Int. J. Nonlinear Anal. Appl. 14 (2023) 1, 309–324
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.26239.3271

General bivariational inclusions and iterative methods

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(Communicated by Abasalt Bodaghi)

Abstract

In this paper, we consider some new classes of general bivariational inclusions. It is shown that the general bivariational inclusions are equivalent to the fixed point problems, resolvent equations and dynamical systems. We have discussed the existence of a solution of the general bivariational inequalities. Some new iterative methods for solving general bivariational inclusions and related optimization problems are suggested by using resolvent methods, resolvent equations and dynamical systems coupled with finite difference technique. Convergence analysis of these methods is investigated under monotonicity. Some special cases are also discussed as applications of the main results.

Keywords: Variational inclusions, existence results, resolvent method, resolvent equations, dynamical systems, convergence
2020 MSC: Primary: 49J40, Secondary: 90C33

1 Introduction

Variational inclusion theory contains a wealth of new ideas and techniques. which can be viewed as a novel extension and generalization of the variational inequalities. It is amazing that a wide class of unrelated problems can be studied in the unified framework of variational inclusions. The resolvent equations were introduced and studied by Noor [25, 26]. Noor [25, 26] proved that the variational inclusions are equivalent to the resolvent equations using the resolvent operator technique. This equivalent alternative formulation has been used to study the existence of a solution as well as to develop various iterative methods for solving the variational inclusions. One of the most difficult and important problems in variational inequality theory is the development of efficient numerical methods. In this direction, several numerical methods have been developed for solving the variational inclusions and their variant forms. Noor [21, 24] suggested and analyzed some three-step forward-backward splitting algorithms for solving variational inequalities and quasi variational inclusions by using the updating techniques of the solution. These forward-backward splitting algorithms are similar to those of Glowinski et al. [9], which they suggested by using the Lagrangian technique. It is known that three-step schemes are versatile and efficient. These three-step schemes are a natural generalization of the splitting methods for solving partial differential equations. For applications of the splitting techniques to partial differential equations, see Ames [2] and the references therein. For novel applications of the three-step methods, see Ashish et al [3]. These methods include the Mann and Ishikawa iterative schemes and modified forward-backward splitting methods of Tseng [47], Noor [21, 24] and Noor et al.[26] as special cases.

Related to variational inclusions, we have problem of dynamical systems. Dynamical systems arise naturally in numerous applied and theoretical fields including celestial mechanics, financial forecasting, environmental applications,

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Received: February 2022 *Accepted:* July 2022

neuroscience, brain modeling. It is known that the variational inequalities are equivalent to the fixed point problems. Dupuis et al.[8] suggested the projected dynamical system using the fixed point technique. This approach is used to study the asymptotic stability of the solution of the variational inequalities. Noor et al, [39] used this technique to suggest some efficient iterative schemes for solving variational inequalities. Noor et al. [33] has proved that variational inclusions are equivalent to the dynamically systems. This equivalence has been used to study the existence and stability of the solution of variational inclusions. For the applications and numerical methods of the dynamical systems, see [7, 15, 22, 23, 24, 27, 36, 39] and the references therein.

Alvarez [1] used the inertial type projection methods for solving variational inequalities, the origin of which can be traced back to Polyak [43]. Noor [24] suggested and investigated inertial type projection methods for solving general variational inequalities. These inertial type methods have been modified in various directions for solving variational inequalities and related optimization problems. Recently Shehu et al [45], Noor et al [41, 43, 44, 45] and Jabeen et al [11] analyzed some inertial projection methods for some classes of general quasi variational inequalities. Convergence analysis of these inertial type methods has been considered under some mild conditions.

In this paper, we consider some new classes of general bivariational inclusions. It have been shown that the system of absolute value equations, complementarity problems, general variational inequalities, difference of two monotone operators and sum of two monotone operators can be obtained as special cases of general bivariational inclusions. We prove that the general bivariational inclusions are equivalent to fixed point problems. This alternative formulation is used to suggest and investigate some new three step implicit and explicit iterative methods for solving general bivariational inclusions. These new iterative methods can be viewed as significant generalization of the three-step methods of Noor [21, 24] and Tseng [47]. We have also used the dynamical systems technique coupled with finite difference schemes to propose some new iterative methods for solving the general bidirectional inclusions. The convergence criteria of the proposed implicit methods is discussed under some mild conditions. Several important special cases are discussed as applications of our results. We have only considered the theoretical aspects of the proposed methods. It is still an open problem to implement these methods and compare with other techniques. It is expected the techniques and ideas of this paper may be starting point for further research.

2 Formulations and basic facts

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$ respectively. Let $\mathcal{T}, \mathcal{B}, A, g : \mathcal{H} \to \mathcal{H}$ be nonlinear operators. Let $\Phi(.,.) : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a continuous bifunction.
We consider the problem of finding $\mu \in \mathcal{H}$ such that

$$0 \in \Phi(\mathcal{T}u, \mathcal{B}(\mu)) + \mathcal{A}(g(\mu)). \tag{2.1}$$

Inclusion of the type (2.1) is called the general bivariational inclusion. We would like to emphasize that the operator $\mathcal{T}$ is strongly monotone, the operator $\mathcal{B}$ is Lipschitz continuous and $\mathcal{A}(.)$ is a maximal monotone operator. Several important problems arising in pure and applied sciences can be studied in the frame work of the form (2.1).

We now discuss several interesting problems, which are special cases of the general bivariational inclusions (2.1).

(I). If $g = I$, the identity operator, then problem (2.1) reduces to finding $\mu \in \mathcal{H}$ such that

$$0 \in \Phi(\mathcal{T}u, \mathcal{B}(\mu)) + \mathcal{A}(\mu), \tag{2.2}$$

which is known bidirectional inclusion.
(II). If $\Phi(\mathcal{T}, \mathcal{B}) = \mathcal{T}$, the problem (2.1) collapses to finding $\mu \in \mathcal{H}$ such that

$$0 \in \mathcal{T}u + \mathcal{A}(g(\mu)), \tag{2.3}$$

is known as finding the zeros of the sum of two composite monotone operators.
(III). If $\mathcal{A}(g(\mu)) = 0$, and $\Phi(\mathcal{T}u, \mathcal{B}(\mu)) = \mathcal{T}u + \mathcal{B}(\mu)$, , then problem (2.1) collapses to finding $\mu \in \mathcal{H}$ such that

$$0 \in \mathcal{T}u + \mathcal{B}(\mu), \tag{2.4}$$

which can be considered as finding the zeros of the sum of two monotone operators. Problem (2.4) can be interpreted as variational inclusion involving difference of two monotone operators, which is itself a very difficult problem. This problem can be viewed as a problem of finding the minimum of two difference of convex functions, known DC-problem. Such type of problems have applications in optimization theory and imaging process in medical sciences and earthquake.

(IV). We note that if $\mathcal{A}(g(.)) = \partial\varphi(g(.))$, where $\partial\varphi(.)$ is the subdifferential of a proper, general convex and lower-semicontinuous function $\varphi(.) : \mathcal{H} \to \mathcal{R} \cup \{+\infty,\}$ then problem (2.1) is equivalent to finding $\mu \in \mathcal{H}$ such that.

$$\langle \Phi(\mathcal{T}u, \mathcal{B}(\mu)), g(\nu) - g(\mu)\rangle + \varphi(g(\nu)) - \varphi(g(\mu)) \geq 0, \quad \forall \mu \in \mathcal{H}. \tag{2.5}$$

The problem of the type (2.5) is called the mixed general variational inequality problem, which has many important and significant applications in regional, physical, mathematical, pure and applied sciences.

(V). If $\varphi(g(.))$ is the indicator function of a closed convex set Ω in $\mathcal{H}$, then problem (2.5) is equivalent to finding $\mu \in \Omega$ such that

$$\langle \Phi(\mathcal{T}u, \mathcal{B}(\mu)), g(\nu) - g(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega, \tag{2.6}$$

which is called the general bidirectional inequality.

(VI). If $g = I$ is identity operator and $\Phi(\mathcal{T}u, \mathcal{B}(\mu)) = \mathcal{T}\mu + \mathcal{B}(\mu)$, then problem (2.6) reduces to finding $\mu \in \Omega$ such that

$$\langle \mathcal{T}u + \mathcal{B}(\mu), \nu - \mu\rangle \geq 0, \quad \forall \nu \in \Omega, \tag{2.7}$$

which is called the mildly nonlinear variational inequalities, see Noor [17].

(VII). If $\Omega = \mathcal{H}, \quad \Phi(\mathcal{T}u, \mathcal{B}(\mu)) = \mathcal{T}u + \mathcal{B}|\mu|)$, then the general variational inequality (2.6) reduces to finding $\mu \in \mathcal{H}$ such that

$$\langle \mathcal{T}u + \mathcal{B}|\mu|, g(\nu)\rangle = 0, \quad \forall \nu \in \mathcal{H}, \tag{2.8}$$

is known as the system of general absolute values equations.

(VIII). For $g = I$, problem (2.8) reduces to find $\mu \in \mathcal{H}$ such that

$$\mathcal{T}u + \mathcal{B}|\mu| = b, \quad \forall \nu \in \mathcal{H}, \tag{2.9}$$

which is known as the system of absolute value equations, introduced and studied by Mangasarian [14]. It is worth mentioning that problem (2.9) is a special case of mildly nonlinear variational inequalities, which was introduced and studied by Noor [17] in 1975.

(IX). If $\Omega^* = \{\mu \in \mathcal{H} : \langle \mu, \nu\rangle \geq 0, \quad \forall \nu \in\}$ is a polar (dual) cone of a cone Ω in $\mathcal{H}$, then problem (2.6) is equivalent to finding $\mu \in \mathcal{H}$ such that

$$g(\mu) \in \Omega, \quad \Phi(\mathcal{T}u, \mathcal{B}(\mu)) \in \Omega^* \quad \text{and} \quad \langle \Phi(\mathcal{T}u, \mathcal{B}(\mu)), g(\mu)\rangle = 0, \tag{2.10}$$

which is known as the general bicomplementarity problems. Obviously general bicomplementarity problems include the complementarity problems. See also Noor [20], Cottle [4], and Cottle et al. [5] for applications in mathematical and engineering sciences.

(X). If $\Phi(\mathcal{T}u, \mathcal{B}(\mu)) = \mathcal{T}\mu$, then problem (2.7) collapses to finding $\mu \in \Omega$ such that

$$\langle \mathcal{T}\mu, \nu - \mu\rangle \geq 0, \quad \forall \nu \in \Omega, \tag{2.11}$$

which is called the classical variational inequalities, introduced and studied by Stampacchia [46]. We would like to emphasize that the variational inequalities are the natural and novel extension of the variational principles. For the applications, formulations, generalizations, numerical methods, sensitivity analysis, dynamical systems and other aspects of variational inequalities, complementarity problems, see [1, 6, 7, 8, 9, 10, 11, 12, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 37, 38, 39, 40, 41, 42, 44, 45, 47] and the references therein.

Remark 2.1. It is worth mentioning that for appropriate and suitable choices of the bifunction $\Phi(.,.)$, operators $\mathcal{T}, \quad \mathcal{B}, g, \quad \mathcal{A}$, convex set Ω and the spaces, one can obtain several classes of variational inclusions, variational inequalities, complementarity problems and optimization problems as special cases of the general bivariational inclusion (2.1). This shows that the problem (2.1) is quite general and unifying one. It is interesting problem to develop efficient and implementable numerical methods for solving the general bivariational inclusions and their variant forms.

Definition 2.2. The bifunction $\Phi(.,.) : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is said to be:

(i) Strongly monotone with respect to the first argument , if there exist a constant $\alpha > 0$, such that

$$\langle \Phi(\mathcal{T}\mu, .) - \Phi(\mathcal{T}\nu, .), \mu - \nu \rangle \geq \alpha \|\mu - \nu\|^2, \quad \forall \mu, \nu \in \mathcal{H}.$$

(ii) Lipschitz continuous with respect to the first argument, if there exist a constant $\beta > 0$, such that

$$\Phi(\mathcal{T}\mu, .) - \Phi(\mathcal{T}\nu, .) \leq \beta \|\mu - \nu\|, \quad \forall \mu, \nu \in \mathcal{H}.$$

(iii) Monotone with respect to the first argument, if

$$\langle \Phi(\mathcal{T}\mu, .) - \Phi(\mathcal{T}\nu, .), \mu - \nu \rangle \geq 0, \quad \forall \mu, \nu \in \mathcal{H}.$$

(iv) Pseudo monotone with respect to the first argument, if

$$\langle \Phi(\mathcal{T}\mu, .), \nu - \mu \rangle \geq 0 \quad \Rightarrow \quad \langle \Phi(\mathcal{T}\nu, .), \nu - \mu \rangle \geq 0, \quad \forall \mu, \nu, \in \mathcal{H}.$$

Remark 2.3. Every strongly monotone bifunction $\Phi(.,.)$ is a monotone bifunction $\Phi(.,.)$ and monotone bifunction $\Phi(.,.)$ is a pseudo monotone bifunction $\Phi(.,.)$, but the converse is not true.

3 Iterative resolvent methods

In this section, we prove that the problem (2.1) is equivalent to the fixed point problem using the resolvent operator technique. we use this alternative fixed point formulation to study the existences of solution as well as to suggest and analyze some new implicit methods for solving the general bivariational inclusions (2.1).

Lemma 3.1. The function $\mu \in \mathcal{H}$ is a solution of the general bivariational inclusion (2.1), if and only if, $\mu \in \mathcal{H}$ satisfies the relation

$$g(\mu) = \mathcal{J}_{\mathcal{A}}[g(\mu) - \rho\Phi(\mathcal{T}\mu, \mathcal{B}(\mu))], \tag{3.1}$$

where $\mathcal{J}_{\mathcal{A}}$ is the resolvent operator and $\rho > 0$ is a constant.

Proof . Let $\mu \in \mathcal{H}$ be a solution of (2.1), then, for a constant $\rho > 0$,

$$\begin{aligned} \rho\Phi(\mathcal{T}\mu, \Phi(\mathcal{B})(\mu)) \quad &+ \quad \rho\mathcal{A}(\mu) \ni 0. \\ &\Longleftrightarrow \\ -g(\mu) + \rho\Phi(\mathcal{T}\mu, \mathcal{B}(\mu)) \quad &+ \quad (I + \rho\mathcal{A})(g(\mu)) \ni 0 \\ &\Longleftrightarrow \\ g(\mu) = \quad &\mathcal{J}_{\mathcal{A}}[g(\mu) - \rho\Phi(\mathcal{T}\mu, \rho\mathcal{B}(\mu))]. \end{aligned}$$

the required (3.1). □ Lemma 3.1 implies that the general bivariational inclusion (2.1) is equivalent to the fixed point problem (3.1).

We use this fixed point formulation to study the existence of a solution of the problem (2.1). We define the mapping Φ associated with (3.1) as:

$$\Phi(\mu) = \mu - g(\mu) + \mathcal{J}_{\mathcal{A}}[g(\mu) - \rho\Phi)(\mathcal{T}\mu, \mathcal{B}(\mu))], \tag{3.2}$$

To prove the existence of the solution of problem (2.1), it is enough that the mapping Φ defined by (3.2) is a contraction mapping.

Theorem 3.2. Let the bifunction $\Phi(\mathcal{T},\mathcal{B})$ be strongly monotone with respect to the first argument with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, respectively. If the operator $\Phi(\mathcal{T},\mathcal{B})$ is Lipschitz continuous with respect to second argument with constant $\gamma > 0$ and there exists a constant $\rho > 0$, such that

$$\left\|\rho - \frac{\alpha-\gamma(1-k)}{\beta^2-\gamma^2}\right\| < \frac{\sqrt{(\alpha-\gamma(1-k))^2-(\beta^2-\gamma^2)k(2-k)}}{\beta^2-\gamma^2}, \quad k<1, \tag{3.3}$$
$$\alpha > \gamma(1-k)+\sqrt{(\beta^2-\gamma^2)k(2-k)}, \quad \rho < \frac{1-k}{\gamma},$$

where

$$k = 2\sqrt{(1-2\delta+\sigma^2)}+\eta, \tag{3.4}$$

then there exists a solution $\mu \in \mathcal{H}$ satisfying problem (2.1).

Proof . Let $\nu \neq \mu \in H$ be two solutions of problem (2.1). Then, from (3.2), we have

$$\begin{aligned}
\|\Phi(\nu)-\Phi(\mu)\| &\leq \|\nu-\mu-(g(\nu)-g(\mu))\| \\
&\quad +\|\mathcal{J}_{\mathcal{A}}[g(\nu)-\rho\Phi(\mathcal{T}\nu,\mathcal{B}(\nu))]-\mathcal{J}_{\mathcal{A}}[g(\mu)-\rho\Phi(\mathcal{T}\mu,\mathcal{B}(\mu)))]\| \\
&\leq 2\|\nu-\mu-(g(\nu)-g(\mu))\| \\
&\quad +\|\nu-\mu-\rho\Phi(\mathcal{T}\nu,\mathcal{B}(\nu))-\rho\Phi(\mathcal{T}\mu,\mathcal{B}(\mu))\| \\
&\leq 2\|\nu-\mu-(g(\nu)-g(\mu))\|+\|\mu-\nu-\rho(\Phi(\mathcal{T}\mu,\mathcal{B}(\mu))-\Phi(\mathcal{T}\nu,\mathcal{B}(\mu)))\| \\
&\quad +\rho\|\Phi(\mathcal{T}\nu,\mathcal{B}(\mu))-\Phi(\mathcal{T}\nu,\mathcal{B}(\nu))\| \\
&\leq 2\|\nu-\mu-(g(\nu)-g(\mu))\| \\
&\quad +\|\mu-\nu-\rho(\Phi(\mathcal{T}\mu,\mathcal{B}(\mu))-\Phi(\mathcal{T}\nu,\mathcal{B}(\mu)))\|+\rho\gamma\|\nu)-\mu\|,
\end{aligned} \tag{3.5}$$

where $\gamma > 0$ is the Lipschitz continuity constant of the operator $\mathcal{B}$.

Since bifunction $\Phi(\mathcal{T},\mathcal{B})$ is strongly monotone with respect to the first argument with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, so

$$\begin{aligned}
\|\mu-\nu-\rho(\Phi(\mathcal{T}\mu,\mathcal{B}(\mu))-\Phi(\mathcal{T}\nu,\mathcal{B}(\mu)))\|^2 &= \|\mu-\nu\|^2 \\
&\quad -\rho\langle\Phi(\mathcal{T}\mu,\mathcal{B}(\mu))-\Phi(\mathcal{T}\nu,\mathcal{B}(\mu)),\mu-\nu\rangle \\
&\quad +\rho^2\|\Phi(\mathcal{T}\mu,\mathcal{B}(\mu))-\Phi(\mathcal{T}\nu,\mathcal{B}(\mu)))\|^2, \\
&\leq (1-2\alpha\rho+\beta^2\rho^2)\|\mu-\nu\|^2.
\end{aligned} \tag{3.6}$$

In a similar way, using the strongly monotonicity and Lipschitz continuity of the operator g with constants $\delta > 0$ and $\sigma > 0$, respectively, we have

$$\|\mu-\nu-\rho(g(\mu)-g(\nu))\|^2 \leq \sqrt{1-2\delta+\sigma^2}\|\mu-\nu\|^2. \tag{3.7}$$

Combining (3.3), (3.6) and (3.7), we have

$$\begin{aligned}
\|\Phi(v)-\Phi(u)\| &\leq \{\sqrt{(1-2\alpha\rho+\beta^2\rho^2)}+\rho\gamma+2\sqrt{1-2\delta+\sigma^2}\}\|\mu-\nu\| \\
&= \theta\|\mu-\nu\|,
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
\theta &= \{\sqrt{(1-2\alpha\rho+\beta^2\rho^2)}+\rho\gamma+2\sqrt{1-2\delta+\sigma^2}\} \\
&= \{\sqrt{(1-2\alpha\rho+\beta^2\rho^2)}+\rho\gamma+k\},
\end{aligned} \tag{3.9}$$

and k is defined by (3.4). From (3.3). it follows that $\theta < 1$. Thus it follows that the mapping $\Phi(\mu)$ defined (3.2) is a contraction mapping and consequently, the mapping $\Phi(\mu)$ has a fixed point $\Phi(\mu) = \mu \in \mathcal{H}$ satisfying (2.1), the required result. □

This alternative equivalent formulation (3.1) is used to suggest the following iterative methods for solving the problem (2.1).

Algorithm 3.1. For a given $\mu_0 \in H$, compute the approximate solutions $\{\mu_n\}$, $\{w_n\}$ and $\{y_n\}$ by the iterative schemes

$$\begin{aligned} g(y_n) &= \mathcal{J}_{\mathcal{A}}[g(\mu_n) - \rho\Phi(\mathcal{T}\mu_n, \mathcal{B}(\mu_n))] \\ g(w_n) &= \mathcal{J}_{\mathcal{A}}[g(y_n) - \rho\Phi(\mathcal{T}y_n, \mathcal{B}(y_n))] \\ g(\mu_{n+1}) &= \mathcal{J}_{\mathcal{A}}[g(w_n) - \rho\Phi(\mathcal{T}w_n, \mathcal{B}(w_n))], \quad n = 0, 1, 2, \ldots. \end{aligned}$$

Algorithm 3.1 is a three step forward-backward splitting algorithm for solving general bivariational inclusions (2.1). This method is very much similar to that of Glowinski and Le Tallec [7], which they suggested by using the Lagrangian technique.

We now suggested another three step scheme for solving the general bivariational inclusion (2.1).

Algorithm 3.2. For a given $\mu_0 \in H$, compute the approximate solution $\{\mu_{n+1}\}$ by the iterative schemes

$$y_n = (1-\gamma_n)\mu_n + \gamma_n\{\mu_n - g(\mu_n) + \mathcal{J}_{\mathcal{A}}[g(\mu_n) - \rho\Phi(\mathcal{T}\mu_n, \mathcal{B}(\mu_n))]\} \tag{3.10}$$

$$w_n = (1-\beta_n)\mu_n + \beta_n\{y_n - g(y_n) + \mathcal{J}_{\mathcal{A}}[g(y_n) - \rho\Phi(\mathcal{T}y_n, \mathcal{B}(y_n))\} \tag{3.11}$$

$$\mu_{n+1} = (1-\alpha_n)\mu_n + \alpha_n\{w_n - g(w_n) + \mathcal{J}_{\mathcal{A}}[g(w_n) - \rho\Phi(\mathcal{T}w_n, \mathcal{B}(w_n))]\}. \tag{3.12}$$

For $\gamma_n = 0$, Algorithm 3.2 reduces to:

Algorithm 3.3. For a given $\mu_0 \in \mathcal{H}$, compute $\{\mu_{n+1}\}$ by the iterative schemes

$$\begin{aligned} w_n &= (1-\beta_n)\mu_n + \beta_n\{\mu_n - g(\mu_n) + \mathcal{J}_{\mathcal{A}}[g(\mu_n) - \rho\Phi(\mathcal{T}\mu_n, \mathcal{B}(\mu_n))\} \\ \mu_{n+1} &= (1-\alpha_n)u_n + \alpha_n\{w_n - g(w_n) + \mathcal{J}_{\mathcal{A}}[g(w_n) - \rho\Phi(\mathcal{T}w_n, \mathcal{B}(w_n))\}, \end{aligned}$$

which is known as the Ishikawa iterative scheme for the general bivariational inclusion (2.1). Note that for $\gamma_n = 0 \quad and \quad \beta_n = 0$, Algorithm 3.1 collapses to:

Algorithm 3.4. For a given $\mu_0 \in \mathcal{H}$, compute $\{\mu_{n+1}\}$ by the iterative schemes

$$\mu_{n+1} = (1-\alpha_n)\mu_n + \alpha_n\{\mu_n - g(\mu_n) + \mathcal{J}_{\mathcal{A}}[g(\mu_n) - \rho\Phi(\mathcal{T}\mu_n, \mathcal{B}(\mu_n))\},$$

is called the Mann iterative method. Now we suggest a perturbed iterative scheme for solving the general bivariational inclusion (2.1).

Algorithm 3.5. For a given $\mu_o \in \mathcal{H}$, compute the approximate solution $\{\mu_n\}$ by the iterative schemes

$$\begin{aligned} y_n &= (1-\gamma_n)\mu_n + \gamma_n\{\mu_n - g(\mu_n) + \mathcal{J}_{\mathcal{A}_n}[g(\mu_n) - \rho\Phi(\mathcal{T}\mu_n, \mathcal{B}(\mu_n))]\} + \gamma_n h_n \\ w_n &= (1-\beta_n)u_n + \beta_n\{y_n - g(y_n) + \mathcal{J}_{\mathcal{A}_n}[g(y_n) - \rho\Phi(\mathcal{T}y_n, \rho\mathcal{B}(y_n))]\} + \beta_n f_n \\ \mu_{n+1} &= (1-\alpha_n)u_n + \alpha_n\{w_n - g(w_n)\mathcal{J}_{\mathcal{A}_n}[g(w_n) - \rho\Phi(\mathcal{T}w_n, \mathcal{B}(w_n))]\} + \alpha_n e_n, \end{aligned}$$

where $\{e_n\}$, $\{f_n\}$, and $\{h_n\}$ are the sequences of the elements of H introduced to take into account possible inexact computations and $\mathcal{J}_{\mathcal{A}_n}$ is the corresponding perturbed resolvent operator; and the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1$; $\forall n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

For $\gamma_n = 0$, we obtain the perturbed Ishikawa iterative method and for $\gamma_n = 0$ and $\beta_n = 0$, we obtain the perturbed Mann iterative schemes for solving general bivariational inclusion (2.1).

We now study the convergence analysis of Algorithm 3.2, which is the main motivation of our next result.

Theorem 3.3. Let the operators $\mathcal{T}, g$ satisfy all the assumptions of Theorem 3.1. If the condition (3.20) holds, then the approximate solution $\{u_n\}$ obtained from Algorithm 3.2 converges to the exact solution u of the general bivariational inclusion (2.1) strongly in $\mathcal{H}$.

Proof . From Theorem 3.2, we see that there exists a unique solution $u \in H$ of the general bivariational inclusion (2.1). Let $\mu \in \mathcal{H}$ be the unique solution of (2.1). Then, using Lemma 3.1, we have

$$\begin{aligned} \mu &= (1-\alpha_n)\mu + \alpha_n\{\mu - g(\mu) + \mathcal{J}_{\mathcal{A}}[g(\mu) - \rho\Phi(\mathcal{T}\mu, \mathcal{B}(\mu))]\} && (3.13)\\ &= (1-\beta_n)\mu + \beta_n\{\mu - g(\mu) + \mathcal{J}_{\mathcal{A}}[g(\mu) - \rho\Phi(\mathcal{T}\mu, \mathcal{B}(\mu))]\} && (3.14)\\ &= (1-\gamma_n)\mu + \gamma_n\{\mu - g(\mu) + \mathcal{J}_{\mathcal{A}}[g(\mu) - \rho\Phi(\mathcal{T}\mu, \mathcal{B}(\mu))]\}. && (3.15) \end{aligned}$$

From (3.12),(3.13) and (3.9), we have

$$\begin{aligned} \|\mu_{n+1} - \mu\| &= \|(1-\alpha_n)(\mu_n - \mu) + \alpha_n(w_n - \mu - (g(w_n) - g(\mu)))\\ &+ \alpha_n\{\mathcal{J}_{\mathcal{A}}[g(w_n) - \rho\Phi(\mathcal{T}w_n, \mathcal{B}(w_n))] - \mathcal{J}_{\mathcal{A}}[g(\mu) - \rho\Phi(\mathcal{T}\mu, \mathcal{B}(\mu))]\}\|\\ &\leq (1-\alpha_n)\|\mu_n - \mu\| + 2\alpha_n\|w_n - \mu - (g(w_n) - g(\mu))\|\\ &+ \alpha_n\|w_n - \mu - \rho(\Phi(\mathcal{T}w_n, \mathcal{B}(w_n)) - \Phi(\mathcal{T}\mu, \mathcal{B}((w_n)))\|\\ &\quad +\alpha_n\rho\|\Phi(\mathcal{T}\mu, \mathcal{B}(w_n)) - \Phi(\mathcal{T}\mu, \mathcal{B}(\mu))\|\\ &\leq (1-\alpha_n)||\mu_n - \mu|| + \alpha_n(k + \rho\gamma + t(\rho))||w_n - \mu|| + \alpha_n\gamma\|w_n - \mu\|,\\ &= (1-\alpha_n)\|u_n - \mu\| + \alpha_n\theta\|w_n - \mu\|. && (3.16) \end{aligned}$$

In a similar way, from (3.10),(3.14) and (3.9), we have

$$\begin{aligned} \|w_n - \mu\| &\leq (1-\beta_n)\|\mu_n - \mu\| + 2\beta_n\theta\|y_n - \mu - (g(y_n) - g(\mu))\|\\ &+ \beta_n\|y_n - \mu - \rho(\Phi(\mathcal{T}y_n, \mathcal{B}(y_n)) - \Phi(\mathcal{T}\mu, \mathcal{B}(y_n)))\|\\ &\quad +\beta_n\rho\|\Phi(\mathcal{T}\mu, \mathcal{B}(y_n)) - \Phi(\mathcal{T}\mu, \mathcal{B}(\mu))\|\\ &\leq (1-\beta_n)\|\mu_n - \mu\| + \beta_n(k + \rho\gamma + t(\rho))\|y_n - \mu\|,\\ &\leq (1-\beta_n)\|\mu_n - \mu|| + \beta_n\theta\|y_n - \mu\| && (3.17) \end{aligned}$$

Also from (3.10), (3.15) and (3.9), we obtain

$$\begin{aligned} \|y_n - \mu\| &\leq (1-\gamma_n)\|\mu_n - \mu\| + \gamma_n\theta\|\mu_n - \mu\|, \quad \text{using (3.9).}\\ &\leq (1-(1-\theta)\gamma_n)\|\mu_n - \mu\|\\ &\leq ||\mu_n - \mu||. && (3.18) \end{aligned}$$

From (3.17) and (3.18), we obtain

$$\begin{aligned} \|w_n - \mu\| &\leq (1-\beta_n)\|\mu_n - \mu\| + \beta_n\theta\|\mu_n - \mu\|\\ &= (1-(1-\theta)\beta_n)\|\mu_n - \mu\|\\ &\leq ||\mu_n - \mu||. && (3.19) \end{aligned}$$

Form the above equations, have

$$\begin{aligned} \|\mu_{n+1} - \mu\| &\leq (1-\alpha_n)\|\mu_n - \mu\| + \alpha_n\theta\|\mu_n - \mu\|\\ &= [1-(1-\theta)\alpha_n]\|\mu_n - \mu\|\\ &\leq \prod_{i=0}^{n}[1-(1-\theta)\alpha_1]\|\mu_0 - \mu\|. \end{aligned}$$

Since $\sum_{n=0}^{\infty}\alpha_n$ diverges and $1-\theta > 0$, we have $\prod_{i=0}^{n}[1-(1-\theta)\alpha_i] = 0$. Consequently the sequence $\{u_n\}$ converges strongly to μ. From (3.18), and (3.19), it follows that the sequences $\{y_n\}$ and $\{w_n\}$ also converge to μ strongly in H. This completes the proof. □

We now suggest some new iterative methods for solving general bivariational inclusions of type (2.1).

Algorithm 3.6. For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$g(\mu_{n+1}) = \mathcal{J}_{\mathcal{A}}[g(\mu_n) - \rho\Phi(\mathcal{T}\mu_n, \mathcal{B}(\mu_n))], \quad n = 0, 1, 2, ...$$

which is known as the resolvent iterative method.

Algorithm 3.7. For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$g(\mu_{n+1}) = \mathcal{J}_{\mathcal{A}}[g(\mu_n) - \rho\Phi(\mathcal{T}\mu_{n+1}, \mathcal{B}(\mu_{n+1})], \quad n = 0, 1, 2, ...$$

which is known as the implicit resolvent method and is equivalent to the following two-step method.

Algorithm 3.8. For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} g(\omega_n) &= \mathcal{J}_{\mathcal{A}}[g(\mu_n) - \rho\Phi(\mathcal{T}\mu_n, \mathcal{B}(\mu_n))] \\ g(\mu_{n+1}) &= \mathcal{J}_{\mathcal{A}}[g(\mu_n) - \rho\Phi(\mathcal{T}\omega_n, \mathcal{B}(\omega_n))], \quad n = 0, 1, 2, ... \end{aligned}$$

We can rewrite the equation (3.1) as:

$$g(\mu) = \mathcal{J}_{\mathcal{A}}[\frac{g(\mu) + g(\mu)}{2} - \rho\Phi(\mathcal{T}\mu, \mathcal{B}(\mu))].$$

This fixed point formulation was used to suggest the following implicit method.

Algorithm 3.9. [29]. For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$g(\mu_{n+1}) = \mathcal{J}_{\mathcal{A}}\big[\frac{g(\mu_n) + g(\mu_{n+1})}{2} - \rho\Phi(\mathcal{T}\mu_{n+1}, \mathcal{B}(\mu_{n+1}))\big].$$

The predictor-corrector technique is applied to suggest the following inertial iterative method for solving the problem (2.1).

Algorithm 3.10. For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} g(\omega_n) &= \mathcal{J}_{\mathcal{A}}[g(\mu_n) - \rho\Phi(\mathcal{T}\mu_n, \mathcal{B}(\mu_n))] \\ g(\mu_{n+1}) &= \mathcal{J}_{\mathcal{A}}[\frac{g(\omega_n) + g(\mu_n)}{2} - \rho\Phi(\mathcal{T}\omega_n, \mathcal{B}(\omega_n))], \quad \lambda \in [0, 1]. \end{aligned}$$

From equation (3.1), we have

$$g(\mu) = \mathcal{J}_{\mathcal{A}}[g(\mu) - \rho\Phi(\mathcal{T}(\frac{\mu + \mu}{2}), \mathcal{B}(\frac{\mu + \mu}{2}))]. \tag{3.20}$$

This fixed point formulation (3.20) is used to suggest the implicit method for solving the problem (2.1) as

Algorithm 3.11. For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$g(\mu_{n+1}) = \mathcal{J}_{\mathcal{A}}[g(\mu_n) - \rho\Phi(\mathcal{T}(\frac{\mu_n + \mu_{n+1}}{2}), \mathcal{B}(\frac{\mu_n + \mu_{n+1}}{2}))].$$

We can use the predictor-corrector technique to rewrite Algorithm 3.11 as:

Algorithm 3.12. For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} g(\omega_n) &= \mathcal{J}_{\mathcal{A}}[g(\mu_n) - \rho\Phi(\mathcal{T}\mu_n, \mathcal{B}(\mu_n))], \\ g(\mu_{n+1}) &= \mathcal{J}_{\mathcal{A}}[g(\mu_n) - \rho\Phi(\mathcal{T}(\frac{\mu_n + \omega_n}{2}), \mathcal{B}(\frac{\mu_n + \omega_n}{2}))], \quad n = 0, 1, 2, ... \end{aligned}$$

is known as the mid-point implicit method for solving the problem (2.1).
We again use the above fixed formulation to suggest some following implicit iterative methods.

Algorithm 3.13. For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$g(\mu_{n+1}) = \mathcal{J}_{\mathcal{A}}[g(\mu_{n+1}) - \rho\Phi(\mathcal{T}(\frac{\mu_n + \mu_{n+1}}{2}), \mathcal{B}(\frac{\mu_n + \mu_{n+1}}{2}))].$$

Using the predictor-corrector technique, Algorithm 3.13 can be written as:

Algorithm 3.14. For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} g(\omega_n) &= \mathcal{J}_{\mathcal{A}}[g(\mu_n) - \rho\Phi(\mathcal{T}\mu_n, \mathcal{B}(\mu_n))], \\ g(\mu_{n+1}) &= \mathcal{J}_{\mathcal{A}}[g(\omega_n) - \rho\Phi(\mathcal{T}(\frac{\mu_n + \omega_n}{2}), \mathcal{B}(\frac{\mu_n + \omega_n}{2}))]. \end{aligned}$$

which appears to be new one. We now use the fixed point formulation to suggest a hybrid implicit method for solving the problem (2.1) and related optimization problems, which is the main motivation of this paper. One can rewrite (3.1) as

$$g(\mu) = \mathcal{J}_{\mathcal{A}}[g(\frac{\mu + \mu}{2}) - \rho\Phi(\mathcal{T}(\frac{\mu + \mu}{2}), \mathcal{B}(\frac{\mu + \mu}{2}))]. \tag{3.21}$$

This equivalent fixed point formulation enables us to suggest the following implicit method for solving the problem (2.1).

Algorithm 3.15. For a given $\mu_0 \in H$, compute μ_{n+1} by the iterative scheme

$$g(\mu_{n+1}) = \mathcal{J}_{\mathcal{A}}\left[g(\frac{\mu_n + \mu_{n+1}}{2}) - \rho\Phi(\mathcal{T}(\frac{\mu_n + \mu_{n+1}}{2}), \mathcal{B}(\frac{\mu_n + \mu_{n+1}}{2}))\right].$$

To implement the implicit method, one uses the predictor-corrector technique. We use Algorithm 3.11 as the predictor and Algorithm 3.15 as corrector. Thus, we obtain a new two-step method for solving the problem (2.1).

Algorithm 3.16. For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} g(\omega_n) &= \mathcal{J}_{\mathcal{A}}[g(\mu_n) - \rho\Phi(\mathcal{T}\mu_n, \mathcal{B}(\mu_n))] \\ g(\mu_{n+1}) &= \mathcal{J}_{\mathcal{A}}\left[g(\frac{\omega_n + \mu_n}{2}) - \rho\Phi(\mathcal{T}\left(\frac{\omega_n + \mu_n}{2}\right), \mathcal{B}\left(\frac{\omega_n + \mu_n}{2})\right)\right]. \end{aligned}$$

which is a predictor-corrector two-step method. For a parameter ξ, one can rewrite the equation (3.1) as

$$g(\mu) = \mathcal{J}_{\mathcal{A}}[(1 - \xi)g(\mu) + \xi g(\mu) - \rho\Phi(\mathcal{T}\mu, \mathcal{B}(\mu))].$$

This equivalent fixed point formulation enables to suggest the following inertial method for solving the problem (2.1).

Algorithm 3.17. For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$g(\mu_{n+1}) = \mathcal{J}_{\mathcal{A}}[(1 - \xi)g(\mu_n) + \xi g(\mu_{n-1}) - \rho\Phi(\mathcal{T}\mu_n, \mathcal{B}(\mu_n))], \quad n = 0, 1, 2, ...$$

It is noted that Algorithm 3.17 is equivalent to the following two-step method.

Algorithm 3.18. For a given $\mu_0 \in H$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} g(\omega_n) &= (1 - \xi)u_n + \xi u_{n-1} \\ g(\mu_{n+1}) &= \mathcal{J}_{\mathcal{A}}[g(\omega_n) - \rho\Phi(\mathcal{T}\mu_n, \mathcal{B}(\mu_n))]. \end{aligned}$$

Algorithm 3.18 is known as the inertial resolvent method.

Using this idea, we can suggest the following iterative methods for solving general bivariational inclusions.

Algorithm 3.19. For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} g(\omega_n) &= (1-\xi)u_n + \xi u_{n-1} \\ g(\mu_{n+1}) &= \mathcal{J}_{\mathcal{A}}[g(\omega_n) - \rho\Phi(\mathcal{T}\omega_n, \mathcal{B}(\omega_n))]. \end{aligned}$$

Using the technique of Noor et al. [34], Shehu et al.[45] and Jabeen et al [11], one can investigate the convergence analysis of these inertial resolvent methods. One can again use the equation (3.1) to suggest a wide class of inertial methods for solving the general bivariational inclusions. We have only conveyed the main ideas and the techniques. To develop the efficient methods and comparison with other techniques is the open problem.

4 Resolvent equations technique

In this section, we discuss the resolvent equations associated with the general bivariational inclusions (2.1). It is worth mentioning that the resolvent equations associated with variational inclusions were introduced and studied by Noor [25, 26, 27] and Noor et al. [28] proved that the quasi variational inclusions are equivalent to the implicit resolvent equations to study the sensitivity analysis.

Related to the general bidirectional inclusion (2.1), we consider the problem of finding $z, \mu \in \mathcal{H}$ such that

$$\Phi(\mathcal{T}\mathcal{J}_{\mathcal{A}}z, \mathcal{B}\mathcal{J}_{\mathcal{A}}z,) + \rho^{-1}\mathcal{R}_{\mathcal{A}}z = 0, \tag{4.1}$$

where $\rho > 0$ is a constant and $\mathcal{R}_{\mathcal{A}} = I - \mathcal{J}_{\mathcal{A}}$. Here I is the identity operator and $\mathcal{J} = (1+\rho\mathcal{A})^{-1}$ is the resolvent operator. The equation of the type (4.1) are called the implicit resolvent equations.

Lemma 4.1. The general bivariational inclusion (2.1) has a solution $\mu \in \mathcal{H}$, if and only if, the resolvent equations (4.1) have a solution $z, \mu \in \mathcal{H}$, where

$$g(\mu) = \mathcal{J}_{\mathcal{A}}z \tag{4.2}$$

and

$$z = g(\mu) - \rho\Phi(\mathcal{T}\mu, \mathcal{B}(\mu)). \tag{4.3}$$

Proof . Let $\mu \in \mathcal{H}$ be a solution of (2.1), then, for a constant ρ,

$$\begin{aligned} \rho\Phi(\mathcal{T}\mu, \mathcal{B}(\mu)) \quad &+ \quad \rho\mathcal{A}(g(\mu)) \ni 0 \\ &\Longleftrightarrow \\ -g(\mu) + \rho\Phi(\mathcal{T}\mu, \mathcal{B}(\mu)) \quad &+ \quad g(\mu) + \rho\mathcal{A}(g(\mu)) \ni 0 \\ &\Longleftrightarrow \\ g(\mu) \quad &= \mathcal{J}_{\mathcal{A}}[g(\mu) - \rho\Phi(\mathcal{T}\mu, \mathcal{B}(\mu))]. \end{aligned}$$

Take $z = g(\mu) - \rho\Phi(\mathcal{T}\mu, \mathcal{B}(\mu))$, then $z = \mathcal{J}_{\mathcal{A}}z$. Thus

$$z = \mathcal{J}_{\mathcal{A}}z - \rho\Phi(\mathcal{T}_{J_A}z, \mathcal{B}_{J_A}z),$$

that is

$$\Phi(\mathcal{T}\mathcal{J}_{\mathcal{A}(}z, \mathcal{B}\mathcal{J}_{\mathcal{A}}z) + \rho^{-1}\mathcal{R}_{\mathcal{A}}z = 0,$$

the required (4.1). □ From Lemma 4.1, we see that the general bivariational inclusion (2.1) and the resolvent equations (4.1) are equivalent. This alternative equivalent formulation has been used to suggest and analyze a wide class of efficient and robust iterative methods for solving the general bivariational inclusions and related optimization problems.

We use the resolvent equations (4.1) to suggest some new iterative methods for solving the general bivariational inclusions. From (4.2) and (4.3), we have

$$\begin{aligned} z &= \mathcal{J}_{\mathcal{A}} z - \rho\Phi(\mathcal{T}\mathcal{J}_{\mathcal{A}} z, \mathcal{B}(\mathcal{J}_{\mathcal{A}} z)) \\ &= \mathcal{J}_{\mathcal{A}}[g(\mu) - \rho\Phi(\mathcal{T}\mu, \mathcal{B}(\mu))] - \rho\mathcal{T}\mathcal{J}_{\mathcal{A}(}[g(\mu) - \rho\Phi(\mathcal{T}\mu, \mathcal{J}_{\mathcal{B}}(\mu))]. \end{aligned}$$

Thus, we have

$$g(\mu) = -\rho\Phi(\mathcal{T}\mu, \mathcal{B}(\mu)) + \big[\mathcal{J}_{\mathcal{A}(}[g(\mu) - \rho\Phi(\mathcal{T}\mu, \mathcal{B}(\mu))] - \rho\mathcal{T}\mathcal{J}_{\mathcal{A}}[g(\mu) - \rho\Phi(\mathcal{T}\mu, \mathcal{B}(\mu))].$$

Consequently, for a constant $\alpha_n > 0$, we have

$$\begin{aligned} \mu &= (1-\alpha_n)\mu + \alpha_n \mathcal{J}_{\mathcal{A}}\{\mathcal{J}_{\mathcal{A}}[g(\mu) - \rho\Phi(\mathcal{T}u, \mathcal{B}(\mu))] + \rho\Phi(\mathcal{T}\mu, \mathcal{B}(\mu)) \\ &\quad -\rho\mathcal{T}\mathcal{J}_{\mathcal{A}}[g(\mu) - \rho\mathcal{T}\mu - \rho\mathcal{B}(\mu)]\} \\ &= (1-\alpha_n)\mu + \alpha_n \mathcal{J}_{\mathcal{A}}\{g(\omega) - \rho\mathcal{T}\omega - \rho\Phi(\mathcal{T}\mu, \mathcal{B}(\mu))\}, \end{aligned} \tag{4.4}$$

where

$$g(\omega) = \mathcal{J}_{\mathcal{A}}[g(\mu) - \rho\Phi(\mathcal{T}\mu, \mathcal{B}(\mu))]. \tag{4.5}$$

Using (4.4) and (4.5), we can suggest the following new predictor-corrector method for solving the general bivariational inclusion (2.1).

Algorithm 4.1. For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} g(\omega_n) &= \mathcal{J}_{\mathcal{A}}[g(\mu_n) - \rho\Phi(\mathcal{T}\mu_n, \mathcal{B}(\mu_n))] \\ g(\mu_{n+1}) &= (1-\alpha_n)\mu_n + \alpha_n \mathcal{J}_{\mathcal{A}}\Big\{g(\omega) - \rho\mathcal{T}\omega - \rho\Phi(\mathcal{T}\mu, \mathcal{B}(\mu))\Big\}. \end{aligned}$$

If $\alpha_n = 1$, then Algorithm 4.1 reduces to

Algorithm 4.2. For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} g(\omega_n) &= \mathcal{J}_{\mathcal{A}}[g(\mu_n) - \rho\Phi(\mathcal{T}\mu_n, \mathcal{B}(\mu))] \\ g(\mu_{n+1}) &= \mathcal{J}_{\mathcal{A}}[g(\omega_n) - \rho\mathcal{T}\omega_n + \rho\Phi(\mathcal{T}\mu_n, \mathcal{B}(\mu_n))], \end{aligned}$$

which appears to be a new one. In a similar way, we can suggest and analyse the predictor-corrector inertial method for solving the general bivariational inclusion(2.1), which involve only one resolvent.

Algorithm 4.3. For given $u_0, u_1 \in H$, compute u_{n+1} by the iterative scheme

$$\begin{aligned} g(\omega_n) &= (1-\xi)g(\mu_n) + \xi g(\mu_{n-1}) \\ g(\mu_{n+1}) &= \mathcal{J}_{\mathcal{A}}[g(\omega_n) - \rho\mathcal{T}\omega_n + \rho\Phi(\mathcal{T}\mu_n, \mathcal{B}(\mu_n))]. \end{aligned}$$

One can study the convergence of the Algorithm 4.3 using the technique of Jabeen et al [11] and Noor et al. [34].

Remark 4.2. We have only given some glimpse of the technique of the resolvent equations for solving the general bivariational inclusions. One can explore the applications of the resolvent equations in developing efficient numerical methods for solving general bivariational inclusions and related nonlinear optimization problems.

5 Dynamical Systems Technique

In this section, we consider the dynamical systems technique for solving general bivariational inclusions. The dynamical systems associated with variational inequalities using the fixed point problems introduced and studied by Dupuis and Nagurney [7]. Thus it is clear that the variational inequalities are equivalent to a first order initial value problem. Variational inequalities, equilibrium and nonlinear problems arising in various branches in pure and applied sciences can now be studied in the setting of dynamical systems. Noor et al. [33, 35, 39] have been shown that the dynamical systems are useful in developing some efficient numerical techniques for solving variational inequalities and related optimization problems. We consider some iterative methods for solving the general bivariational inclusions using the dynamical system.

We now define the residue vector $R(\mu)$ by the relation

$$R(\mu) = g(\mu) - \mathcal{J}_{\mathcal{A}}[g(\mu) - \rho\Phi(\mathcal{T}\mu, \mathcal{B}(\mu))]. \tag{5.1}$$

Invoking Lemma 3.1, one can easily conclude that $\mu \in \mathcal{H}$ is a solution of the problem(2.1), if and only if, $\mu \in \mathcal{H}$ is a zero of the equation

$$R(\mu) = 0. \tag{5.2}$$

We now consider a dynamical system associated with the general bidirectional inclusions. Using the equivalent formulation (3.1), we suggest a class of project dynamical systems as

$$\frac{dg(\mu)}{dt} = \lambda\{\mathcal{J}_{\mathcal{A}}[g(\mu) - \rho\Phi(\mathcal{T}u, \mathcal{B}(\mu)] - g(\mu))\}, \quad g(\mu(t_0)) = \alpha, \tag{5.3}$$

where λ is a parameter. The system of type (5.3) is called the resolvent dynamical system associated with the problem (2.1). Here the right hand is related to the resolvent and is discontinuous on the boundary. From the definition, it is clear that the solution of the dynamical system always stays in $\mathcal{H}$. This implies that the qualitative results such as the existence, uniqueness and continuous dependence of the solution of the general bivariational inclusions (5.1) can be studied.

We use the resolvent dynamical system (5.1) to suggest some iterative for solving the general bidirectional inclusion (2.1). These methods can be viewed in the sense of Korpelevich [13] and Noor [24].

For simplicity, we take $\lambda = 1$. Thus the dynamical system (5.1) becomes

$$\frac{dg(\mu)}{dt} + g(\mu) = \mathcal{J}_{\mathcal{A}}\big[g(\mu) - \rho\Phi(\mathcal{T}u, \mathcal{B}(\mu))\big], \quad g(\mu(t_0)) = \alpha. \tag{5.4}$$

The forward difference scheme is used to construct the implicit iterative method.
Discretizing (5.4), we have

$$\frac{g(\mu_{n+1}) - g(\mu_n)}{h} + g(\mu_n) = \mathcal{J}_{\mathcal{A}}[g(\mu_n) - \rho\Phi(\mathcal{T}\mu_{n+1}, \mathcal{B}(\mu_{n+1}))], \tag{5.5}$$

where h is the step size. For $h = 1$, we can suggest the following implicit iterative method for solving the general bivariational inclusion (2.1).

Algorithm 5.1. For a given μ_0,, compute μ_{n+1} by the iterative scheme

$$g(\mu_{n+1}) = \mathcal{J}_{\mathcal{A}}\Big[g(\mu_n) - \rho\Phi(\mathcal{T}\mu_{n+1}, \mathcal{B}(\mu_{n+1}))\Big],$$

This is an implicit method, which is quite different from the implicit method of [4].
Algorithm 5.1 is equivalent to the following two-step method.

Algorithm 5.2. For a given μ_0,, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} g(\omega_n) &= \mathcal{J}_{\mathcal{A}}[g(\mu_n) - \rho\Phi(\mathcal{T}\mu_n, \mathcal{B}(\mu_n))] \\ g(\mu_{n+1}) &= \mathcal{J}_{\mathcal{A}}\big[g(\mu_n) - \rho\Phi(\mathcal{T}\omega_n, \mathcal{B}(\omega_n))\big]. \end{aligned}$$

Discretizing (5.5), we now suggest an other implicit iterative method for solving (2.1).

$$\frac{g(\mu_{n+1})-g(\mu_n)}{h}+g(\mu_n)=\mathcal{J}_{\mathcal{A})}[g(\mu_{n+1})-\rho\Phi(\mathcal{T}\mu_{n+1},\mathcal{B}(\mu_{n+1}))],$$

where h is the step size. For $h=I$, this formulation enables us to suggest the two-step iterative method.

Algorithm 5.3. For a given $\mu_0\in\mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} g(\omega_n) &= \mathcal{J}_{\mathcal{A}}[g(\mu_n)-\rho\mathcal{T}\mu_n-\rho\mathcal{B}(\mu_n)] \\ g(\mu_{n+1}) &= \mathcal{J}_{\mathcal{A}}\Big[(1-\zeta)g(\omega_n)+\zeta g(\mu_n)-\rho\Phi(\mathcal{T}\omega_n,\mathcal{B}(\omega_n)\Big], \end{aligned}$$

where $\zeta\in[0,1]$ is a constant. Again using the resolvent dynamical systems, we can suggest some iterative methods for solving the general bivariational inclusion and related optimization problems.

Algorithm 5.4. For a given $\mu_0\in\mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$g(\mu_{n+1})=\mathcal{J}_{\mathcal{A}}\Big[(1-\lambda)g(\mu_n)+\lambda g(\mu_{n+1})-\rho\Phi(\mathcal{T}\mu_n,\mathcal{B}(\mu_n))\Big],\quad \lambda\in[0,1]$$

or equivalently

Algorithm 5.5. For a given $\mu_0\in\mathcal{H}$, compute μ_{n+1} by the iterative scheme

$$\begin{aligned} g(\omega_n) &= \mathcal{J}_{\mathcal{A}}[g(\mu_n)-\rho\Phi(\mathcal{T}\mu_n,\mathcal{B}(\mu_n))] \\ g(\mu_{n+1}) &= \mathcal{J}_{\mathcal{A}}\Big[(1-\lambda)g(\mu_n)+\lambda g(\omega_n)-\rho\Phi(\mathcal{T}u_n,\mathcal{B}(\mu_n))\Big]. \end{aligned}$$

In a similar way, we have

$$\frac{dg(\mu)}{dt}+g(\mu)=\mathcal{J}_{\mathcal{A}}[g((1-\alpha)\mu+\alpha\mu)-\rho\Phi(\mathcal{T}((1-\alpha)\mu+\alpha\mu),\mathcal{B}((1-\alpha)\mu+\alpha\mu)))],\tag{5.6}$$

where $\alpha\in[0,1]$ is a constant. Discretizating (5.6) and taking $h=1$, we have

$$g(\mu_{n+1})=\mathcal{J}_{\mathcal{A}}\big[g(1-\alpha)\mu_n)+\alpha\mu_{n-1})-\rho\Phi(\mathcal{T}((1-\alpha)\mu_n+\alpha\mu_{n-1}),\mathcal{B}((1-\alpha)\mu_n+\alpha\mu_{n-1}))\big],$$

which is an inertial type iterative method for solving the general bivariational inclusion (2.1). Using the predictor-corrector techniques, we have

Algorithm 5.6. For a given $\mu_0\in H$, compute μ_{n+1} by the iterative schemes

$$\begin{aligned} \omega_n &= (1-\alpha)\mu_n-\alpha\mu_{n-1} \\ g(\mu_{n+1}) &= \mathcal{J}_{\mathcal{A}}\big[g(\omega_n)-\rho\Phi(\mathcal{T}(\omega_n),\mathcal{B}(\omega_n))\big], \end{aligned}$$

which is known as the inertial two-step iterative method.
One can study the convergence criteria of Algorithm (5.6) using essentially the technique of Jabeen et al. [11], Noor et al. [34] and Shehu et al. [45].

Computational Aspects

In this paper, we have suggested several new iterative methods for solving general bivariational inclusions and related problems using the techniques of are resolvent operators, resolvent equations and dynamical systems. The inertial type iterations need one resolvent only as compared with known methods. Due to these facts, the newly methods perform better than the other techniques. To the best of our knowledge, no implementable numerical methods are available. This is a relatively new field and may be starting point for further applications in various fields.

Conclusion

In this paper, we have introduced and studied some new classes of general bivariational inclusions. Some interesting and important known and new classes of variational inequalities and optimizations are discussed. We have proved that the general bivariational inclusions are equivalent to this fixed point problems, resolvent equations and dynamical systems. These alternative formulations are used to discuss the existence of a solution of the general bivariational inclusions and suggest some new iterative methods for solving the general bivariational inclusions. These new methods include extraresolvent method, modified double resolvent methods and inertial type are suggested using the techniques of resolvent method, resolvent equations and dynamical systems. Convergence analysis of the proposed method is discussed for monotone operators. We have given only the glimpse of the applications of the dynamical systems. This technique is quite flexible and unified one. Using the ideas and techniques of this paper, one can suggest and investigate several new implicit methods for solving various classes of general bivariational inclusions and related problems. The implementation and comparison of these methods with other methods needs further efforts.

Acknowledgements

We wish to express our deepest gratitude to our teachers, colleagues, students, collaborators, and friends, who have direct or indirect contributions in the process of this paper. Authors are grateful to the referees for their valuable comments and suggestions.

References

[1] F. Alvarez, *Weak convergence of a relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert space,* SIAM J. Optim. **14** (2003), 773–782.

[2] W.F. Ames, *Numerical Methods for Partial Differential Equations,* 3rd ed., Academic Press, New York, 1992

[3] K. Ashish, M. Rani and R. Chugh, *Julia sets and Mandelbrot sets in Noor orbit,* Appl. Math. Comput. **228** (2014), no. 1, 615–631.

[4] R.W. Cottle, *Nonlinear programs with positively bounded Jacobians,* J. Soc. Indust. Appl. Math. **1** (1966), 147–158.

[5] R.W. Cottle , J,-S. Pang and R, E. Stone, *The Linear Complementarity Problem,* SIAM Publ. Pjiladelphia, USA, 2009.

[6] P. Daniele, F. Giannessi and A. Maugeri, *Equilibrium Problems and Variational Models,* Kluwer Acadamic, London, 2003.

[7] P. Dupuis and A. Nagurney, *Dynamical systems and variational inequalities,* Ann. Oper. Res. **44** (1993), 7–42.

[8] R. Glowinski, J.L. Lions and R. Tremolieres, *Numerical Analysis of Variational Inequalities,* NorthHolland, Amsterdam, 1981.

[9] R. Glowinski and P. Le Tallec, *Augmented Lagrangian and Operator Splitting Methods in Nonlinear Mechanics*, SIAM, Philadelphia, Pennsylvania, 1989.

[10] S. Haubruge, V.H. Nguyen and J.J. Strodiot, *Convergence analysis and applications of the Glowinski-Le Tallec splitting method for finding a zero of the sum of two maximal monotone operators,* J. Optim. Theory Appl. **97** (1998), 645–673.

[11] S. Jabeen, B.B.Mohsin , M.A. Noor and K.I. Noor. *Inertial projection methods for solving general quasi-variational inequalities,* AIMS Math. **6** (2021), no. 2, 1075–1086.

[12] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications,* SIAM, Philadelphia, 2000

[13] G.M. Korpelevich, *The extragradient method for finding saddle points and other problems,* Ekonomika Mat. Metody **12** (1976), 747–756.

[14] O.L. Mangasarian, *A generalized Newton method for absolute value equations,* Optim. Lett. **3** (2009), 101–108.

[15] A. Nagurney and D. Zhang, *Projected Dynamical Systems and Variational Inequalities with Applications,* Kluwer Academic Publishers, Boston, Dordrecht, London 1996.

[16] T, V. Nghi and N. N. Tam,*General variational inequalities: existence of solutions, Tikhonov-Type regularization, and well-posedness,* Acta Math. Vietnam. **47** (2022) https://doi.org/10.1007/s40306-021-00435-0

[17] M.A. Noor, *On Variational Inequalities,* PhD Thesis, Brunel University, London, U. K., 1975.

[18] M.A. Noor, *General variational inequalities,* Appl. Math. Lett. **1** (1988), 119–121.

[19] M.A. Noor, *Quasi variational inequalities,* Appl. Math. Lett. **1** (1988), no. 4, 367–372.

[20] M.A. Noor, *Fixed point approach for complementaritty problems,* J. Math. Anal. Appl. **133** (1988), 437–448.

[21] M.A. Noor, *New approximation schemes for general variational inequalities,* J. Math. Anal. Appl. **251** (2000), 217–230.

[22] M.A. Noor, *Resolvent dynamical systems for mixed variational inequalities,* Korean J. Comput. Appl. Math. **9** (2002), 15–26.

[23] M.A. Noor, *A Wiener-Hopf dynamical system and variational inequalities,* New Zealand J. Math. **31** (2002), 173–182.

[24] M.A. Noor, *Some dvelopments in general variational inequalites,* Appl. Math. Comput. **152** (2004), 199–277.

[25] M.A. Noor, *Generalized set-valued variational inclusions and resolvent equations,* J. Math. Anal. Appl. **228** (1998), 206–220.

[26] M.A. Noor, *Three-step iterative algorithms for multivalued quasi variational inclusions,* J. Math. Anal. Appl. **255** (2001), 589-604.

[27] M.A. Noor, *Implicit dynamical systems and quasi variational inequalities,* Appl. Math. Comput. **134** (2003), 69–81.

[28] M.A. Noor and K.I. Noor, *Sensitivity analysis of quasi variational inclusions,* J. Math. Anal. Appl. **236** (1999), 290–299.

[29] M.A. Noor and K.I. Noor, *From representaion theorems to variational inequalities,* Comput. Math. Variat. Anal. (Edits: N. J. Daras, T. M. Rassias), Springer Optim. Appl. **159** (2020), 261–277.

[30] M.A. Noor and K. . Noor, *Some new classes of strongly generalized preinvex functions,* TWMS J. Pure Appl. Math. **12** (2021), no. 2, 181–192.

[31] M.A. Noor and K.I. Noor, *New novel iterative schemes for solving general absolute value equations,* J. Math. Anal. **13** (2022), no. 4, 15–29.

[32] M.A. Noor and K.I. Noor, *Iterstive schemes for solving new system of general equations,* U.P.B. Sci. Bull., Series A, **84** (2022), no. 1, 59–70.

[33] M.A. Noor and K.I. Noor, *Absolute value variational inclusions,* Earthlin. J. Math. Sci. **8**(2022), no. 1, 121-153.

[34] M. A. Noor. K.I. Noor and B. B. Mohsen, *Some new classes of general quasi variational inequalities,* AIMS Math. **6** (2021), no. 6, 6406–6421.

[35] M.A. Noor, K.I. Noor and A. Bnouhachem, *Some new iterative methods for variational inequalities,* Canad. J. Appl. Math. **3** (2021), no. 1, 1–17.

[36] M.A. Noor, K.I. Noor, and M.T. Rassias, *New trends in general variational inequalities,* Acta Appl. Math. **170** (2021), no. 1, 981–1046.

[37] M.A. Noor, K.I. Noor and Th.M. Rassias, *Some aspects of variational inequalities,* J. Comput. Appl. Math. **47** (1993), 285–312.

[38] M.A. Noor, K.I. Noor and S. Batool, *On generalized absolute value equations,* U.P.B. Sci. Bull. Series A. **80** (2018), no. 4, 63–70.

[39] M.A. Noor, K.I. Noor and R. Latif, *Dynamical systems and variational inequalities,* J. Inequal. Special Funct. **8** (2017), no. 5, 22–29.

[40] M.A. Noor, J. Iqbal, K.I. Noor and E. Al-Said, *On an iterative method for solving absolute value equations,* Optim. Lett. **6** (2012), 1027–1033.

[41] M.A. Noor, K.I. Noor, A. Hamdi and E.H. El-Shemas, *On difference of two monotone operators,* Optim. Lett. **3** (2009), 329–335.

[42] M. Patriksson, *Nonlinear Programming and Variational Inequalities: A Unified Approach,* Kluwer Acadamic publishers, Drodrecht, 1998.

[43] B.T. Polyak, *Some methods of speeding up the convergence of iterative methods,* USSR Comput. Math. Math. Phys. **4** (1964), no. 5, 1–17.

[44] P. Shi, *Equivalence of variational inequalities with Wiener-Hopf equations,* Proc. Amer.Math. Soc. **111** (1991), 339–346.

[45] Y. Shehu, A. Gibali and S. Sagratella, *Inertial projection-type method for solving quasi variational inequalities in real Hilbert space,* J. Optim. Theory Appl. **184** (2020), 877–894.

[46] G. Stampacchia, *Formes bilineaires coercitives sur les ensembles convexes,* C. R. Acad. Sci. Paris **258** (1964), 4413–4416.

[47] P. Tseng, *A modified forward-backward splitting method for maximal monotone mappings,* SIAM J. Control Optim. **38** (2000), no.2, 431–446