \alpha$.

Corollary 3.9. Let (X, N_1) and (Y, N_2) be fuzzy Banach spaces such that N_2 satisfies (N7), $0 < p < 1$ and $\theta, \delta \in F^+(\mathbb{R})$. Moreover, let $f : X \longrightarrow Y$ be a function such that for every $\alpha \in (0,1)$, $x, y \in X$ and $s, t \geq 0$, $N_2(2f((x+y)/2) - f(x) - f(y), \delta_\alpha^- + \theta_\alpha^-(t^p + s^p)) \geq \alpha$, where $N_1(x,t) \geq \alpha$ and $N_1(y,s) \geq \alpha$. Then, there exists a unique additive function $A : X \longrightarrow Y$ such that for every $\alpha \in (0,1)$, $x \in X$ and $s,t \geq 0$,

$$N_2(f(x) - A(x), \delta_\alpha^- + s + (2^{1-p}-1)^{-1}2\theta_\alpha^- t^p) \geq \alpha,$$

where $N_1(x,t) \geq \alpha$ and $N_2(f(0), s) \geq \alpha$.

Corollary 3.10. Let (X, N_1) and (Y, N_2) be fuzzy Banach spaces such that N_2 satisfies (N7), $0 < p < 1$ and $\theta, \delta \in F^+(\mathbb{R})$. Moreover, let $f : X \longrightarrow Y$ be a function such that for every $\alpha \in (0,1)$, $x, y \in X$ and $s, t \geq 0$, $N_2(2f((x+y)/2) - f(x) - f(y), \delta_\alpha^- + \theta_\alpha^+(t^p + s^p)) \geq \alpha$, where $N_1(x,t) \geq \alpha$ and $N_1(y,s) \geq \alpha$. Then, there exists a unique additive function $A : X \longrightarrow Y$ such that for every $\alpha \in (0,1)$, $x \in X$ and $s,t \geq 0$,

$$N_2(f(x) - A(x), \delta_\alpha^- + s + (2^{1-p}-1)^{-1}2\theta_\alpha^+ t^p) \geq \alpha,$$

where $N_1(x,t) \geq \alpha$ and $N_2(f(0), s) \geq \alpha$.

Corollary 3.11. Let (X, N_1) and (Y, N_2) be fuzzy Banach spaces such that N_2 satisfies (N7), $0 < p < 1$ and $\{\theta_\alpha\}_{\alpha\in[0,1]}, \{\delta_\alpha\}_{\alpha\in[0,1]} \subseteq \mathbb{R}^+$. Moreover, let $f : X \longrightarrow Y$ be a function such that for every $x, y \in X$,

$$N_2(2f((x+y)/2) - f(x) - f(y), \delta_\alpha + \theta_\alpha(t^p + s^p)) \geq \min\{N_1(x,t), N_1(y,s)\},$$

where $\alpha = \min\{N_1(x,t), N_1(y,s)\}$. Then, there exists a unique additive function $A : X \longrightarrow Y$ such that for every $x \in X$,

$$N_2(f(x) - A(x), \delta_\alpha^- + s + (2^{1-p}-1)^{-1}2\theta_\alpha^- t^p) \geq \min\{N_1(x,t), N_2(f(0), s)\},$$

where $\alpha = \min\{N_1(x,t), N_2(f(0), s)\}$.

Now, we show that the results of Theorem 3.2 and Theorem 3.7 cannot be extended to the case $p = 1$. In fact, we show that fuzzy Banach spaces (X, N_1) and (Y, N_2) exist such that N_2 satisfies (N7), $\{\theta_\alpha\}_{\alpha\in(0,1)}, \{\delta_\alpha\}_{\alpha\in(0,1)} \subseteq \mathbb{R}^+$ and a function $f : X \longrightarrow Y$ exists such that for every $\alpha \in (0,1)$, $x, y \in X$ and $s, t \geq 0$, $N_2(2f((x+y)/2) - f(x) - f(y), \delta_\alpha + \theta_\alpha(t+s)) \geq \alpha$, where $N_1(x,t) \geq \alpha$ and $N_1(y,s) \geq \alpha$. But, $\{\mu_\alpha\}_{\alpha\in(0,1)} \subseteq \mathbb{R}^+$ and an additive function $A : X \longrightarrow Y$ cannot be found such that for every $\alpha \in (0,1)$, $x \in X$ and $t \geq 0$, $N_2(f(x) - A(x), \mu_\alpha t) \geq \alpha$, where $N_1(x,t) \geq \alpha$.

Example 3.12. Let $\theta > 0$ and $\delta = \theta/6$ be fixed. Suppose that a function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$\phi(x) = \begin{cases} \delta, & x \in [1, +\infty), \\ \delta x, & x \in (-1, 1), \\ -\delta, & x \in (-\infty, -1]. \end{cases}$$

Then, ϕ is continuous and $|\phi(x)| \le \delta$ for all $x \in \mathbb{R}$.

Now, we define $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = \sum_{n=1}^{\infty} \phi(2^n x)/2^n$ for all $x \in \mathbb{R}$. Then, by [13, Example],

$$|f((x+y)) - f(x) - f(y)| \le \theta(|x| + |y|), \text{ for all } x, y \in \mathbb{R}.$$

We obtain $2f((x+y)/2) = \sum_{n=1}^{\infty} \phi(2^{n-1}(x+y))/2^{n-1} = \phi(x+y) + f(x+y)$. Hence,

$$\begin{aligned}|2f((x+y)/2) - f(x) - f(y)| =& |\phi(x+y) + f(x+y) - f(x) - f(y)|\\ \le& |\phi(x+y)| + |f(x+y) - f(x) - f(y)|\\ \le& \delta + \theta(|x| + |y|), \text{ for all } x, y \in \mathbb{R}.\end{aligned}$$

We define a fuzzy norm N on X by

$$N(x,t) = \begin{cases} t/|x|, & 0 < t \le |x|,\\ 1, & |x| < t,\\ 0, & t \le 0,\end{cases}$$

where $x \in X$ and $t \in \mathbb{R}$.

Let $\{\alpha\delta\}_{\alpha\in(0,1)} \subseteq \mathbb{R}^+$. Suppose that $x, y \in X$, $s, t \in \mathbb{R}$, $\alpha \in (0,1)$, $N(x,t) \ge \alpha$ and $N(y,s) \ge \alpha$. Now, we show that $N(2f((x+y)/2) - f(x) - f(y), \alpha\delta + \theta(t+s)) \ge \alpha$.

Case 1: If $|2f((x+y)/2) - f(x) - f(y)| < \alpha\delta + \theta(t+s)$, then

$$N(2f((x+y)/2) - f(x) - f(y), \alpha\delta + \theta(t+s)) = 1 \ge \alpha.$$

Case 2: $\alpha\delta + \theta(t+s) \le |2f((x+y)/2) - f(x) - f(y)|$. Since $N(x,t) \ge \alpha$ and $N(y,s) \ge \alpha$, we deduce that $t/|x| \ge \alpha$ and $s/|y| \ge \alpha$. Hence, $\alpha|x| \le t$ and $\alpha|y| \le s$. So,

$$\alpha|2f((x+y)/2) - f(x) - f(y)| \le \alpha\delta + \alpha\theta(|x| + |y|) \le \alpha\delta + \theta(t+s).$$

Thus,

$$N(2f((x+y)/2) - f(x) - f(y), \alpha\delta + \theta(t+s)) = [\alpha\delta + \theta(t+s)]/|2f((x+y)/2) - f(x) - f(y)| \ge \alpha.$$

Assume that a unique additive function $A : X \longrightarrow Y$ exists such that $N(f(x) - A(x), \mu_\alpha t) \ge \alpha$ for every $\alpha \in (0,1)$, $x \in \mathbb{R}$ and $t \ge 0$, where $N(x,t) \ge \alpha$. Then, for every $\alpha \in (0,1)$, $x \in \mathbb{R}$ and $t \ge 0$, $\alpha \le \mu_\alpha t/|f(x) - A(x)|$, where $N(x,t) \ge \alpha$. Therefore, for every $\alpha \in (0,1)$, $x \in \mathbb{R}$ and $t \ge 0$, $|f(x) - A(x)| \le \mu_\alpha t/\alpha$, where $N(x,t) \ge \alpha$. Thus, for every $\alpha \in (0,1)$, $x \in \mathbb{R}$ and $t \ge 0$, $|f(x) - A(x)| \le \mu_\alpha t/\alpha$, where $|x| \le t/\alpha$. Hence,

$$|f(x) - A(x)| \le \mu_\alpha |x|, \text{ for all } x \in \mathbb{R}.$$

By [13, Example], this is a contradiction.

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