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Hyperstability of bi-Cauchy-Jensen functional equations

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Abstract

In this paper, we prove some hyperstability results of the bi-Cauchy-Jensen functional equation: $2f\left(x+y, \frac{z+w}{2}\right) = f(x,z)+f(x,w)+f(y,z)+f(y,w)$ in Banach spaces by using fixed point method.

Keywords: Hyperstability, Bi-Cauchy-Jensen functional equation, Fixed point theorem
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1 Introduction and preliminaries

Throughout this paper, let $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$ be the set of natural numbers, the set of real numbers, the set of non-negative real numbers, respectively, and let $\mathbb{N}_{m_0} = \{n \in \mathbb{N} : n \geq m_0\}$.

The motivation for researching on stability theory of functional equations was initiated by Ulam in 1940. In [41], Ulam proposed some unsloved problems and one of them is stability problem of functional equation on the stability of group homomorphisms as follows:

"*Let $(G_1, \cdot)$ be a group and $(G_2, \star, d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given a real number $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality*

$$d(h(x\cdot y), h(x) \star h(y)) < \delta$$

for all $x, y \in G_1$, then there exists a homomorphism $g : G_1 \to G_2$ with $d(h(x), g(x)) < \varepsilon$ for all $x \in G_1$?".

Later, in 1941, Hyers [29] provided first an affirmative partial answer to Ulam's problem for the case of an approximately additive mapping in Banach spaces. In 1978, Rassias [39] presented a generalization of Hyers's theorem for a linear mapping by considering an unbounded Cauchy differences.

In 1994, Găvruţa [21] generalized Rassias's results by replacing the unbounded Cauchy difference with a general control function. For more results on Ulam's stabitiy and Găvruţa's results, refer to [14] and [15]. Further, recently, many mathematicians have extended and developed the stability of functional equations in many directions (see, for example, [16, 17, 18, 37]).

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One way to develop the stability of functional equations is the fixed point method, which is a kind of the approximation method, instead of the direct method.

Recently, since the first stability problem formulated by Ulam, some authors have considered some kinds of the stability, for example, stability, b-stability, hyperstability, orthogonal stability, inverse stability and some others (see, for example, [17, 35, 26, 27]). One of interesting types of the stability is the hyperstability. We say that a functional equation $\mathcal{D}$ is *hyperstable* if any functional f satisfying the equation $\mathcal{D}$ approximately is an actual solution of the equation $\mathcal{D}$.

In 1949, the first hyperstability result was published in [34] by concerning ring homomorphisms. However, the term hyperstability was used first by Bourgin in [8]. For more results on the hyperstability, refer to many interesting papers on the hyperstability (see [1, 2, 3, 4, 7, 11, 9, 12, 10, 16, 18, 20, 22, 23, 24, 25, 28, 34, 38, 40])

Let X and Y be vector spaces. A mapping $f : X \times X \to Y$ is called the *bi-Cauchy-Jensen functional equation* (bi-CJE, shortly) if f satisfies the system of the following equations:

$$f(x + y, z) = f(x, z) + f(y, z)$$

and

$$2f\left(x, \frac{y+z}{2}\right) = f(x, y) + f(x, z) \tag{1.1}$$

for all $x, y, z \in X$. In particular, For $X = Y = \mathbb{R}$, The solution of the system (1.1) is given by the function

$$f(x, y) = axy + bx,$$

where $x, y \in \mathbb{R}$ and a, b are constant. In fact, the mapping $f : X \times X \to Y$ satisfies

$$2f\left(x + y, \frac{z+w}{2}\right) = f(x, z) + f(x, w) + f(y, z) + f(y, w) \tag{1.2}$$

for all $x, y, z, w \in X$.

In 2006, Park and Bae [36] showed that the mapping $f : X \times X \to Y$ satisfies the system (1.1) if and only if it satisfies the equation (1.2) and gave the general solution of the equation (1.2) given by

$$f(x, y) = B(x, y) + A(x)$$

for all $x, y \in X$, where $B : X \times X \to Y$ is a bi-additive mapping and $A : X \to Y$ is an additive mapping. Moreover, they proved the stability of the functional equations (1.1) and (1.2) in the sense of Găvruţa by using the direct method. In 2012, Bae and Park [5] investigated the stability of the functional equation (1.2) by using the fixed point method which has derived from Diaz and Mogolis [19]. For more results on the stability of the functional equation (1.2), see [6, 30, 31, 33].

Especially, in 2017, Fassi et al. [20] presented some hyperstability results of the biadditive functional equation

$$f(x + y, z - w) + f(x - y, z + w) = 2f(x, z) - 2f(y, w)$$

on restricted domain by using the fixed point theorem of Brzdęk et al. [13] (Theorem 1) and obtained some inequalities characterizing bi-additive mappings and inner product spaces.

In this paper, we present some results on the hyperstability of the functional equation (1.2) by using the fixed point theorem in function spaces, which have been derived from Brzdęk et al. [13].

Before proving our main results, we state the fixed point theorem which is a useful tool for proving our main results.

Let A, B be nonempty sets. We denote the family of all mappings of B into A by A^B and use the following three conditions:

(H1) W is a nonempty set, $f_1, f_2, \cdots, f_k : W \to W$ and $L_1, L_2, \cdots, L_k : W \to \mathbb{R}_+$ are given mappings;

(H2) Y is a Banach space and $\mathcal{T} : Y^W \to Y^W$ is a mapping satisfying the inequality:

$$\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \le \sum_{i=1}^{k} L_i(x)\|\xi(f_i(x)) - \mu(f_i(x))\|$$

for all $\xi, \mu \in Y^W$ and $x \in W$;

(H3) $\Lambda : \mathbb{R}_+^W \to \mathbb{R}_+^W$ is a mapping defined by

$$\Lambda\delta(x) := \sum_{i=1}^{k} L_i(x)\delta(f_i(x))$$

for all $\delta \in \mathbb{R}_+^W$ and $x \in W$.

The following theorem was proved in complete metric spaces by Brzdęk et al. [13]:

Theorem 1.1. [13] Assume that the condition (H1)–(H3) are satisfied and the functions $\varepsilon : X \to \mathbb{R}_+$ and $\varphi : W \to Y$ fulfil the following two conditions:

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \le \varepsilon(x)$$

for all $x \in W$ and

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) < \infty$$

for all $x \in W$. Then there exists a unique fixed point $\psi \in Y^W$ of $\mathcal{T}$ such that

$$\|\varphi(x) - \psi(x)\| \le \varepsilon^*(x)$$

for all $x \in X$. Moreover, we have

$$\psi(x) := \lim_{n\to\infty} \mathcal{T}^n \varphi(x)$$

for all $x \in X$.

2 The Hyperstability Result I

Let X be a normed space, Y be a Banach space and let $X^* = X\backslash\{0\}$. First, we give some lemmas for our main results.

Lemma 2.1. Let $m, l \in \mathbb{N}$. Define a mapping $\mathcal{T} : Y^{X^*\times X^*} \to Y^{X^*\times X^*}$ by

$$\begin{aligned}\mathcal{T}\xi(x,y) =& \frac{1}{2}\xi(mx, 2ly) + \frac{1}{2}\xi(mx, (2-2l)y) + \frac{1}{2}\xi((1-m)x, 2ly)\\ &+ \frac{1}{2}\xi((1-m)x, (2-2l)y),\end{aligned} \tag{2.1}$$

and define a mapping $\Lambda : \mathbb{R}_+^{X^*\times X^*} \to \mathbb{R}_+^{X^*\times X^*}$ by

$$\begin{aligned}\Lambda\delta(x,y) =& \frac{1}{2}\delta(mx, 2ly) + \frac{1}{2}\delta(mx, (2-2l)y) + \frac{1}{2}\delta((1-m)x, 2ly)\\ &+ \frac{1}{2}\delta((1-m)x, (2-2l)y)\end{aligned} \tag{2.2}$$

for all $x, y \in X^*$, $\xi \in Y^{X^*}$ and $\delta \in \mathbb{R}_+^{X^*\times X^*}$. Then the conditions (H1)–(H3) hold for the mappings $\mathcal{T}$ and Λ.

Proof . For any $\xi, \mu \in Y^{X^*\times X^*}$ and $x, y \in X^*$, we obtain that

$$\begin{aligned}&\|\mathcal{T}\xi(x,y) - \mathcal{T}\mu(x,y)\|_Y\\ &= \Big\|\frac{1}{2}\xi(mx, 2ly) + \frac{1}{2}\xi(mx, (2-2l)y) + \frac{1}{2}\xi((1-m)x, 2ly)\\ &\quad + \frac{1}{2}\xi((1-m)x, (2-2l)y) - \Big(\frac{1}{2}\mu(mx, 2ly) + \frac{1}{2}\mu(mx, (2-2l)y)\\ &\quad + \frac{1}{2}\mu((1-m)x, 2ly) + \frac{1}{2}\mu((1-m)x, (2-2l)y)\Big)\Big\|_Y\end{aligned}$$

$$
\begin{aligned}
&= \Big\| \frac{1}{2}\xi(mx,2ly) - \frac{1}{2}\mu(mx,2ly) + \frac{1}{2}\xi(mx,(2-2l)y) \\
&\quad - \frac{1}{2}\mu(mx,(2-2l)y) + \frac{1}{2}\xi((1-m)x,2ly) - \frac{1}{2}\mu((1-m)x,2ly) \\
&\quad + \frac{1}{2}\xi((1-m)x,(2-2l)y) - \frac{1}{2}\mu((1-m)x,(2-2l)y) \Big\|_Y \\
&= \Big\| \frac{1}{2}(\xi-\mu)(mx,2ly) + \frac{1}{2}(\xi-\mu)(mx,(2-2l)y) \\
&\quad + \frac{1}{2}(\xi-\mu)((1-m)x,2ly) + \frac{1}{2}(\xi-\mu)((1-m)x,(2-2l)y) \Big\|_Y \\
&\le \frac{1}{2}\|(\xi-\mu)f_1(x,y)\|_Y + \frac{1}{2}\|(\xi-\mu)f_2(x,y)\|_Y \\
&\quad + \frac{1}{2}\|(\xi-\mu)f_3(x,y)\|_Y + \frac{1}{2}\|(\xi-\mu)f_4(x,y)\|_Y \\
&\le \sum_{i=1}^{4} L_i(x,y)\|(\xi-\mu)f_i(x,y)\|_Y,
\end{aligned}
$$

where

$$
(\xi-\mu)(x,y) = \xi(x,y) - \mu(x,y), \quad f_1(x,y) = (mx,2ly),
$$
$$
f_2(x,y) = (mx,(2-2l)y), \quad f_3(x,y) = ((1-m)x,2ly),
$$
$$
f_4(x,y) = ((1-m)x,(2-2l)y)
$$

and

$$
L_1(x,y) = L_2(x,y) = L_3(x,y) = L_4(x,y) = \frac{1}{2}
$$

for all $x, y \in X^*$. So, the condition $(H2)$ is valid for $\mathcal{T}$ with $k = 4$ and $W = X^* \times X^*$.

It easy to show that the mapping Λ has the form described in the condition $(H3)$ with $k = 4$ and $W = X^* \times X^*$ by above notation. This completes the proof. □

Lemma 2.2. If $f : X^* \times X^* \to Y$ satisfies

$$
\begin{aligned}
&\left\| 2f\left(x+y, \frac{z+w}{2}\right) - f(x,z) - f(x,w) - f(y,z) - f(y,w) \right\|_Y \\
&\le \theta \|x\|_X^p \|y\|_X^q \|z\|_X^r \|w\|_X^s
\end{aligned} \tag{2.3}
$$

for some $\theta, p, q, r, s \in \mathbb{R}$ with $\theta \ge 0$, then , for any $m, l \in \mathbb{N}\backslash\{1\}$,

$$
\|f(x,y) - \mathcal{T}_m f(x,y)\|_Y \le \varepsilon_m(x,y) \quad \text{and} \quad \Lambda_m^n \varepsilon_m(x,y) \le \eta_m^n \varepsilon_m(x,y)
$$

for all $x, y \in X^*$ and $n \in \mathbb{N} \cup \{0\}$ such that $\mathcal{T}_m, \Lambda_m$ satisfies (2.1), (2.2), respectively, where

$$
\varepsilon_m(x,y) = \frac{\theta}{2} m^p (m-1)^q (2l)^r (2l-2)^s \|x\|_X^{p+q} \|y\|_X^{r+s} \tag{2.4}
$$

and

$$
\begin{aligned}
\eta_m =& m^{p+q}(2l)^{r+s} + m^{p+q}(2l-2)^{r+s} + (m-1)^{p+q}(2l)^{r+s} \\
&+ (m-1)^{p+q}(2l-2)^{r+s}.
\end{aligned} \tag{2.5}
$$

Proof . Replacing $(x,y,z,w) = (mx,(1-m)x,2ly,(2-2l)y)$ where for any $m, l \in \mathbb{N}\backslash\{1\}$ in (2.3), we obtain

$$
\begin{aligned}
&\Big\| 2f\Big(mx + (1-m)x, \frac{2my + (2-2l)y}{2}\Big) \\
&\quad - f(mx,2ly) - f(mx,(2-2l)y) - f((1-m)x,2ly) - f((1-m)x,(2-2l)y) \Big\|_Y \\
&\le \theta \|mx\|_X^p \|(1-m)x\|_X^q \|2ly\|_X^r \|(2-2l)y\|_X^s
\end{aligned}
$$

and

$$\begin{aligned}&\Big\|f(x,y)-\frac{1}{2}f(mx,2ly)-\frac{1}{2}f(mx,(2-2l)y)\\&\quad-\frac{1}{2}f((1-m)x,2ly)-\frac{1}{2}f((1-m)x,(2-2l)y)\Big\|_Y\\&\leq\frac{\theta}{2}\|mx\|_X^p\|(1-m)x\|_X^q\|2ly\|_X^r\|(2-2l)y\|_X^s\\&=\frac{\theta}{2}m^p(m-1)^q(2l)^r(2l-2)^s\|x\|_X^{p+q}\|y\|_X^{r+s}\end{aligned}\tag{2.6}$$

for all $x,y\in X^*$. Define an operator $\mathcal{T}_m : Y^{X^*\times X^*}\to Y^{X^*\times X^*}$ by

$$\begin{aligned}\mathcal{T}_m\xi(x,y)=&\frac{1}{2}\xi(mx,2ly)+\frac{1}{2}\xi(mx,(2-2l)y)+\frac{1}{2}\xi((1-m)x,2ly)\\&+\frac{1}{2}\xi((1-m)x,(2-2l)y)\end{aligned}\tag{2.7}$$

for all $\xi\in Y^{X^*}$ and $x,y\in X^*$. It follows from (2.4), (2.6) and (2.7) that we have

$$\|f(x,y)-\mathcal{T}_mf(x,y)\|_Y\leq\varepsilon_m(x,y)$$

for all $x,y\in X^*$ and $m,l\in\mathbb{N}\backslash\{1\}$. Define an operator $\Lambda_m:\mathbb{R}_+^{X^*\times X^*}\to\mathbb{R}_+^{X^*\times X^*}$ by

$$\begin{aligned}\Lambda_m\delta(x,y)=&\frac{1}{2}\delta(mx,2ly)+\frac{1}{2}\delta(mx,(2-2l)y)+\frac{1}{2}\delta((1-m)x,2ly)\\&+\frac{1}{2}\delta((1-m)x,(2-2l)y)\end{aligned}\tag{2.8}$$

for all $x,y\in X^*$ and $\delta\in\mathbb{R}_+^{X^*\times X^*}$. Next, we will show that, for any $x,y\in X^*$ and $m,l\in\mathbb{N}\backslash\{1\}$,

$$\Lambda_m^n\varepsilon_m(x,y)\leq\eta_m^n\varepsilon_m(x,y)\tag{2.9}$$

for all $n\in\mathbb{N}\cup\{0\}$. It is clear that the inequality (2.9) holds for $n=0$. Next, assume that (2.9) holds for some $n=k\in\mathbb{N}$, that is,

$$\Lambda_m^k\varepsilon_m(x,y)\leq\eta_m^k\varepsilon_m(x,y).$$

Then it follows that

$$\begin{aligned}&\Lambda_m^{k+1}\varepsilon_m(x,y)\\&=\Lambda_m(\Lambda_m^k\varepsilon_m(x,y))\\&=\frac{1}{2}\Lambda_m^k\varepsilon_m(mx,2ly)+\frac{1}{2}\Lambda_m^k\varepsilon_m(mx,(2-2l)y)+\frac{1}{2}\Lambda_m^k\varepsilon_m((1-m)x,2ly)\\&\quad+\frac{1}{2}\Lambda_m^k\varepsilon_m((1-m)x,(2-2l)y)\\&\leq\frac{1}{2}\eta_m^k\varepsilon_m(mx,2ly)+\frac{1}{2}\eta_m^k\varepsilon_m(mx,(2-2l)y)+\frac{1}{2}\eta_m^k\varepsilon_m((1-m)x,2ly)\\&\quad+\frac{1}{2}\eta_m^k\varepsilon_m((1-m)x,(2-2l)y)\\&=\frac{1}{2}\eta_m^k\Big(\varepsilon_m(mx,2ly)+\varepsilon_m(mx,(2-2l)y)+\varepsilon_m((1-m)x,2ly)\\&\quad+\varepsilon_m((1-m)x,(2-2l)y)\Big)\\&=\frac{1}{2}\eta_m^k\Big(\frac{\theta}{2}m^p(m-1)^q(2l)^r(2l-2)^s\|mx\|_X^{p+q}\|2ly\|_X^{r+s}\\&\quad+\frac{\theta}{2}m^p(m-1)^q(2l)^r(2l-2)^s\|mx\|_X^{p+q}\|(2-2l)y\|_X^{r+s}\\&\quad+\frac{\theta}{2}m^p(m-1)^q(2l)^r(2l-2)^s\|(1-m)x\|_X^{p+q}\|2ly\|_X^{r+s}\end{aligned}$$

$$\begin{aligned}
&+\frac{\theta}{2}m^p(m-1)^q(2l)^r(2l-2)^s\|(1-m)x\|_X^{p+q}\|(2-2l)y\|_X^{r+s}\Big)\\
=&\frac{1}{2}\cdot\frac{\theta}{2}m^p(m-1)^q(2l)^r(2l-2)^s\eta_m^k(\|mx\|_X^{p+q}\|2ly\|_X^{r+s}\\
&+\|mx\|_X^{p+q}\|(2-2l)y\|_X^{r+s}+\|(1-m)x\|_X^{p+q}\|2ly\|_X^{r+s}\\
&+\|(1-m)x\|_X^{p+q}\|(2-2l)y\|_X^{r+s})\\
=&\frac{1}{2}\cdot\frac{\theta}{2}m^p(m-1)^q(2l)^r(2l-2)^s\eta_m^k(m^{p+q}\|x\|_X^{p+q}(2l)^{r+s}\|y\|_X^{r+s}\\
&+m^{p+q}\|x\|_X^{p+q}(2l-2)^{r+s}\|y\|_X^{r+s}+(m-1)^{p+q}\|x\|_X^{p+q}(2l)^{r+s}\|y\|_X^{r+s}\\
&+(m-1)^{p+q}\|x\|_X^{p+q}(2l-2)^{r+s}\|y\|_X^{r+s})\\
=&\frac{1}{2}\cdot\frac{\theta}{2}m^p(m-1)^q(2l)^r(2l-2)^s\|x\|_X^{p+q}\|y\|_X^{r+s}\eta_m^k(m^{p+q}(2l)^{r+s}\\
&+m^{p+q}(2l-2)^{r+s}+(m-1)^{p+q}(2l)^{r+s}+(m-1)^{p+q}(2l-2)^{r+s})\\
=&\frac{1}{2}\varepsilon_m(x)\eta_m^k\eta_m=\frac{1}{2}\eta_m^{k+1}\varepsilon_m(x)\le\eta_m^{k+1}\varepsilon_m(x)
\end{aligned}$$

for all $x\in X^*$. This implies that (2.9) holds for $n=k+1$, that is, (2.9) holds for all $n\in\mathbb{N}\cup\{0\}$. □

By using Lemma 2.1 and Lemma 2.2, we have the following hyperstability result:

Theorem 2.3. If $f:X^*\times X^*\to Y$ satisfies (2.3) such that $p+q<0$ or $r+s<0$, then f is a solution of the functional equation (1.2) on X^*.

Proof . Suppose that $p+q<0$. Replacing $(x,y,z,w)=(mx,(1-m)x,2ly,(2-2l)y)$ where $m,l\in\mathbb{N}$ with $m>2$ and a fixed number $l\in\mathbb{N}\backslash\{1\}$ in (2.3), By the similar step of the proof of Lemma 2.2, we have

$$\|f(x,y)-\mathcal{T}_mf(x,y)\|_Y\le\varepsilon_m(x,y)\quad\text{and}\quad\Lambda_m^n\varepsilon_m(x,y)\le\eta_m^n\varepsilon_m(x,y)$$

for all $x,y\in X^*$ and $n\in\mathbb{N}\cup\{0\}$ where $\mathcal{T}_m,\Lambda_m,\varepsilon_m$ and η_m are defined by (2.7), (2.8), (2.4) and (2.5), respectively. By Lemma 2.1, we obtain that conditions (H1)–(H3) hold for the mappings $\mathcal{T}_m$ and Λ_m. Since $m>2$ and $p+q<0$, we obtain $m^{p+q}<1$ and $(m-1)^{p+q}<1$, Indeed, we have

$$m-1>2-1=1\implies(m-1)^{p+q}<1.$$

Then we have $\lim_{m\to\infty}m^{p+q}=0$ and $\lim_{m\to\infty}(m-1)^{p+q}=0$. Therefore, it follows that

$$\begin{aligned}
\lim_{m\to\infty}\eta_m=\lim_{m\to\infty}&[m^{p+q}(2ly)^{r+s}+m^{p+q}(2l-2)^{r+s}+(m-1)^{p+q}(2l)^{r+s}\\
&+(m-1)^{p+q}(2l-2)^{r+s}]=0.
\end{aligned}$$

Then there exists $m_0\in\mathbb{N}_3$ such that

$$m^{p+q}(2ly)^{r+s}+m^{p+q}(2l-2)^{r+s}+(m-1)^{p+q}(2l)^{r+s}+(m-1)^{p+q}(2l-2)^{r+s}<1$$

for all $m\ge m_0$. It follows from (2.9) that

$$\varepsilon_m^*(x,y):=\sum_{n=0}^{\infty}\Lambda_m^n\varepsilon_m(x,y)\le\varepsilon_m(x,y)\sum_{n=0}^{\infty}\eta_m^n=\frac{\varepsilon_m(x,y)}{1-\eta_m}$$

for all $x,y\in X^*$ and $m\ge m_0$. It follows from Theorem 1.1 that, for each $m\ge m_0$, there exists a unique solution $F_m:X^*\times X^*\to Y$ of the following equation:

$$\begin{aligned}
F_m(x,y)=&\frac{1}{2}F_m(mx,2ly)+\frac{1}{2}F_m(mx,(2-2l)y)+\frac{1}{2}F_m((1-m)x,2ly)\\
&+\frac{1}{2}F_m((1-m)x,(2-2l)y)
\end{aligned}$$

for all $x,y\in X^*$ such that

$$\|f(x,y)-F_m(x,y)\|_Y\le\frac{\varepsilon_m(x,y)}{1-\eta_m}$$

for all $x, y \in X^*$ and, moreover,

$$F_m(x,y) = \lim_{n\to\infty} \mathcal{T}_m^n f(x,y)$$

for all $x, y \in X^*$.

Now, we will show that F_m satisfies the equation (1.2) for all $x, y \in X^*$ and $m \geq m_0$. First, we show that, for each $m \geq m_0$,

$$\begin{aligned}&\Big\|2\mathcal{T}_m^n f\Big(x+y, \frac{z+w}{2}\Big) - \mathcal{T}_m^n f(x,z) - \mathcal{T}_m^n f(x,w) - \mathcal{T}_m^n f(y,z) - \mathcal{T}_m^n f(y,w)\Big\|_Y \\ &\leq \eta_m^n \theta \|x\|_X^p \|y\|_X^q \|z\|_X^r \|w\|_X^s \end{aligned} \tag{2.10}$$

for all $x, y, z, w \in X^*$ and $n \in \mathbb{N} \cup \{0\}$. It follows from (2.3) that, for all $x, y, z, w \in X^*$, the inequality (2.10) holds in case $n = 0$. Assume that, for all $x, y, z, w \in X^*$, (2.10) holds for some $n = k \in \mathbb{N}$, that is,

$$\begin{aligned}&\Big\|2\mathcal{T}_m^k f\Big(x+y, \frac{z+w}{2}\Big) - \mathcal{T}_m^k f(x,z) - \mathcal{T}_m^k f(x,w) - \mathcal{T}_m^k f(y,z) - \mathcal{T}_m^k f(y,w)\Big\|_Y \\ &\leq \eta_m^k \theta \|x\|_X^p \|y\|_X^q \|z\|_X^r \|w\|_X^s.\end{aligned}$$

Then we have

$$\begin{aligned}
&\Big\|2\mathcal{T}_m^{k+1} f\Big(x+y, \frac{z+w}{2}\Big) - \mathcal{T}_m^{k+1} f(x,z) - \mathcal{T}_m^{k+1} f(x,w) - \mathcal{T}_m^{k+1} f(y,z) - \mathcal{T}_m^{k+1} f(y,w)\Big\|_Y \\
&= \Big\|2\mathcal{T}_m\Big(\mathcal{T}_m^k f\Big(x+y, \frac{z+w}{2}\Big)\Big) - \mathcal{T}_m(\mathcal{T}_m^k f(x,z)) - \mathcal{T}_m(\mathcal{T}_m^k f(x,w)) \\
&\quad - \mathcal{T}_m(\mathcal{T}_m^k f(y,z)) - \mathcal{T}_m(\mathcal{T}_m^k f(y,w))\Big\|_Y \\
&= \Big\|2\Big(\frac{1}{2}\mathcal{T}_m^k f\Big(m(x+y), 2l\Big(\frac{z+w}{2}\Big)\Big) + \frac{1}{2}\mathcal{T}_m^k f\Big(m(x+y), (2-2l)\Big(\frac{z+w}{2}\Big)\Big) \\
&\quad + \frac{1}{2}\mathcal{T}_m^k f\Big((1-m)(x+y), 2l\Big(\frac{z+w}{2}\Big)\Big) \\
&\quad + \frac{1}{2}\mathcal{T}_m^k f\Big((1-m)(x+y), (2-2l)\Big(\frac{z+w}{2}\Big)\Big)\Big) \\
&\quad - \Big(\frac{1}{2}\mathcal{T}_m^k f(mx, 2lz) + \frac{1}{2}\mathcal{T}_m^k f(mx, (2-2l)z) \\
&\quad + \frac{1}{2}\mathcal{T}_m^k f((1-m)x, 2lz) + \frac{1}{2}\mathcal{T}_m^k f((1-m)x, (2-2l)z)\Big) \\
&\quad - \Big(\frac{1}{2}\mathcal{T}_m^k f(mx, 2lw) + \frac{1}{2}\mathcal{T}_m^k f(mx, (2-2l)w) \\
&\quad + \frac{1}{2}\mathcal{T}_m^k f((1-m)x, 2lw) + \frac{1}{2}\mathcal{T}_m^k f((1-m)x, (2-2l)w)\Big) \\
&\quad - \Big(\frac{1}{2}\mathcal{T}_m^k f(my, 2lz) + \frac{1}{2}\mathcal{T}_m^k f(my, (2-2l)z) \\
&\quad + \frac{1}{2}\mathcal{T}_m^k f((1-m)y, 2lz) + \frac{1}{2}\mathcal{T}_m^k f((1-m)y, (2-2l)z)\Big) \\
&\quad - \Big(\frac{1}{2}\mathcal{T}_m^k f(my, 2lw) + \frac{1}{2}\mathcal{T}_m^k f(my, (2-2l)w) \\
&\quad + \frac{1}{2}\mathcal{T}_m^k f((1-m)y, 2lw) + \frac{1}{2}\mathcal{T}_m^k f((1-m)y, (2-2l)w)\Big)\Big\|_Y \\
&= \Big\|\frac{1}{2}\Big(2\mathcal{T}_m^k f\Big(m(x+y), 2l\Big(\frac{z+w}{2}\Big)\Big)\Big) - \frac{1}{2}\mathcal{T}_m^k f(mx, 2lz) - \frac{1}{2}\mathcal{T}_m^k f(mx, 2lw) \\
&\quad - \frac{1}{2}\mathcal{T}_m^k f(my, 2lz) - \frac{1}{2}\mathcal{T}_m^k f(my, 2lw) \\
&\quad + \frac{1}{2}\Big(2\mathcal{T}_m^k f\Big(m(x+y), (2-2l)\Big(\frac{z+w}{2}\Big)\Big)\Big) - \frac{1}{2}\mathcal{T}_m^k f(mx, (2-2l)z) \\
&\quad - \frac{1}{2}\mathcal{T}_m^k f(mx, (2-2l)w) - \frac{1}{2}\mathcal{T}_m^k f(my, (2-2l)z) - \frac{1}{2}\mathcal{T}_m^k f(my, (2-2l)w)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\Big(2\mathcal{T}_m^k f\Big((1-m)(x+y),2l\Big(\frac{z+w}{2}\Big)\Big)\Big)-\frac{1}{2}\mathcal{T}_m^k f((1-m)x,2lz)\\
&-\frac{1}{2}\mathcal{T}_m^k f((1-m)x,2lw)-\frac{1}{2}\mathcal{T}_m^k f((1-m)y,2lz)-\frac{1}{2}\mathcal{T}_m^k f((1-m)y,2lw)\\
&+\frac{1}{2}\Big(2\mathcal{T}_m^k f\Big((1-m)(x+y),(2-2l)\Big(\frac{z+w}{2}\Big)\Big)\Big)-\frac{1}{2}\mathcal{T}_m^k f((1-m)x,(2-2l)z)\\
&-\frac{1}{2}\mathcal{T}_m^k f((1-m)x,(2-2l)w)-\frac{1}{2}\mathcal{T}_m^k f((1-m)y,(2-2l)z)\\
&-\frac{1}{2}\mathcal{T}_m^k f((1-m)y,(2-2l)w)\Big\|_Y\\
\le{}&\frac{1}{2}\Big\|\Big(2\mathcal{T}_m^k f\Big(m(x+y),2l\Big(\frac{z+w}{2}\Big)\Big)\Big)-\mathcal{T}_m^k f(mx,2lz)-\mathcal{T}_m^k f(mx,2lw)\\
&-\mathcal{T}_m^k f(my,2lz)-\mathcal{T}_m^k f(my,2lw)\Big\|_Y\\
&+\frac{1}{2}\Big\|\Big(2\mathcal{T}_m^k f\Big(m(x+y),(2-2l)\Big(\frac{z+w}{2}\Big)\Big)\Big)-\mathcal{T}_m^k f(mx,(2-2l)z)\\
&-\mathcal{T}_m^k f(mx,(2-2l)w)-\mathcal{T}_m^k f(my,(2-2l)z)-\mathcal{T}_m^k f(my,(2-2l)w)\Big\|_Y\\
&+\frac{1}{2}\Big\|\Big(2\mathcal{T}_m^k f\Big((1-m)(x+y),2l\Big(\frac{z+w}{2}\Big)\Big)\Big)-\mathcal{T}_m^k f((1-m)x,2lz)\\
&-\mathcal{T}_m^k f((1-m)x,2lw)-\mathcal{T}_m^k f((1-m)y,2lz)-\mathcal{T}_m^k f((1-m)y,2lw)\Big\|_Y\\
&+\frac{1}{2}\Big\|\Big(2\mathcal{T}_m^k f\Big((1-m)(x+y),(2-2l)\Big(\frac{z+w}{2}\Big)\Big)\Big)\\
&-\mathcal{T}_m^k f((1-m)x,(2-2l)z)-\mathcal{T}_m^k f((1-m)x,(2-2l)w)\\
&-\mathcal{T}_m^k f((1-m)y,(2-2l)z)-\mathcal{T}_m^k f((1-m)y,(2-2l)w)\Big\|_Y\\
\le{}&\frac{1}{2}\eta_m^k\theta\|mx\|_X^p\|my\|_X^q\|2lz\|_X^r\|2lw\|_X^s\\
&+\frac{1}{2}\eta_m^k\theta\|mx\|_X^p\|my\|_X^q\|(2-2l)z\|_X^r\|(2-2l)w\|_X^s\\
&+\frac{1}{2}\eta_m^k\theta\|(1-m)x\|_X^p\|(1-m)y\|_X^q\|2lz\|_X^r\|2lw\|_X^s\\
&+\frac{1}{2}\eta_m^k\theta\|(1-m)x\|_X^p\|(1-m)y\|_X^q\|(2-2l)z\|_X^r\|(2-2l)w\|_X^s\\
\le{}&\eta_m^k\theta\|x\|_X^p\|y\|_X^q\|z\|_X^r\|w\|_X^s\Big(m^{p+q}(2l)^{r+s}+m^{p+q}(2l-2)^{r+s}\\
&+(m-1)^{p+q}(2l)^{r+s}+(m-1)^{p+q}(2l-2)^{r+s}\Big)\\
={}&\eta_m^k\theta\|x\|_X^p\|y\|_X^q\|z\|_X^r\|w\|_X^s\eta_m=\eta_m^{k+1}\theta\|x\|_X^p\|y\|_X^q\|z\|_X^r\|w\|_X^s.
\end{aligned}
$$

Thus the inequality (2.10) holds for all $x,y,z,w\in X^*$ and $n\in\mathbb{N}\cup\{0\}$. Letting $n\to\infty$ in (2.10), it follows that

$$2F_m\Big(x+y,\frac{z+w}{2}\Big)=F_m(x,z)+F_m(x,w)+F_m(y,z)+F_m(y,w)$$

for all $x,y,z,w\in X^*$ and $m\ge m_0$. Therefore, we obtain a sequence $\{F_m\}_{m\ge m_0}$ for the bi-Cauchy-Jensen functional equations on $X\backslash\{0\}$ such that

$$\|f(x,y)-F_m(x,y)\|_Y\le\frac{\varepsilon_m(x,y)}{1-\eta_m} \tag{2.11}$$

for all $x,y,z,w\in X^*$ and $m\ge m_0$. Since

$$\lim_{m\to\infty}\eta_m=0,\quad \lim_{m\to\infty}\varepsilon_m(x,y)=0$$

for all $x,y\in X^*$, taking $m\to\infty$ in (2.11), f satisfies the equation (1.2) on X^*.

In the case $r+s<0$, we have the same result for each $l\in\mathbb{N}$ with $l>2$ and a fixed number $m\in\mathbb{N}\backslash\{1\}$. This completes the proof. □

3 The Hyperstability Result II

Let X be a normed space, Y be a Banach space and let $X^* = X\backslash\{0\}$. First, we give some lemmas for our main results.

Lemma 3.1. If a mapping $f : X \times X \to Y$ satisfies

$$\begin{aligned}&\left\|2f\Big(x+y, \frac{z+w}{2}\Big) - f(x,z) - f(x,w) - f(y,z) - f(y,w)\right\|_Y \\ \le&\theta(\|x\|_X^p + \|y\|_X^q + \|z\|_X^r + \|w\|_X^s),\end{aligned} \tag{3.1}$$

where $p, q, r, s \in \mathbb{R}$, then , for any $m \in \mathbb{N}$,

$$\|f(x,y) - \mathcal{T}_m f(x,y)\|_Y \le \varepsilon_m(x,y) \quad \text{and} \quad \Lambda_m^n \varepsilon_m(x,y) \le \eta_m^n \varepsilon_m(x,y)$$

for all $x, y \in X^*$ and $n \in \mathbb{N}\cup\{0\}$ such that $\mathcal{T}_m, \Lambda_m$ satisfies (2.1), (2.2), respectively, where

$$\varepsilon_m(x,y) = \frac{\theta}{2}(\|mx\|_X^p + \|(1-m)x\|_X^q + \|2my\|_X^r + \|(2-2m)y\|_X^s) \tag{3.2}$$

and

$$\eta_m = 2(m-1)^{p_0} \quad \text{for} \quad p_0 = \max\{p,q,r,s\}. \tag{3.3}$$

Proof . Replacing $(x,y,z,w) = (mx, (1-m)x, 2my, (2-2m)y)$ with $m \in \mathbb{N}$ in (3.1), we obtain

$$\begin{aligned}&\left\|2f\Big(mx + (1-m)x, \frac{2my + (2-2m)y}{2}\Big) - f(mx, 2my) - f(mx, (2-2m)y)\right. \\ &\quad\left. - f((1-m)x, 2my) - f((1-m)x, (2-2m)y)\right\|_Y \\ &\le \theta(\|mx\|_X^p + \|(1-m)x\|_X^q + \|2my\|_X^r + \|(2-2m)y\|_X^s)\end{aligned}$$

and

$$\begin{aligned}&\left\|f(x,y) - \frac{1}{2}f(mx, 2my) - \frac{1}{2}f(mx,(2-2m)y)\right. \\ &\quad\left. - \frac{1}{2}f((1-m)x, 2my) - \frac{1}{2}f((1-m)x,(2-2m)y)\right\|_Y \\ &\le \frac{\theta}{2}(\|mx\|_X^p + \|(1-m)x\|_X^q + \|2my\|_X^r + \|(2-2m)y\|_X^s)\end{aligned} \tag{3.4}$$

for all $x, y \in X^*$. Define an operator $\mathcal{T}_m : Y^{X^*\times X^*} \to Y^{X^*\times X^*}$ by

$$\begin{aligned}\mathcal{T}_m\xi(x,y) =&\frac{1}{2}\xi(mx, 2my) + \frac{1}{2}\xi(mx,(2-2m)y) \\ &+ \frac{1}{2}\xi((1-m)x, 2my) + \frac{1}{2}\xi((1-m)x,(2-2m)y),\end{aligned} \tag{3.5}$$

for all $\xi \in Y^{X^*}$ and $x, y \in X^*$. It follows from (3.4) and (3.5) that we have the following:

$$\|f(x,y) - \mathcal{T}_m f(x,y)\|_Y \le \varepsilon_m(x,y)$$

for all $x, y \in X^*$ and $m \in \mathbb{N}$. Similarly, we define a mapping $\Lambda_m : \mathbb{R}_+^{X^*\times X^*} \to \mathbb{R}_+^{X^*\times X^*}$ by

$$\begin{aligned}\Lambda_m\delta(x,y) =&\frac{1}{2}\delta(mx, 2my) + \frac{1}{2}\delta(mx,(2-2m)y) + \frac{1}{2}\delta((1-m)x, 2my) \\ &+ \frac{1}{2}\delta((1-m)x,(2-2m)y)\end{aligned} \tag{3.6}$$

for all $x, y \in X^*$ and $\delta \in \mathbb{R}_+^{X^*\times X^*}$. Next, we will show that, for any $x, y \in X^*$ and $m \in \mathbb{N}$,

$$\Lambda_m^n \varepsilon_m(x,y) \le \eta_m^n \varepsilon_m(x,y) \tag{3.7}$$

for each $n \in \mathbb{N} \cup \{0\}$. It is clear that the inequality (3.7) holds for $n = 0$. Next, assume that (3.7) holds for some $n = k \in \mathbb{N}$, that is,

$$\Lambda_m^k \varepsilon_m(x,y) \le \eta_m^k \varepsilon_m(x,y).$$

Then it follows that

$$\begin{aligned}
&\Lambda_m^{k+1}\varepsilon_m(x,y)\\
&= \Lambda_m(\Lambda_m^k\varepsilon_m(x,y))\\
&= \frac{1}{2}\Lambda_m^k\varepsilon_m(mx,2my) + \frac{1}{2}\Lambda_m^k\varepsilon_m(mx,(2-2m)y) + \frac{1}{2}\Lambda_m^k\varepsilon_m((1-m)x,2my)\\
&\quad + \frac{1}{2}\Lambda_m^k\varepsilon_m((1-m)x,(2-2m)y)\\
&\le \frac{1}{2}\eta_m^k\varepsilon_m(mx,2my) + \frac{1}{2}\eta_m^k\varepsilon_m(mx,(2-2m)y) + \frac{1}{2}\eta_m^k\varepsilon_m((1-m)x,2my)\\
&\quad + \frac{1}{2}\eta_m^k\varepsilon_m((1-m)x,(2-2m)y)\\
&\le \frac{1}{2}\eta_m^k(\varepsilon_m(mx,2my) + \varepsilon_m(mx,(2-2m)y) + \varepsilon_m((1-m)x,2my)\\
&\quad + \varepsilon_m((1-m)x,(2-2m)y))\\
&= \frac{1}{2}\eta_m^k\Big(\frac{\theta}{2}(\|m(mx)\|_X^p + \|(1-m)(mx)\|_X^q + \|2m(2my)\|_X^r\\
&\quad + \|(2-2m)(2my)\|_X^s)\\
&\quad + \frac{\theta}{2}(\|m(mx)\|_X^p + \|(1-m)(mx)\|_X^q + \|2m((2-2m)y)\|_X^r\\
&\quad + \|(2-2m)((2-2m)y)\|_X^s)\\
&\quad + \frac{\theta}{2}(\|m((1-m)x)\|_X^p + \|(1-m)((1-m)x)\|_X^q + \|2m(2my)\|_X^r\\
&\quad + \|(2-2m)(2my)\|_X^s) + \frac{\theta}{2}(\|m((1-m)x)\|_X^p + \|(1-m)((1-m)x)\|_X^q\\
&\quad + \|2m((2-2m)y)\|_X^r + \|(2-2m)((2-2m)y)\|_X^s)\Big)\\
&= \frac{1}{2}\eta_m^k\frac{\theta}{2}(m^p\|mx\|_X^p + m^q\|(1-m)x\|_X^q\\
&\quad + (2m)^r\|2my\|_X^r + (2m)^s\|(2-2m)y\|_X^s + m^p\|mx\|_X^p + m^q\|(1-m)x\|_X^q\\
&\quad + (2m-2)^r\|2my\|_X^r + (2m-2)^s\|(2-2m)y\|_X^s\\
&\quad + (m-1)^p\|mx\|_X^p + (m-1)^q\|(1-m)x\|_X^q + (2m)^r\|2my\|_X^r\\
&\quad + (2m)^s\|(2-2m)y\|_X^s + (m-1)^p\|mx\|_X^p + (m-1)^q\|(1-m)x\|_X^q\\
&\quad + (2m-2)^r\|2my\|_X^r + (2m-2)^s\|(2-2m)y\|_X^s)\\
&= \frac{1}{2}\eta_m^k\frac{\theta}{2}(m^p\|mx\|_X^p + m^q\|(1-m)x\|_X^q + (2m)^r\|2my\|_X^r\\
&\quad + (2m)^s\|(2-2m)y\|_X^s + m^p\|mx\|_X^p + m^q\|(1-m)x\|_X^q + (2m)^r\|2my\|_X^r\\
&\quad + (2m)^s\|(2-2m)y\|_X^s + (m-1)^p\|mx\|_X^p + (m-1)^q\|(1-m)x\|_X^q\\
&\quad + (2m-2)^r\|2my\|_X^r + (2m-2)^s\|(2-2m)y\|_X^s\\
&\quad + (m-1)^p\|mx\|_X^p + (m-1)^q\|(1-m)x\|_X^q\\
&\quad + (2m-2)^r\|2my\|_X^r + (2m-2)^s\|(2-2m)y\|_X^s)\\
&= \eta_m^k\frac{\theta}{2}(m^p\|mx\|_X^p + m^q\|(1-m)x\|_X^q + (2m)^r\|2my\|_X^r + (2m)^s\|(2-2m)y\|_X^s\\
&\quad + (m-1)^p\|mx\|_X^p + (m-1)^q\|(1-m)x\|_X^q + (2m-2)^r\|2my\|_X^r\\
&\quad + (2m-2)^s\|(2-2m)y\|_X^s)\\
&\le \eta_m^k\frac{\theta}{2}(m^{p_0}\|mx\|_X^p + m^{p_0}\|(1-m)x\|_X^q + m^{p_0}\|2my\|_X^r + m^{p_0}\|(2-2m)y\|_X^s\\
&\quad + (m-1)^{p_0}\|mx\|_X^p + (m-1)^{p_0}\|(1-m)x\|_X^q
\end{aligned}$$

$$
\begin{aligned}
&\quad + (m-1)^{p_0}\|2my\|_X^r + (m-1)^{p_0}\|(2-2m)y\|_X^s)\\
&\le \eta_m^k \frac{\theta}{2}((\|mx\|_X^p + \|(1-m)x\|_X^q + \|2my\|_X^r + \|(2-2m)y\|_X^s)m^{p_0}\\
&\quad + (\|mx\|_X^p + \|(1-m)x\|_X^q + \|2my\|_X^r + \|(2-2m)y\|_X^s)(m-1)^{p_0})\\
&\le \eta_m^k \frac{\theta}{2}(\|mx\|_X^p + \|(1-m)x\|_X^q + \|2my\|_X^r + \|(2-2m)y\|_X^s)(m^{p_0} + (m-1)^{p_0})\\
&\le \eta_m^k \varepsilon_m(x,y)(m^{p_0} + (m-1)^{p_0}) \le \eta_m^k \varepsilon_m(x,y)(2(m-1)^{p_0})\\
&= \eta_m^{k+1}\varepsilon_m(x,y)
\end{aligned}
$$

for all $x, y \in X^*$. This implies that (3.7) holds for $n = k+1$, that is, (3.7) holds for each $n \in \mathbb{N} \cup \{0\}$. □

Theorem 3.2. If a mapping $f : X \times X \to Y$ satisfies (3.1) where $p, q, r, s < 0$. Then f is a solution of the equation (1.2) on X^*.

Proof . Replacing $(x, y, z, w) = (mx, (1-m)x, 2my, (2-2m)y)$ with $m \in \mathbb{N}, m > 2$ in (3.1) and the similar step of the proof of Lemma 3.1, we have

$$\|f(x,y) - \mathcal{T}_m f(x,y)\|_Y \le \varepsilon_m(x,y) \quad \text{and} \quad \Lambda_m^n \varepsilon_m(x,y) \le \eta_m^n \varepsilon_m(x,y)$$

for all $x, y \in X^*$ and $n \in \mathbb{N} \cup \{0\}$ where $\mathcal{T}_m, \Lambda_m, \varepsilon_m$ and η_m are defined by (3.5), (3.6), (3.2) and (3.3), respectively. By Lemma 2.1, we obtain that conditions (H1)–(H3) hold for the mappings $\mathcal{T}_m$ and Λ_m.

Since $\lim_{m\to\infty} 2(m-1)^{p_0} = 0$, it follows that there exists $m_0 \in \mathbb{N}$ such that

$$\eta_m < 1$$

for all $m \ge m_0$. From (3.7), for each $m \ge m_0$, we have

$$\varepsilon_m^*(x,y) := \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x,y) \le \varepsilon_m(x,y) \sum_{n=0}^{\infty} \eta_m^n = \frac{\varepsilon_m(x,y)}{1-\eta_m}$$

for all $x, y \in X^*$. It follows from Theorem 1.1 that, for each $m \ge m_0$, there exists a unique solution $F_m : X^* \times X^* \to Y$ of the following equation:

$$
\begin{aligned}
F_m(x,y) =& \frac{1}{2}F_m(mx, 2my) + \frac{1}{2}F_m(mx, (2-2m)y) + \frac{1}{2}F_m((1-m)x, 2my)\\
&+ \frac{1}{2}F_m((1-m)x, (2-2m)y)
\end{aligned}
$$

for all $x, y \in X^*$ such that

$$\|f(x,y) - F_m(x,y)\|_Y \le \frac{\varepsilon_m(x,y)}{1-\eta_m}$$

for all $x, y \in X^*$ and, moreover, we have

$$F_m(x,y) = \lim_{n\to\infty} \mathcal{T}_m^n f(x,y)$$

for all $x, y \in X^*$.

Next, we will show that F_m satisfies the equation (1.2) for all $x, y \in X^*$ and $m \ge m_0$. First, we show that, for any $m \ge m_0$,

$$
\begin{aligned}
&\left\|2\mathcal{T}_m^n f\Big(x+y, \frac{z+w}{2}\Big) - \mathcal{T}_m^n f(x,z) - \mathcal{T}_m^n f(x,w) - \mathcal{T}_m^n f(y,z) - \mathcal{T}_m^n f(y,w)\right\|_Y\\
\le& \eta_m^n(\theta\|x\|_X^p + \|y\|_X^q + \|z\|_X^r + \|w\|_X^s) \qquad (3.8)
\end{aligned}
$$

for all $x, y, z, w \in X^*$ and $n \in \mathbb{N} \cup \{0\}$. It follows from (3.1) that, for all $x, y, z, w \in X^*$, the inequality (3.8) holds in case $n=0$. Assume that (3.8) holds for some $n=k\in\mathbb{N}$, that is,

$$
\begin{aligned}
&\Big\|2\mathcal{T}_m^{k+1}f\Big(x+y,\frac{z+w}{2}\Big)-\mathcal{T}_m^{k+1}f(x,z)-\mathcal{T}_m^{k+1}f(x,w)-\mathcal{T}_m^{k+1}f(y,z)-\mathcal{T}_m^{k+1}f(y,w)\Big\|_Y\\
&=\Big\|2\mathcal{T}_m\Big(\mathcal{T}_m^kf\Big(x+y,\frac{z+w}{2}\Big)\Big)-\mathcal{T}_m(\mathcal{T}_m^kf(x,z))-\mathcal{T}_m(\mathcal{T}_m^kf(x,w))\\
&\quad-\mathcal{T}_m(\mathcal{T}_m^kf(y,z))-\mathcal{T}_m(\mathcal{T}_m^kf(y,w))\Big\|_Y\\
&=\Big\|2\Big(\frac{1}{2}\mathcal{T}_m^kf\Big(m(x+y),2l\Big(\frac{z+w}{2}\Big)\Big)+\frac{1}{2}\mathcal{T}_m^kf\Big(m(x+y),(2-2l)\Big(\frac{z+w}{2}\Big)\Big)\\
&\quad+\frac{1}{2}\mathcal{T}_m^kf\Big((1-m)(x+y),2l\Big(\frac{z+w}{2}\Big)\Big)\\
&\quad+\frac{1}{2}\mathcal{T}_m^kf\Big((1-m)(x+y),(2-2l)\Big(\frac{z+w}{2}\Big)\Big)\Big)\\
&\quad-\Big(\frac{1}{2}\mathcal{T}_m^kf(mx,2lz)+\frac{1}{2}\mathcal{T}_m^kf(mx,(2-2l)z)+\frac{1}{2}\mathcal{T}_m^kf((1-m)x,2lz)\\
&\quad+\frac{1}{2}\mathcal{T}_m^kf((1-m)x,(2-2l)z)\Big)\\
&\quad-\Big(\frac{1}{2}\mathcal{T}_m^kf(mx,2lw)+\frac{1}{2}\mathcal{T}_m^kf(mx,(2-2l)w)\\
&\quad+\frac{1}{2}\mathcal{T}_m^kf((1-m)x,2lw)+\frac{1}{2}\mathcal{T}_m^kf((1-m)x,(2-2l)w)\Big)\\
&\quad-\Big(\frac{1}{2}\mathcal{T}_m^kf(my,2lz)+\frac{1}{2}\mathcal{T}_m^kf(my,(2-2l)z)\\
&\quad+\frac{1}{2}\mathcal{T}_m^kf((1-m)y,2lz)+\frac{1}{2}\mathcal{T}_m^kf((1-m)y,(2-2l)z)\Big)\\
&\quad-\Big(\frac{1}{2}\mathcal{T}_m^kf(my,2lw)+\frac{1}{2}\mathcal{T}_m^kf(my,(2-2l)w)\\
&\quad+\frac{1}{2}\mathcal{T}_m^kf((1-m)y,2lw)+\frac{1}{2}\mathcal{T}_m^kf((1-m)y,(2-2l)w)\Big)\Big\|_Y\\
&=\Big\|\frac{1}{2}\Big(2\mathcal{T}_m^kf\Big(m(x+y),2l\Big(\frac{z+w}{2}\Big)\Big)\Big)-\frac{1}{2}\mathcal{T}_m^kf(mx,2lz)-\frac{1}{2}\mathcal{T}_m^kf(mx,2lw)\\
&\quad-\frac{1}{2}\mathcal{T}_m^kf(my,2lz)-\frac{1}{2}\mathcal{T}_m^kf(my,2lw)\\
&\quad+\frac{1}{2}\Big(2\mathcal{T}_m^kf\Big(m(x+y),(2-2l)\Big(\frac{z+w}{2}\Big)\Big)\Big)-\frac{1}{2}\mathcal{T}_m^kf(mx,(2-2l)z)\\
&\quad-\frac{1}{2}\mathcal{T}_m^kf(mx,(2-2l)w)-\frac{1}{2}\mathcal{T}_m^kf(my,(2-2l)z)-\frac{1}{2}\mathcal{T}_m^kf(my,(2-2l)w)\\
&\quad+\frac{1}{2}\Big(2\mathcal{T}_m^kf\Big((1-m)(x+y),2l\Big(\frac{z+w}{2}\Big)\Big)\Big)-\frac{1}{2}\mathcal{T}_m^kf((1-m)x,2lz)\\
&\quad-\frac{1}{2}\mathcal{T}_m^kf((1-m)x,2lw)-\frac{1}{2}\mathcal{T}_m^kf((1-m)y,2lz)-\frac{1}{2}\mathcal{T}_m^kf((1-m)y,2lw)\\
&\quad+\frac{1}{2}\Big(2\mathcal{T}_m^kf\Big((1-m)(x+y),(2-2l)\Big(\frac{z+w}{2}\Big)\Big)\Big)-\frac{1}{2}\mathcal{T}_m^kf((1-m)x,(2-2l)z)\\
&\quad-\frac{1}{2}\mathcal{T}_m^kf((1-m)x,(2-2l)w)-\frac{1}{2}\mathcal{T}_m^kf((1-m)y,(2-2l)z)\\
&\quad-\frac{1}{2}\mathcal{T}_m^kf((1-m)y,(2-2l)w)\Big\|_Y\\
&\le\frac{1}{2}\Big\|\Big(2\mathcal{T}_m^kf\Big(m(x+y),2l\Big(\frac{z+w}{2}\Big)\Big)\Big)-\mathcal{T}_m^kf(mx,2lz)-\mathcal{T}_m^kf(mx,2lw)\\
&\quad-\mathcal{T}_m^kf(my,2lz)-\mathcal{T}_m^kf(my,2lw)\Big\|_Y\\
&\quad+\frac{1}{2}\Big\|\Big(2\mathcal{T}_m^kf\Big(m(x+y),(2-2l)\Big(\frac{z+w}{2}\Big)\Big)\Big)-\mathcal{T}_m^kf(mx,(2-2l)z)\\
&\quad-\mathcal{T}_m^kf(mx,(2-2l)w)-\mathcal{T}_m^kf(my,(2-2l)z)-\mathcal{T}_m^kf(my,(2-2l)w)\Big\|_Y
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\Big\|\Big(2\mathcal{T}_m^k f\Big((1-m)(x+y),2l\Big(\frac{z+w}{2}\Big)\Big)\Big)-\mathcal{T}_m^k f((1-m)x,2lz)\\
&-\mathcal{T}_m^k f((1-m)x,2lw)-\mathcal{T}_m^k f((1-m)y,2lz)-\mathcal{T}_m^k f((1-m)y,2lw)\Big\|_Y\\
&+\frac{1}{2}\Big\|\Big(2\mathcal{T}_m^k f\Big((1-m)(x+y),(2-2l)\Big(\frac{z+w}{2}\Big)\Big)\Big)-\mathcal{T}_m^k f((1-m)x,(2-2l)z)\\
&-\mathcal{T}_m^k f((1-m)x,(2-2l)w)-\mathcal{T}_m^k f((1-m)y,(2-2l)z)\\
&-\mathcal{T}_m^k f((1-m)y,(2-2l)w)\Big\|_Y\\
\le{}&\frac{1}{2}(\eta_m^k\theta(\|mx\|_X^p+\|my\|_X^q+\|2mz\|_X^r+\|2mw\|_X^s))\\
&+\frac{1}{2}(\eta_m^k\theta(\|mx\|_X^p+\|my\|_X^q+\|(2-2m)z\|_X^r+\|(2-2m)w\|_X^s))\\
&+\frac{1}{2}(\eta_m^k\theta(\|(1-m)x\|_X^p+\|(1-m)y\|_X^q+\|2mz\|_X^r+\|2mw\|_X^s))\\
&+\frac{1}{2}(\eta_m^k\theta(\|(1-m)x\|_X^p+\|(1-m)y\|_X^q+\|(2-2m)z\|_X^r+\|(2-2m)w\|_X^s))\\
\le{}&\frac{1}{2}\eta_m^k\theta[m^p\|x\|_X^p+m^q\|y\|_X^q+(2m)^r\|z\|_X^r+(2m)^s\|w\|_X^s\\
&+m^p\|x\|_X^p+m^q\|y\|_X^q+(2m-2)^r\|z\|_X^r+(2m-2)^s\|w\|_X^s]\\
&+(m-1)^p\|x\|_X^p+(m-1)^q\|y\|_X^q+(2m)^r\|z\|_X^r+(2m)^s\|w\|_X^s\\
&+(m-1)^p\|x\|_X^p+(m-1)^q\|y\|_X^q+(2m-2)^r\|z\|_X^r+(2m-2)^s\|w\|_X^s]\\
\le{}&\frac{1}{2}\eta_m^k\theta[m^p\|x\|_X^p+m^q\|y\|_X^q+(2m)^r\|z\|_X^r+(2m)^s\|w\|_X^s\\
&+m^p\|x\|_X^p+m^q\|y\|_X^q+(2m)^r\|z\|_X^r+(2m)^s\|w\|_X^s]\\
&+(m-1)^p\|x\|_X^p+(m-1)^q\|y\|_X^q+(2m-2)^r\|z\|_X^r+(2m-2)^s\|w\|_X^s\\
&+(m-1)^p\|x\|_X^p+(m-1)^q\|y\|_X^q+(2m-2)^r\|z\|_X^r+(2m-2)^s\|w\|_X^s]\\
={}&\eta_m^k\theta[(m^p\|x\|_X^p+m^q\|y\|_X^q+(2m)^r\|z\|_X^r+(2m)^s\|w\|_X^s)\\
&+((m-1)^p\|x\|_X^p+(m-1)^q\|y\|_X^q+(2m-2)^r\|z\|_X^r+(2m-2)^s\|w\|_X^s)]\\
\le{}&\eta_m^k\theta[(m^{p_0}\|x\|_X^p+m^{p_0}\|y\|_X^q+m^{p_0}\|z\|_X^r+m^{p_0}\|w\|_X^s)\\
&+((m-1)^{p_0}\|x\|_X^p+(m-1)^{p_0}\|y\|_X^q+(m-1)^{p_0}\|z\|_X^r+(m-1)^{p_0}\|w\|_X^s)]\\
\le{}&\eta_m^k\theta(\|x\|_X^p+\|y\|_X^q+\|z\|_X^r+\|w\|_X^s)(m^{p_0}+(m-1)^{p_0})\\
\le{}&\eta_m^k\theta(\|x\|_X^p+\|y\|_X^q+\|z\|_X^r+\|w\|_X^s)(2(m-1)^{p_0})\\
={}&\eta_m^k\theta(\|x\|_X^p+\|y\|_X^q+\|z\|_X^r+\|w\|_X^s)\eta_m\\
={}&\eta_m^{k+1}\theta(\|x\|_X^p+\|y\|_X^q+\|z\|_X^r+\|w\|_X^s).
\end{aligned}
$$

Therefore, the inequality (3.8) holds for all $x,y,z,w\in X^*$ and $n\in\mathbb{N}\cup\{0\}$. Letting $n\to\infty$ in (3.8), it follows that

$$2F_m\Big(x+y,\frac{z+w}{2}\Big)=F_m(x,z)+F_m(x,w)+F_m(y,z)+F_m(y,w)$$

for all $x,y,z,w\in X^*$ and $m\ge m_0$. Therefore, we obtain a sequence $\{F_m\}_{m\ge m_0}$ for the bi-Cauchy-Jensen functional equation on $X\backslash\{0\}$ such that

$$\|f(x,y)-F_m(x,y)\|_Y\le\frac{\varepsilon_m(x,y)}{1-\eta_m} \tag{3.9}$$

for all $x,y,z,w\in X^*$ and $m\ge m_0$. Since

$$\lim_{m\to\infty}\eta_m=0,\quad \lim_{m\to\infty}\varepsilon_m(x,y)=0$$

for all $x,y\in X^*$, taking $m\to\infty$ in (3.9), the mapping f satisfies the equation (1.2) on X^*. This completes the proof. □

Corollary 3.3. Let $F:X^{*4}\to Y$ be a mapping such that $F(x_0,y_0,z_0,w_0)\neq 0$ for some $x_0,y_0,z_0,w_0\in X^*$,

$$\|F(x,y,z,w)\|_Y\le\theta\|x\|_X^p\|y\|_X^q\|z\|_X^r\|w\|_X^s \tag{3.10}$$

and

$$\|F(x,y,z,w)\|_Y \leq \theta(\|x\|_X^p + \|y\|_X^q + \|z\|_X^r + \|w\|_X^s) \tag{3.11}$$

for all $x, y, z, w \in X^*$, where $\theta \geq 0$ and $p, q, r, s \in \mathbb{R}$. Assume that p, q, r, s with $p + q < 0$ or $r + s < 0$ in (3.10) and $p, q, r, s < 0$ in (3.11). Then the functional equation

$$g(x,z) + g(x,w) + g(y,z) + g(y,w) + F(x,y,z,w) = 2g\Big(x+y, \frac{z+w}{2}\Big) \tag{3.12}$$

for all $x, y, z, w \in X^*$ has no any solution in the class of mappings $g : X^* \times X^* \to Y$.

Proof . Suppose that $g : X^* \times X^* \to Y$ is a solution of the equation (3.12), that is,

$$2g\Big(x+y, \frac{z+w}{2}\Big) - g(x,z) - g(x,w) - g(y,z) - g(y,w) = F(x,y,z,w).$$

Then (3.10) or (3.11) holds. Indeed, we have

$$\begin{aligned}&\Big\|2g\Big(x+y, \frac{z+w}{2}\Big) - g(x,z) - g(x,w) - g(y,z) - g(y,w)\Big\|_Y\\ =&\|F(x,y,z,w)\|_Y\\ \leq&\theta\|x\|_X^p\|y\|_X^q\|z\|_X^r\|w\|_X^s\end{aligned}$$

or

$$\begin{aligned}&\Big\|2g\Big(x+y, \frac{z+w}{2}\Big) - g(x,z) - g(x,w) - g(y,z) - g(y,w)\Big\|_Y\\ =&\|F(x,y,z,w)\|_Y\\ \leq&\theta(\|x\|_X^p + \|y\|_X^q + \|z\|_X^r + \|w\|_X^s)\end{aligned}$$

for all $x, y, z, w \in X^*$. From Theorem 2.3 and Theorem 3.2, it follows that the mapping g is the solution of the equation (1.2) on X^*. Thus it follows that

$$2g\Big(x+y, \frac{z+w}{2}\Big) = g(x,z) + g(x,w) + g(y,z) + g(y,w)$$

for all $x, y, z, w \in X^*$, that is,

$$\|F(x,y,z,w)\|_Y = \Big\|2g\Big(x+y, \frac{z+w}{2}\Big) - g(x,z) - g(x,w) - g(y,z) - g(y,w)\Big\|_Y = 0$$

for all $x, y, z, w \in X^*$, which implies that $F(x_0, y_0, z_0, w_0) = 0$. This is a contradiction. This completes the proof. □

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