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Construction confidence interval for a linear combination of parameters of the two-parameter exponential distribution under type II progressive censoring

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Abstract

In this paper, to establish general and shortest confidence interval, for a linear combination of parameters of the two-parameter exponential distribution, we introduce a pivotal quantity. In the case of two populations, we use the method of variance estimate recovery and generalized pivotal quantities to construct a confidence interval for the difference of them. Based on the shortest confidence interval, we present a simple method to obtain its percentiles, and which a shorter confidence interval can be constructed. Also, the performances of the presented methods are studied by real data examples and simulations.

Keywords: Percentiles, Modified based-normal asymptotic, Generalized pivotal quantity, Two-parameter exponential, Shortest confidence interval
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1 Introduction

Suppose that the random variable X has a two-parameter exponential distribution, $Exp(\mu, \sigma)$ with a distribution function

$$F_X(x) = 1 - e^{-\frac{x-\mu}{\sigma}}; \quad x > \mu,$$

where $\mu \in R$ is the location parameter and $\sigma > 0$ is the parameter of the scale. The exponential distribution is due to the lack of memory and the possibility of creating a block distribution in issues related to location transformations in reliability and survival analysis of numerous statistical articles [4]. The exponential distribution of two parameters arises in engineering, biological studies, epidemiology, medical studies, etc. is widely used. As an example, the widespread use of this distribution can be cited in modelling the fail over time for reliability [3]. In engineering and reliability, the location parameter is called threshold value or average warranty period and scale parameter called average time after warranty. In medical sciences, as an example in experiments related to the dose rate, these two parameters are known as the duration of the effect of the medication and the median time of the effect of the drug. Also, in biology and epidemiology, these two parameters are known under the heading of the period of illness and the average incidence of disease.

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The parameter discussed in this article, the linear combination of the location parameters and scale of the two-parameter exponential distribution, is given in the form $\theta_a=\mu+a\sigma$ for different values of a. By putting $a = 1$, the average two-parameter exponential distribution is obtained. Considering the value $a = -\ln(1-p)$, p-th quantile for $0 < p < 1$, and also $a = 0$, the location parameter μ is obtained. Inference about the average distributions of a statistical population is commonplace amongst statisticians. Li et al. [17] proposed a simultaneous confidence interval for the mean k of the two-parameter exponential distribution of populations. Paolino [21] and Balakrishnan et al. [4] introduced a pivotal quantity to create a confidence interval with equal (general) and shortest Two-parameter exponential distribution [16] used a modified base-normal approximation (MNA) concept to create a confidence interval for the linear combination of parameters of two-parameter distribution in one and two populations. Fernandez [6], Krishnamoorthy et al. [14] and Roy et al. [24], proposed one-sided confidence level for a reliability function and quantile of two-parameter exponential distribution based on the generalized pivotal quantity. The test for the assumption of the equal of the location parameters of several exponential distributions was designed by [9], and [18] introduced exact statistics for inference on the common location parameters of this distribution. Also, a simultaneous confidence interval for the difference in location parameters exponential distribution provided by various authors such as [10, 11, 19, 20]. Rozbeh and Najarian [23] identified which parts of the generalized exponential distribution have more information about the distribution parameters.

In this paper, we use the results of n observation with type II progressive censoring [2] to establish the confidence interval for the parameter θ_a. The growing of type II progressive censoring of n observation is thus observed, with the observation of the first failure at the time $X_{(1)}$, R_1 of units of $n-1$ remaining units of the test process are set aside. After observing the second failure data at the time $X_{(2)}$, R_2 units of $n-R_1-2$ units are left out of the test process, and similarly, at the m-th time of failure, $X_{(m)}$, all remaining units are

$$R_m = n - m - R_1 - R_2 - \cdots - R_{m-1}.$$

In this type of censored, the values m and $R=(R_1,\ldots,R_m)$ are predetermined.

Remark 1.1. If $R_1=R_2=\cdots=R_{m-1}=0$ and $R_m=n-m$, then censor of the second type right is obtained. Also, if $R_1=R_2=\cdots=R_m=0$, we have $m=n$, then we will have an uncensored scheme (complete sample).

In the second part of the paper, we use a pivotal quantity to construct the two-sided confidence intervals and the shortest for the θ_a parameter in a population. In two independent exponential populations, the subject is considered to provide a confidence interval for the difference between these parameters. To this end, consider the generalized pivotal value provided by [16]. By using the modified base-normal approximation method to estimate its distribution percentile of the distribution. Also, by using the Method of Variance Estimates Recovery (MOVER), we provide a general and short confidence interval for the third, for this parameters difference. In part 4, we present the proposed methods using the Monte Carlo simulation regarding both the reliability and the coverage probability. We also examine their performance based on the data from the actual examples in Section 5.

2 Confidence Interval in Case One Population

Assume that $X_1,\ldots,X_n$ is a random sample of exponential distribution $Exp\ (\mu,\ \sigma)$, so that based on the values of m and $R_1,\ldots,R_m$ are preset, $X_{(1)}<\cdots<X_{(m)}$ be the observed values derived from this sample based on the progressive type-II censoring. The maximum likelihood estimators of the parameters of this distribution, to form

$$\hat{\mu}=X_{(1)},\quad \hat{\sigma}=\frac{\left(\sum_{i=1}^{m}X_{(i)}+(n-m)X_{(m)}-nX_{(1)}\right)}{m}=\frac{1}{m}\sum_{i=1}^{m}\left(R_i+1\right)\left(X_{(i)}-X_{(1)}\right),$$

[25]. We know these two estimators are independent of each other, and also for $m\geq 2$

$$\frac{2n\left(X_{(1)}-\mu\right)}{\sigma}\sim\chi_2^2,\qquad \frac{2m\hat{\sigma}}{\sigma}\sim\chi_{2m-2}^2.$$

where χ_r^2 is the chi-square random variable with r degrees of freedom. For different values of a, the parameter θ_a behaves like a location parameter. Therefore, we can develop pivotal quantity [4] in the following form,

$$T_{n,m,a}=\frac{X_{(1)}-\theta_a}{\hat{\sigma}}=\frac{\left(X_{(1)}-\mu\right)-a\sigma}{\hat{\sigma}}\quad\xrightarrow{d}\quad\frac{m}{n}\frac{\left(\chi_2^2-2na\right)}{\chi_{2m-2}^2},\tag{2.1}$$

where random variables are independent of each other with the chi-square distribution, and $\xrightarrow{d}$ it means being distributable. The support of the pivotal quantity $T_{n,m,a}$ for positive value of a is real line and for negative value of a is the positive part of the real line. The distribution function $T_{n,m,a}$ is determined based on the following theorem.

Theorem 2.1. If $X_1,\ldots,X_n$ is a random sample of exponential distribution $Exp(\mu,\sigma)$, so that $X_{(1)}<\cdots<X_{(m)}$ the observed values derived from type II progressive censoring with values $R_1,\ldots,R_m$ are preset. Then the distribution function of the random variable $T_{n,m,a}$ is obtained in the following from. For positive values of a,

$$P\left(T_{n,\ m,a}\ \leq t\right)=\begin{cases}\overline{\mathrm{F}}_{2m-2}\left(-\frac{2ma}{t}\right)-\mathrm{e}^{-na}\left(\frac{m}{m+nt}\right)^{m-1}\overline{\mathrm{F}}_{2m-2}\left(-2a\frac{m+nt}{m}\right); & t<0\\ 1-\mathrm{e}^{-na}\left(\frac{m}{m+nt}\right)^{m-1}; & t\geq 0\end{cases} \tag{2.2}$$

where $\overline{\mathrm{F}}_r\,(.)=1-\mathrm{F}_r\,(.)$ and F_r is the chi-square distribution function with r degrees of freedom. Also, for negative values of a, we have

$$P\left(T_{n,\ m,a}\ \leq t\right)=\begin{cases}0; & t<0\\ \overline{\mathrm{F}}_{2m-2}\left(-\frac{2ma}{t}\right)-\mathrm{e}^{-na}\left(\frac{m}{m+nt}\right)^{m-1}\overline{\mathrm{F}}_{2m-2}\left(-2a\frac{m+nt}{m}\right); & t\geq 0\end{cases} \tag{2.3}$$

Proof . Included in the appendix. □

If we let $a\ =-\ln\left(1-p\right)$, the distribution function [4] is obtained. Now, with the inverse of the distribution function, we can easily obtain the percentile of the probability distribution of the pivotal quantity $T_{n,m,a}$. By defining $t_{n,m,a;\beta}$ as the percentile of βth this pivotal quantity, a general confidence interval of 100 $(1-\alpha)$ % for the linear combination $\theta_a=\mu{+}a\sigma$ is obtained as follows

$$\left(X_{(1)}-\hat{\sigma}t_{n,m,a;1-\frac{\alpha}{2}}\ ,\ \ X_{(1)}-\hat{\sigma}t_{n,m,a;\alpha/2}\right). \tag{2.4}$$

In order to establish the shortest confidence interval at 100 $(1-\alpha)$%, it is only necessary to choose the value of α^* in the interval $(0,\ \alpha)$ such that $L=t_{n,m,a;1-\alpha+\alpha^*}-t_{n,m,a;\alpha^*}$ accepts its minimum value. Then, the shortest confidence interval for θ_a equals with

$$\left(X_{(1)}-\hat{\sigma}t_{n,m,a;1-\alpha+\alpha^*},\ \ X_{(1)}-\hat{\sigma}t_{n,m,a;\alpha^*}\right). \tag{2.5}$$

3 Confidence Interval of Parameters Difference in Two Populations

Assume that $X_{i1},...,X_{in_i}$ is a random sample of exponential distribution $Exp(\mu_i,\sigma_i)$, so that based on the preset values of m_i and $R_{i1},...,R_{im_i}$, the observed values of these samples are based on the progressive type-II censoring for $i=1,2$. In order to establish the confidence interval for the difference in parameters, i.e., $\theta_{1a}-\theta_{2a}$, we use the method of variance estimates recovery and the generalized pivotal quantity. for this purpose, we do not have any presuppositions on the scale parameters such as their equality or their definiteness. Necessary values of the percentile for the generalized pivotal quantity require the use of numerical methods, such as the Monte Carlo, in order to facilitate this, we also propose a solution based on the method modified based-normal asymptotic of [13]. In both methods, the confidence interval with shorter lengths will also be considered.

3.1 The method of variance estimates recovery

This method was introduced by [27] to establish an approximate confidence interval for the linear composition of the parameters based on their single confidence intervals. This method is used by writers like [15, 18, 22, 28] and etc. In order to get familiar with this method, assume that $\theta'=(\theta_1,\ldots,\theta_k)$ is unknown parameters and their unbiased estimation vector $\hat{\theta}=(\hat{\theta}_1\ \ldots\hat{\theta}_k)'$, so that the confidence interval$(l_i.\ u_i)$ is 100 $(1-\alpha)$ % of the parameter θ_i

for $i = 1, \ldots, k$. Then, the confidence interval of 100 $(1-\alpha)$ % for the linear combination $\delta = \sum_{i=1}^{k} c_i\theta_i$ will be based on the method of variance estimates recovery into the form $(L,\ U)$, in which c_i are the values and we have

$$\begin{cases} L = \sum_{i=1}^{k} c_i\hat{\theta}_{\mathrm{i}} - \sqrt{\sum_{i=1}^{k} c_i^2\left(\hat{\theta}_{\mathrm{i}} - l_i^*\right)^2} \\ U = \sum_{i=1}^{k} c_i\hat{\theta}_{\mathrm{i}} + \sqrt{\sum_{i=1}^{k} c_i^2\left(\hat{\theta}_{\mathrm{i}} - u_i^*\right)^2} \end{cases}$$

where for $c_i \geq 0$, $l_i^* = l_i$ and $u_i^* = u_i$ and for $c_i < 0$, $l_i^* = u_i$ and $u_i^* = l_i$ are defined. For more details, see [27].

We consider the general confidence interval obtained from relation (2.4) in the form $(l_i,\ u_i)$ for the parameter $\theta_{ia} = \mu_i + a\sigma_i$. Using the method of variance estimates recovery, the general confidence interval of 100 $(1-\alpha)$ % for $\theta_{1a} - \theta_{2a}$will be given to form$(L,\ U)$, where

$$\begin{cases} L = \hat{\theta}_{1a} - \hat{\theta}_{2a} - \sqrt{\left(\hat{\theta}_{1a} - X_{1(1)} + \hat{\sigma}_1 t_{n_1,m_1,a;1-\alpha/2}\right)^2 + \left(\hat{\theta}_{2a} - X_{2(1)} + \hat{\sigma}_2 t_{n_2,m_2,a;\alpha/2}\right)^2} \\ U = \hat{\theta}_{1a} - \hat{\theta}_{2a} + \sqrt{\left(\hat{\theta}_{1a} - X_{1(1)} + \hat{\sigma}_1 t_{n_1,m_1,a;\alpha/2}\right)^2 + \left(\hat{\theta}_{2a} - X_{2(1)} + \hat{\sigma}_2 t_{n_2,m_2,a;1-\alpha/2}\right)^2} \end{cases}$$

where $\hat{\theta}_{ia} = X_{i(1)} + \hat{\sigma}_i\left(a - \frac{1}{n_i}\right)$. We will simplify it

$$\begin{cases} L = \hat{\theta}_{1a} - \hat{\theta}_{2a} - \sqrt{\hat{\sigma}_1^2\left(a_1' + t_{n_1,m_1,a;1-\alpha/2}\right)^2 + \hat{\sigma}_2^2\left(a_2' + t_{n_2,m_2,a;\alpha/2}\right)^2} \\ U = \hat{\theta}_{1a} - \hat{\theta}_{2a} + \sqrt{\hat{\sigma}_1^2\left(a_1' + t_{n_1,m_1,a;\alpha/2}\right)^2 + \hat{\sigma}_2^2\left(a_2' + t_{n_2,m_2,a;1-\alpha/2}\right)^2} \end{cases} . \tag{3.1}$$

where $a_{\mathrm{i}}' = a - \frac{1}{n_i}$.

Remark 3.1. If we put the values obtained from α_i^* (2.5) in the limits provided in (3.1), we have a confidence interval of 100 $(1-\alpha)$% to another form $(L^*,\ U^*)$ in which

$$\begin{cases} L^* = \hat{\theta}_{1a} - \hat{\theta}_{2a} - \sqrt{\hat{\sigma}_1^2\left(a_1' + t_{n_1,m_1,a;1-\alpha+\alpha_1^*}\right)^2 + \hat{\sigma}_2^2\left(a_2' + t_{n_2,m_2,a;\alpha_2^*}\right)^2} \\ U^* = \hat{\theta}_{1a} - \hat{\theta}_{2a} + \sqrt{\hat{\sigma}_1^2\left(a_1' + t_{n_1,m_1,a;\alpha_1^*}\right)^2 + \hat{\sigma}_2^2\left(a_2' + t_{n_2,m_2,a;1-\alpha+\alpha_2^*}\right)^2} \end{cases} . \tag{3.2}$$

3.2 Generalized pivotal quantity

This method was first presented by [26]. Due to the widely use of this method in numerous papers, here we also present the generalized pivotal quantity and then, using the method modified based-normal asymptotic by [13], we introduce a solution to approximate its percentiles. We also provide a shorter confidence interval. First, we assume that $x_{i(1)}$ and $\hat{\sigma}_{i0}$ are the values observed for the estimators $X_{i(1)}$ and $\hat{\sigma}_i$ based on the definitions of [24], then the generalized pivotal quantity (GPQ) for the two parameters μ_i and σ_i are respectively

$$R_{\mu_i} = x_{i(1)} - \frac{m_i}{n_i}\frac{V_i}{U_i}\hat{\sigma}_{i0}, \qquad R_{\sigma_i} = \frac{2m_i\hat{\sigma}_{i0}}{U_i},$$

where $V_i \sim \chi_2^2$ and $U_i \sim \chi_{2m_i-2}^2$ are independent random variables. Also, the generalized pivotal quantity for $\theta_{ia} = \mu_i + a\sigma_i$ is equal to

$$R_{\theta_{ia}} = R_{\mu_i} + aR_{\sigma_i} = x_{i(1)} - \frac{m_i}{n_i}\hat{\sigma}_{i0}\left(\frac{V_i - 2n_i a}{U_i}\right).$$

For $i = 1, 2$. Based on the above process, the generalized pivotal quantity related to the difference in parameters $\theta_{1a} - \theta_{2a}$ is as follows

$$R_{12.a} = R_{\theta_{1a}} - R_{\theta_{2a}} = x_{1(1)} - x_{2(1)} + \frac{m_2}{n_2}\hat{\sigma}_{20}\left(\frac{V_2 - 2n_2 a}{U_2}\right) - \frac{m_1}{n_1}\hat{\sigma}_{10}\left(\frac{V_1 - 2n_1 a}{U_1}\right). \tag{3.3}$$

To find the distribution of $R_{12.a}$, we must generate the random variables V_i and U_i and calculate the value of (3.3) based on them. By repeating this procedure in many times (M order), the distribution of $R_{12.a}$ is obtained approximately as well as an approximate value of its βth percentile, namely, $R_{12.a;\beta}$. Then, the generalized confidence interval of 100 $(1-\alpha)$% for $\theta_{1a} - \theta_{2a}$ based on the generalized pivotal quantity will result as follows

$$\left(R_{12.a;\frac{\alpha}{2}},\ R_{12.a;\ 1-\frac{\alpha}{2}}\right). \tag{3.4}$$

3.3 Modified based-normal approximation

As mentioned in the previous paper, to obtain (3.4), it is necessary to use numerical methods. Here we use the modified based-normal asymptotic method, the form of the package for the generalized pivotal number percentile approximation (3.3). This method is proposed by [13] to approximate the percentiles of the linear distribution of random variables, which we will briefly describe in the following from.

Assume $Y_1 \ldots . Y_k$ are continuous random variables which do not necessarily have the same distribution. Assuming $Y_{i;\alpha}$, the percentile αth is the random variable Y_i for $i = 1. \ldots . k$. We consider $Q = \sum_{i=1}^{k} w_i Y_i$ in which w_i constant and invariant coefficients are. In this case, the percentile approximation is the random variable Q in the form below

$$Q_\alpha \simeq \begin{cases} \sum_{i=1}^{k} w_i E(Y_i) - \sqrt{\sum_{i=1}^{k} w_i^2 (E(Y_i) - Y_i^*)^2}; & 0 < \alpha < .5 \\ \sum_{i=1}^{k} w_i E(Y_i) + \sqrt{\sum_{i=1}^{k} w_i^2 (E(Y_i) - Y_i^*)^2}; & .5 < \alpha < 1 \end{cases}.$$

where $Y_i^* = Y_{i;\alpha}$ if $w_i > 0$ and $Y_i^* = Y_{i;1-\alpha}$ if $w_i < 0$. He introduced this approximation as the modified base-normal approximation (MNA). For more details, see [13].

Here, with care, in (3.3) we can formulate the generalized pivotal quantity displayed as

$$R_{12.a} = x_{1(1)} - x_{2(1)} + [\hat{\sigma}_2 T_{n_2,m_2,a} - \hat{\sigma}_1 T_{n_1,m_1,a}].$$

This means that the pivotal quantity (3.3) is a linear functional of $T_{12.a} = \hat{\sigma}_2 T_{n_2,m_2,a} - \hat{\sigma}_1 T_{n_1,m_1,a}$. Using the modified base-normal approximation method, the percentile approximation of α is the distribution of $T_{12.a}$, i.e. $t_{12.a;\alpha}$, as follows.

$$t_{12.a;\alpha} \simeq \begin{cases} \hat{\sigma}_2 r_{2.a} - \hat{\sigma}_1 r_{1.a} - \left[\hat{\sigma}_2^2 (r_{2.a} - t_{n_2.m_2.a;1-\alpha})^2 + \hat{\sigma}_1^2 (r_{1.a} - t_{n_1.m_1.a;\alpha})^2\right]^{\frac{1}{2}}; & 0 < \alpha \le .5 \\ \hat{\sigma}_2 r_{2.a} - \hat{\sigma}_1 r_{1.a} + \left[\hat{\sigma}_2^2 (r_{2.a} - t_{n_2.m_2.a;\alpha})^2 + \hat{\sigma}_1^2 (r_{1.a} - t_{n_1.m_1.a;1-\alpha})^2\right]^{\frac{1}{2}}; & .5 < \alpha < 1 \end{cases}, \tag{3.5}$$

where $r_{i.a} = \frac{m_i}{n_i} \frac{1-n_i a}{m_i - 2}$. Therefore, the approximation of the generalized pivotal quantity percentile $R_{12.a}$ can be written in the following from

$$R_{12.a;\beta} \simeq x_{1(1)} - x_{2(1)} + t_{12.a;\beta}.$$

And the general confidence interval at the level 100 $(1-\alpha)$% for $\theta_{1a} - \theta_{2a}$will be equal to

$$\left(x_{1(1)} - x_{2(1)} + t_{12.a,\frac{\alpha}{2}},\ \ x_{1(1)} - x_{2(1)} + t_{12.a;1-\alpha/2}\right). \tag{3.6}$$

Remark 3.2. If in the calculation of (3.5), instead of the usual values of α^* for $t_{n_i,m_i,a;\alpha}$ and $t_{n_i,m_i,a;1-\alpha}$ correspond to the shortest confidence interval $t_{n_i,m_i,a;\alpha^*}$ and $t_{n_i,m_i,a;1-\alpha+\alpha^*}$ for $i = 1, 2$ another approximation will be obtained for $t_{12.a;\alpha}$we denote it by $t^*_{12.a;\alpha}$. Putting the values of $t^*_{12.a;\alpha/2}$ and $t^*_{12.a;1-\alpha/2}$in (3.6), the confidence interval can be written as;

$$\left(x_{1(1)} - x_{2(1)} + t^*_{12.a;\frac{\alpha}{2}},\ x_{1(1)} - x_{2(1)} + t^*_{12.a;1-\alpha/2}\right). \tag{3.7}$$

Remark 3.3. It is important to note that [16] did not use exact percentile distributions for the normalized base-approximation $R_{12.a}$, but from the corrected base-normal approximation of the percentile in a state Society benefited.

Remark 3.4. In order to carry out the test, the hypothesis $\begin{cases} H_0 : \ \theta_{1a} = \theta_{2a} \\ H_1 : \theta_{1a} \neq \theta_{2a} \end{cases}$ can be assumed at the level of α. It is possible to assume zero when the value of zero at the confidence interval created at the level 100 $(1-\alpha)$ % does not.

4 Simulation

Using the Monte Carlo simulation, we study the coverage probability (CP) and the average length (AL) of confidence intervals assurance relationships (3.1), (3.2), (3.4), (3.6) and (3.7). We present the methods described above with MOV, SMOV, GCI, MNA, and SMNA. It should be noted that the generalized method requires the use of numerical methods in estimating its percentiles, while the remaining methods are approximations that are easily obtained. We simulated the following steps

Table 1: Simulation scenarios.

n_1, n_2	m_1, m_2	σ_1	Scenarios
10, 10	5, 5	0.01, 1, 5, 10	1-4
	5, 8	0.01, 1, 5, 10	5-8
	10, 10	0.01, 1, 5, 10	9-12
10, 15	5, 5	0.01, 1, 5, 10	13-16
	5, 8	0.01, 1, 5, 10	17-20
	8, 10	0.01, 1, 5, 10	21-24
	10, 15	0.01, 1, 5, 10	25-28
15, 10	5, 5	0.01, 1, 5, 10	29-32
	5, 8	0.01, 1, 5, 10	33-36
	15, 10	0.01, 1, 5, 10	37-40
20, 20	5, 5	0.01, 1, 5, 10	41-44
	5, 8	0.01, 1, 5, 10	45-48
	20, 20	0.01, 1, 5, 10	49-52
30, 30	5, 5	0.01, 1, 5, 10	53-56
	5, 8	0.01, 1, 5, 10	57-60
	20, 20	0.01, 1, 5, 10	61-64

1. Based on the values of θ_{ia}, σ_i, n_i and m_i, we obtain the sample $X_{i1}.\dots.X_{in_i}$ and obtain the observations $X_{i(1)} < \cdots < X_{i(m_i)}$ in terms of the values of $R_1.\dots.R_{m_i}$, and also we calculate the value of $\hat{\sigma}_{i0}$.
2. We obtain the required methods of assuming $R_{12.a;\beta}$ (taking into account the value of 10,000 for M), $t_{12.a;\beta}$ and $t^*_{12.a;\beta}$ to establish the confidence intervals (3.1), (3.2), (3.4), (3.6) and (3.7).
3. Step I and II are repeated 10,000 times. The ratio of the number of times that the difference $\theta_{1a} - \theta_{2a}$ is within these intervals is considered as an estimate for the CP and the average length of the confidence interval generated by all the cases as an estimate for the AL of confidence intervals.

In order to simulate different values of a such as 0,1 ,$-\ln(1-0.05)$ and $-\ln(1-0.95)$, and the number of different samples and distinct quantities of scale parameters, we use it. The values used in this simulation study are summarized in Table 1 under the scenario. In all simulations, the values of θ_{1a} and θ_{2a} were considered equal and the confidence intervals were constructed at 95% confidence level. Figures 1, 2, 3 and 4 respectively, show the simulation results for the scenarios in table 1 for the five available methods and the values of a equal to 0, 1, $-\ln(1-0.05)$ and $-\ln(1-0.95)$ can be made. It is referred a method as a reliable method when CP is less than 0.95. We consider the following reasonable criterion for comparing the methods: Firstly, a method is preferred to the other methods when is not liberal. Secondly, the candidate for the best method must have the smallest AL among reliable methods; see [18]. It should be kept in mind that the likelihood of coatings rarely varies from simulation to simulation. In fact, the estimated coverage probability is an almost normal distribution with an average of 0.95 and a variance of $0.95 \times 0.05/10000$, so a low limit for them is 0.946. This means that if the coverage probability of a confidence interval is less than 0.946, then this confidence interval is liberal. According to the simulation results, the results of the continuation can be expressed.

Figure 1 is related to the confidence interval of the difference location parameters of the two-exponential population, i.e., $\mu_1 - \mu_2$. Based on the simulation results of the 5 methods, we can say that all methods can have a coverage of probability significantly less than 0.95, that is, in $a = 0$, the presented methods are liberal. However, according to the scenarios considered in this simulation, the MOV method is more reliable than the rest of the methods due to the minimum probability value of 0.9430.

For $a = 1$, i.e., the difference in mean distributions, the simulation results show that the SMOV method is the only method which is liberal (Figure 2). Looking at the average length results, it can be said SMNA method has a lower AL, so that this method can be introduced as the shortest confidence interval for this difference parameters. Therefore, we recommend using method SMNA to construct a confidence interval for the difference mean between two-parameter exponential distributions.

In Figures 3 and 4, the simulation results are presented in terms of differences of percentiles 0.05 and 0.95. In examining the coverage probability of reported in these figures, it can be said that SMNA and SMOV methods are liberal methods, while GCI, MNA and MOV methods have the acceptable concluded to coverage probability. Also, the average length confidence intervals created by them are not significantly different. Because of the simplicity of calculations associated with MNA and MOV methods and the impossibility to observe a significant difference between the simulation results obtained from them, it is recommended to use these methods in constructing the confidence interval for the difference between quantiles of two-parameter exponential distributions.

As expected in our simulation results, the length confidence intervals of both SMNA and SMOV methods are lower

Figure 1: Estimation of the CP and the AL provided for $a = 0$.

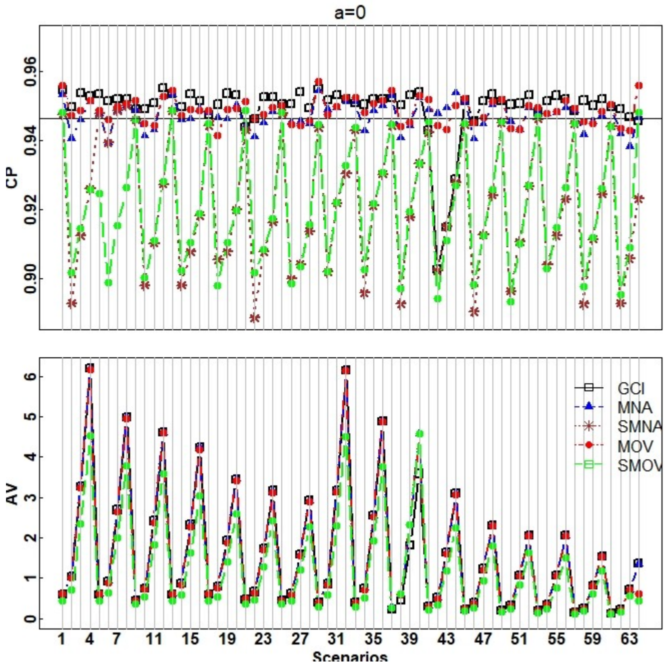


Figure 2: Estimation of the CP and the AL provided for $a = 1$.

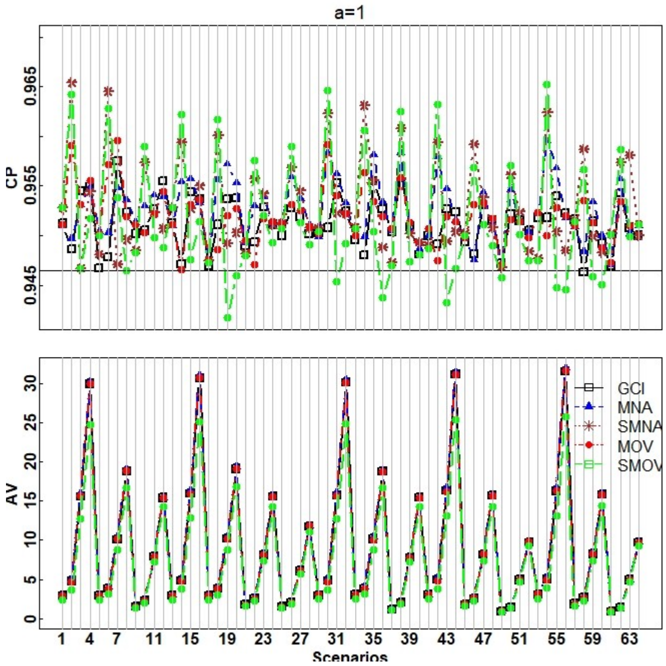


Table 2: Observations of the distance traveled by the military carrier.

162	200	271	302	393	508	539	629	706	777
884	1008	1101	1182	1463	1603	1984	2355	2880	

than the other three methods, even for the states which are not liberal. However, these methods do not have the possibility of coverage probability in all cases. Therefore, according to the results of this simulation study, SMNA method seems that only effective in establishing the confidence interval in the case of the difference between means of the two-parameter exponential distributions.

Figure 3: Estimation of the CP and the AL provided for $a = -\ln(1-0.05)$.

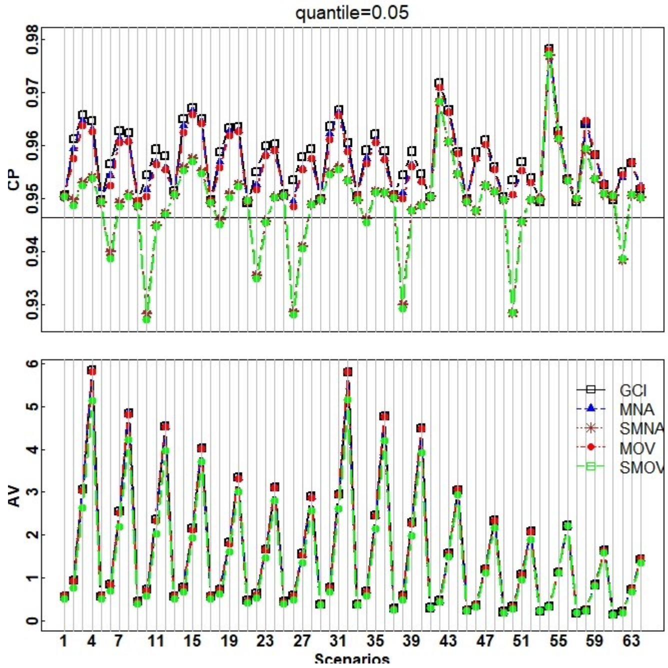
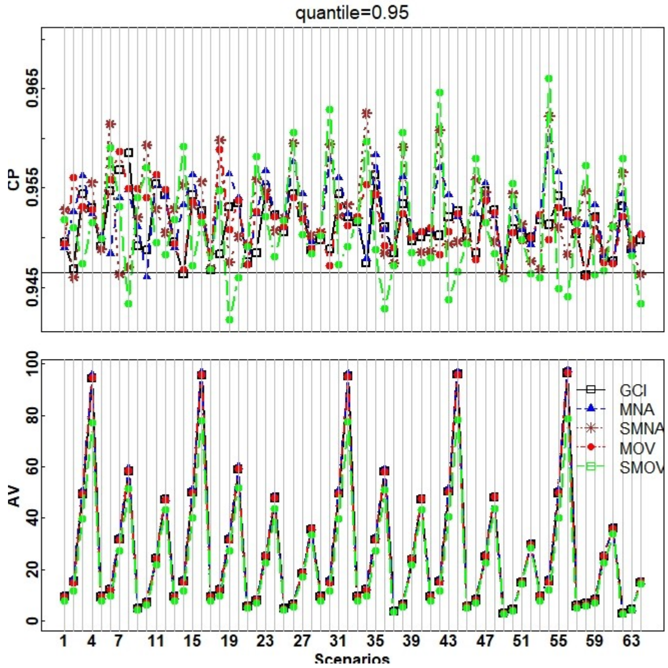


Figure 4: Estimation of the CP and the AL provided for $a = -\ln(1-0.95)$.



5 Real Data

In this section, we will expose the performance of the detailed confidentiality of the article with two examples that are commonly used in the two-parameter exponential distribution. The first example is the data for a population mode, and in example 2, we use the admission time data to simulate the two-parameter exponential populations.

Example 5.1. Here we use data reported by [7] regarding the distance traveled until the failure of 19 military carriers (Table 2). These data have been used by researchers such as [5, 6, 21, 24] as an example of two-parameter exponential distribution data (Anderson-Darling p-value is reported as 0.196).

The data are of a complete type and, by defining different values of m, and $R_1, ..., R_m$, we construct several progressive

Table 3: The precise general and shortest confidence interval for fault data.

m	$R_1, ..., R_m$	Type of confidence interval	location parameter Lower bound	location parameter Upper bound	mean Lower bound	mean Upper bound	5th percentile Lower bound	5th percentile Upper bound	95th percentile Lower bound	95th percentile Upper bound
19	0 . . . 0	general	-28.02	160.83	702.62	1582.93	26.11	217.95	1875.04	4544.54
		shortest	10.66	162.00	658.92	1501.27	55.07	228.93	1736.34	4281.92
8	2,4,0,1,2,2,0.0	general	-334.34	159.52	1051.89	4961.87	-171.98	325.56	2960.12	15016.95
		shortest	-216.40	162.00	855.37	4272.43	-112.61	357.82	2370.67	12896.24
8	1,0,2,2,1,4,0,1	general	-324.65	159.46	1115.72	4668.17	-170.07	319.53	3163.23	14102.25
		shortest	-212.47	162.00	934.97	4097.84	-109.22	350.49	2593.84	12323.67
7	2,0,1,0,2,4,3	general	-363.20	159.39	1104.07	5242.19	-191.71	335.10	3124.04	15884.32
		shortest	-238.48	162.00	900.51	4517.41	-129.12	368.89	2489.00	13630.65
7	0,3,2,2,1,0,4	general	-376.37	159.32	1126.84	5366.65	-200.61	339.33	3195.49	16271.00
		shortest	-248.27	162.00	914.87	4621.32	-136.49	373.57	2546.45	13963.33
5	2,4,2,3,3	general	-754.29	158.16	1389.27	10175.33	-424.77	477.72	4013.54	31196.40
		shortest	-512.36	162.20	1021.34	8234.94	-332.84	533.73	2859.16	25158.13
5	0,4,4,3,3	general	-733.45	158.25	1361.07	9946.05	-412.21	470.93	3925.04	30482.14
		shortest	-496.51	162.00	992.10	8044.44	-322.23	525.77	2809.75	24607.91
5	4,2,3,3,1	general	-683.93	158.45	1295.93	9408.80	-380.33	453.71	3719.83	28814.42
		shortest	-460.17	162.00	947.66	7612.30	-296.26	504.31	2668.24	23266.77

type-II censoring using the observations of Table 2. The distance between general confidence and the shortest precision obtained from these values, and for $0, 1, -\ln(1-0.05)$ and $-\ln(1-0.95)$ are given in Table 3.
The results of Table 3 clearly show the significant difference between the lengths of the confidence interval generated in the shortest state with the general in all states. Noteworthy, the value of 162 is the upper bound for the shortest confidence interval for the location parameter in all states and shows the efficiency of this type of confidence interval in comparison with the general method. Based on the complete sample, we can conclude the all parameters are positive, while with the second-generation censorship of data some results are disturbed. These results are far from expected and show the importance of having complete data.

Example 5.2. In this example, we use data on the survival time of people with lung cancer that cannot be surgically treated. The data of this example are a part of a larger data set given by Hill et al. (1988) and collected by the Veterans Administrative Lung Cancer Study Group in the USA. These data refer to four types of cancer, called Squamous, Small, Adeno and Large. Follow the [10, 11, 12, 16], we only use observations of two types of cancers, including 9 observations, as in Table 4.

Table 4: Observations on the durability of two types of cancers.

Squamous type	10	81	110	100	42	8	25	11	72
Small type	30	13	23	16	21	18	20	27	31

The results of the confidence intervals related to the difference in location parameters ($a = 0$), the mean ($a = 1$), the 5th percentile and the 95th percentile of the first society of the second society are presented in Table 5. The results include full data and incremental censorship data when $m = 5$. Based on the results of Table 5 for the complete

Table 5: Confidence interval for the cancer data.

m_1, m_2	$R_{11}, ..., R_{1m_1}$ / $R_{21}, ..., R_{2m_2}$	parameter	GCI Lower bound	GCI Upper bound	MNA Lower bound	MNA Upper bound	SMNA Lower bound	SMNA Upper bound	MOV Lower bound	MOV Upper bound
9,9	0 . . . 0	Location	-28.27	-1.99	-28.90	-2.62	-23.26	-3.07	-29.05	-2.53
		Mean	4.46	86.23	4.01	86.98	0.79	76.60	4.12	88.14
	0 . . . 0	5th percentile	-25.28	0.26	-25.69	-0.05	-21.21	0.83	-25.77	-0.03
		95th percentile	33.12	285.85	31.02	286.62	21.41	250.05	31.09	290.40
5,5	0,1,1,0,2	Location	-56.55	2.08	-56.42	0.63	-42.19	-0.88	-57.03	1.24
		Mean	-7.59	245.80	-6.03	239.71	-10.91	191.11	-8.94	244.57
	1,2,0,1,0	5th percentile	-47.87	6.74	-47.90	5.77	-38.30	8.04	-48.23	5.89
		95th percentile	-4.03	801.56	-0.80	779.10	-13.23	616.54	-11.02	794.94
	1,0,1,1,1	Location	-58.56	1.42	-61.47	0.16	-45.71	-1.14	-62.08	0.69
		Mean	-3.07	263.88	-1.54	263.92	-8.11	209.85	-3.94	268.70
	2,0,0,1,1	5th percentile	-50.22	7.21	-52.06	6.01	-41.49	8.47	-52.37	6.10
		95th percentile	11.62	855.50	13.95	854.93	-3.71	677.61	5.45	870.45

sample, we can say that the average and 95th percentile of the time of hospitalization of the Squamous type are more than the Small type, while the 5th percentile parameter of these distributions are reversely different. When we use the second-generation censorship of constructing this confidence interval, the results show that in each of the four parameters, these two types of cancer are the same. Perhaps the reason for this could be to reduce the information obtained from censoring data towards the full sample.

Appendix: **Proof of Theorem In order to find the distribution function $\mathbf{T_{n,m,a}}$**

In order to obtain the distribution function $T_{n,m,a}$, we have

$$\begin{aligned}F_{n,\mathrm{m},a}\left(t\right)&=P\left(T_{n,\mathrm{m},a}\leq t\right)=P\left(\tfrac{m}{n}\tfrac{\mathrm{V}-2na}{U}\leq t\right)=P\left(\mathrm{V}-\tfrac{n}{m}tU\leq 2na\right)\\&=\int_0^\infty f_U\left(u\right)F_V\left(2na+\tfrac{n}{m}tu\right)du=\int_0^\infty f_U\left(u\right)\left(1-\exp\left(-\left(na+\tfrac{n}{2m}tu\right)\right)\right)du.\end{aligned}$$

where $U\sim\chi^2_{2m-2}$ and $V\sim\chi^2_2$ are independent and F_V is probability distribution function of V. It should be noted that when $a>0$, its support is $(-\infty,\ \infty)$. For $t<0$, the bound of u should be chosen such that $2na+\frac{ntu}{m}>0$, i.e. $u<-\frac{2ma}{t}$, so

$$\begin{aligned}F_{n,\mathrm{m},a}\left(t\right)&=1-F_U\left(-\tfrac{2ma}{t}\right)-e^{-na}\int_{-\frac{2ma}{t}}^\infty f_U\left(u\right)\exp\left(-\tfrac{n}{2m}tu\right)du\\&=1-F_U\left(-\tfrac{2ma}{t}\right)-e^{-na}\left(\tfrac{m}{m+nt}\right)^{m-1}\int_{-\frac{2a}{t}(nt+m)}^\infty f_U\left(u\right)du\\&=1-F_U\left(-\tfrac{2ma}{t}\right)-\mathrm{e}^{-na}\left(\tfrac{m}{m+nt}\right)^{m-1}\left(1-F_U\left(-\tfrac{2a}{t}\left(nt+m\right)\right)\right)\end{aligned}$$

For $t>0$, the $2na+\frac{ntu}{m}$ is positive and

$$F_{n,\mathrm{m},a}\left(t\right)=1-e^{-na}\int_0^\infty f_U\left(u\right)\exp\left(-\frac{n}{2m}tu\right)du=1-\mathrm{e}^{-na}\left(\frac{m}{m+nt}\right)^{m-1}.$$

It is very clear that both random variables, U and V, take positive values. Therefore, for calculating $P\left(V\leq 2na+\frac{ntu}{m}\right)$. Since $2na+\ ntu/m$ may be lower than zero, we use $\max\left(0,\ 2na+\frac{ntu}{m}\right)$. To calculate $P\left(V-\frac{ntU}{m}\leq 2na\right)$. The integral regions are as follows:

$$\begin{cases}V<0 & if\quad 0<u<-\frac{2ma}{t}\\ V<2na+\frac{n}{m}tu & if\quad u>-\frac{2ma}{t}\end{cases}$$

To obtain the quotient distribution function we perform the following

$$P\left(V\leq\frac{n}{m}tU+2na\right)=\int_0^\infty\int_{max(\ 0.2na-\frac{n}{m}ty)}^\infty f_V(\mathrm{v})\ f_U\left(u\right)dvdu=\int_{-\frac{2ma}{t}}^\infty F_V\left(2na+\frac{n}{m}tu\right)f_U\left(u\right)du.$$

Given that V has an exponential distribution with an average of 2, when $-2ma\ <t\ <0$, we will have the density of the random variable U, we will have

$$P\left(V\leq\frac{n}{m}tU+2na\right)=1-\mathrm{F_{2m-2}}\left(-\frac{2ma}{t}\right)-\mathrm{e}^{-na}\int_{-\frac{2ma}{t}}^\infty\frac{u^{m-2}e^{-\frac{u}{2}\left(1+\frac{n}{m}t\right)}}{2^{m-1}\Gamma\left(m-1\right)}du.$$

Now with consideration

$$\left(1+\frac{n}{m}t\right)u=s\quad\rightarrow u=s\left(\frac{m}{m+nt}\right)$$

we have

$$\begin{aligned}P\left(V\leq\tfrac{n}{m}tU+2na\right)&=1-\mathrm{F_{2m-2}}\left(-\tfrac{2ma}{t}\right)-\mathrm{e}^{-na}\left(\tfrac{m}{m+nt}\right)^{m-1}\left(1-\mathrm{F_{2m-2}}\left(-2a\tfrac{m+nt}{m}\right)\right)\\&=e^{-\frac{ma}{t}}\sum_{j=m-1}^\infty\tfrac{1}{j!}\left(\tfrac{ma}{t}\right)^j-e^{-a\frac{m+n+nt}{m}}\left(\tfrac{m}{m+nt}\right)^{m-1}\sum_{j=m-1}^\infty\tfrac{1}{j!}\left(a\tfrac{m+nt}{m}\right)^j.\end{aligned}$$

If $t\geq 0$, then

$$P\left(V\leq\frac{n}{m}tU+2na\right)=1-\mathrm{e}^{-na}\left(\frac{m}{m+nt}\right)^{m-1}.$$

When $a\ <0$, the support $f_{n,m,a}$ contains positive values and the minimum value is $-2ma$, so for positive values t the distribution function will be in the form below.

$$\begin{aligned}F_{n.\mathrm{m}.a}\left(t\right)&=P\left(V\leq\tfrac{n}{m}tU+2na\right)=\int_{-\frac{2ma}{t}}^\infty F_V\left(2na+\tfrac{n}{m}tu\right)f_U\left(u\right)du\\&=1-F_U\left(-\tfrac{2ma}{t}\right)-\mathrm{e}^{-na}\left(\tfrac{m}{m+nt}\right)^{m-1}\left(1-F_U\left(-\tfrac{2a}{t}\left(nt+m\right)\right)\right),\end{aligned}$$

where

$$F_U(x) = \int_0^x f_U(x)\,dx = 1 - e^{-\frac{x}{2}} \sum_{j=0}^{m-2} \frac{1}{j!}\left(\frac{x}{2}\right)^j = e^{-\frac{x}{2}} \sum_{j=m-1}^{\infty} \frac{1}{j!}\left(\frac{x}{2}\right)^j, \quad x \geq 0.$$

This complete the proof.

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