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Determining optimal rank in reduced rank regression model by the likelihood ratio test

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Abstract

This paper presents a method for determining the actual rank of the coefficients matrix in the reduced rank multivariate regression model. The method is constructed using the singular value decomposition and the Likelihood Ratio Test(LRT). Some illustrative examples are given to verify this method.

Keywords: Multivariate Reduced Rank Regression, Singular Value Decomposition, More-Penrose inverse, Maximum likelihood estimation
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1 Introduction

Consider the multivariate regression model

$$\underline{y}_i^T = \underline{c}^T + \underline{x}_i^T M + \underline{e}_i^T, \quad i = 1, 2, ..., n \tag{1.1}$$

where $\underline{x}_i(r \times 1)$ is the vector of dependent variables, $\underline{y}_i(t \times 1)$ is the vector of dependent variables, $M(r \times t)$ is a matrix whose columns are the individual unknown regression coefficients for each dependent variable on the set of the independent variables, $\underline{c}(t \times 1)$ is a vector of unknown response specific constants and $\underline{e}_i(t \times 1)$ is a vector of stochastic errors assumed to be independently distributed with zero mean and unit variance. T indicates transpose of a matrix or a vector. In matrix form, Equation (1.1) can be written as

$$Y = C + XM + E, \quad where \;\; C = \underline{1}\ \underline{c}^T. \tag{1.2}$$

Suppose the rank of M is s, where $s \leq \mathrm{k} = \min(r, t)$. The full rank regression coefficient matrix occurs when $s = \mathrm{k}$. The reduced rank regression coefficient matrix occurs (due to some linear restrictions on the regression coefficients) when $s < \mathrm{k}$. Such models have been studied by several authors. Izenman (1975) considered the problem of estimating the regression coefficient matrix having (known reduced) rank and showed that the canonical variable and principal component are special cases of a reduced rank regression model [5]. Alvarez et al. (2016) presented a procedure for coefficient matrix estimation in multivariate model for reduced rank in the presence of multicollinearity [1]. Davies et al. (1982) gave a method for estimating the coefficient matrix, which is justified by a least-square analysis employing singular value decomposition and the Eckart-Young theorem [2].

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One problem that arises in the estimation of the reduced rank regression coefficient matrix is the choice of s, the assumed maximum rank of M. Madhi (1981) employed cross validation criterion to determine the rank in reduced rank regression model, RRRM , [6].

Madhi and Abushilah (2021) determined the rank of the coefficients matrix in RRRM using the Akaike's information criterion (AIC) [7].

In this paper, we present a method for determining the actual rank of the coefficient matrix in the reduced rank regression model (RRRM) employing the method of estimation proposed by Davies et al. (1982) [2] and the Likelihood Ratio Test(LRT).

Some numerical examples are given to illustrate the Method.

2 Preliminaries

2.1 Singular Value Decomposition of a Matrix (SVD)

If A is a $m \times n$ matrix, of rank k, it can be expressed as

$$A = UDW^T, \tag{2.1}$$

where
U is an $m \times m$ orthogonal matrix, W is an $n \times n$ orthogonal matrix and D is an $m \times n$ diagonal matrix with non-negative elements, $D = dia(\sigma_1, ..., \sigma_k, 0, ..., 0), \sigma_1 \geq ... \geq \sigma_k > 0$.

If Q and V consist of the first k columns of U and W respectively, and Σ is a $k \times k$ diagonal matrix with positive diagonal elements, $\sigma_1 \geq \cdots \geq \sigma_k > 0$, then

$$A = Q\Sigma V^T, \tag{2.2}$$

where $Q^TQ = V^TV = I_k$ and $\Sigma = \text{diag}\,(\sigma_1, \sigma_2, \ldots, \sigma_k)$ with $\sigma_1 \geq \sigma_2, \ldots \geq \sigma_k > 0$ is called basic diagonal (Green, 2014) [4].

The basic diagonal part of the decomposition is always unique regardless of whether A is of full rank, square, or rectangular. Each one of Equations (2.1) and (2.2) is equivalent to

$$A = \sigma_1 \underline{q}_1 \underline{v}_1^T + \sigma_2 \underline{q}_2 \underline{v}_2^T + \ldots\ldots + \sigma_k \underline{q}_k \underline{v}_k^T \tag{2.3}$$

That is, the sum of k matrices of rank 1. The column vectors $\left\{\underline{q}_i\right\}_{i=1}^{k}$ of Q are orthonormal (orthogonal and each of length 1) and each has m components. The row vectors $\{\underline{v}_i\}_{i=1}^{k}$ of V^T are orthonormal and each has n components.

The numbers $\sigma_1, \sigma_2, \ldots$ are the singular values of A. The vectors $\underline{q}_1, \underline{q}_2, \ldots\ldots$ and $\underline{v}_1, \underline{v}_2, \ldots\ldots$ are respectively the left and right singular vectors. When A is square and symmetric the singular decomposition reduces to known spectral decomposition, where the left and right singular vectors are identical and reduce to eigenvectors.

The orthonormal sets $\left\{\underline{q}_1, \underline{q}_2, \ldots . \underline{q}_k\right\}$ and $\{\underline{v}_1, \underline{v}_2, \ldots . \underline{v}_k\}$ can be completed to sets $\left\{\underline{q}_1, \underline{q}_2, \ldots . \underline{q}_m\right\}$ and $\{\underline{v}_1, \underline{v}_2, \ldots . \underline{v}_n\}$. A complete decomposition of A is then (if $m \leq n$ without loss of generality)

$$\sum_{i=1}^{m} \sigma_i \underline{q}_i \underline{v}_i^T, \qquad \text{with } \sigma_{k+1} = \cdots = \sigma_m = 0. \tag{2.4}$$

The singular decomposition (2.3) is of course equal in numerical value to the complete singular decomposition (2.2). The SVD is closely related to the eigenvalue decomposition, since

$$AA^T = Q\Sigma^2 Q^T, \tag{2.5}$$

where $V^TV = I_k$ and $\sum^2 = \text{diag}\left(\sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2\right)$ and $\sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2$ are the non-zero eigenvalues of the $m \times m$ matrix AA^T and the columns of Q are the corresponding eigenvectors of A^TA. Furthermore,

$$A^TA = V\Sigma^2 V^T, \tag{2.6}$$

where $Q^TQ = I_k$ and $\sum^2 = \text{diag}\left(\sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2\right)$ and $\sigma_1^2, \sigma_2^2, \ldots\ldots, \sigma_k^2$ are also the non-zero eigenvalues of the $n \times n$ matrix A^TA and the columns of V are the corresponding eigenvectors. Hence, the singular values of A are the square roots of the common positive eigenvalues of the $m \times m$ matrix AA^T and the $n \times n$ matrix ${}^TA$.

2.2 Generalized Inverse

Generalized inverse can be very useful in the regression model, where inverses arise naturally. The basic types of generalized inverse (Green, 2014) [4] are:

(a) G-Inverse

Let A be an $m \times n$ matrix of any rank. A generalized inverse (or a G-Inverse) of A is an $n \times m$, denoted by A^-, such that

$$AA^-A = A. \tag{2.7}$$

A generalized inverse always exists, but it is not necessarily unique. One way of illustrating the existence of A^- and its non- uniqueness is by using SVD. For $m \times n$ matrix, write

$$A = U\Sigma V^T,$$

then the general G-Inverse (Good, 1969) [3] is

$$A^- = V\Sigma^- U^T = \sigma_1{}^- \underline{v}_1\underline{u}_1^T + ... + \sigma_k{}^- \underline{v}_k\underline{u}_k^T \tag{2.8}$$

where $k = \min(m, n), \sigma_1, \sigma_2, \ldots, \sigma_k$ are all the singular values of A (non-negative square roots of all the eigenvalues of AA^T if $m \le n$, or A^TA if $m \ge n$), and σ^-means σ^{-1} if $\sigma \ne 0$ and is otherwise arbitrary.

(b) Moore-Penrose Generalized Inverse

Moore and Penrose (Good, 1969) [3] defined a particular generalized inverse often called (the pseudo inverse) as a matrix $A^+(nxm\ )$ to distinguish it from a general g-inverse A^-, satisfying the properties:

i. $AA^+A = A$
ii. $A^+AA^+ = A^+$
iii. $(AA^+)^T = AA^+$
iv. $(A^+A)^T = A^+A$

Such an inverse always exists and is unique. For an arbitrary $m \times n$ matrix A, of rank k, write the SVD as

$$A = U\Sigma V^T,$$

then the Moore-Penrose inverse is

$$A^+ = V\Sigma^{-1}U^T = \sigma_1^{-1}\underline{v}_1\underline{u}_1^T + \sigma_2^{-1}\underline{v}_2\underline{u}_2^T + \ldots\ldots + \sigma_k^{-1}\underline{v}_k\underline{u}_k^T. \tag{2.9}$$

It is obvious that A^+can be uniquely defined as the g-inverse of minimum rank, since the rank of $A^+$is simply the number of σ'^s_j that do not vanish.

2.3 Eckart-Young Theorem

Given $Y(m, n)$, of rank $k = \min(m, n)$, the matrix $H^*(m, n)$ of rank at most$(s < k = \min(m, n))$ that best approximate Y, i.e. H^* satisfies

$$\min_{H \text{ rank } H \le s} \|Y - H\|^2, \tag{2.10}$$

is given by the partial sum of the first s terms of the SVD of Y. That is, if

$$Y = \sum_{i=1}^{k=\min\{m,n\}} \sigma_i \underline{u}_i \underline{v}_i^T, \tag{2.11}$$

then

$$H^* = \sum_{i=1}^{s} \sigma_i \underline{u}_i \underline{v}_i^T. \tag{2.12}$$

3 Estimation of the Coefficient Matrix Rank in RRRM

3.1 Introduction

In this section, we discuss the method of estimation of the coefficient matrix in RRRM which has been developed by Davies and Tso (1982) [2]. A solution employing matrix singular value decomposition was proposed and justified by the Eckart-Young theorem. This solution has the feature of generality as we can use it even when the regression coefficients are under - determined. Generalized inverses have been used to achieve this generality.

3.2 Reduced Rank Regression Model, RRRM

Consider the model (1.1). Suppose that there are $s < \min(r,t)$ linear combinations of the independent variables $n_1 = \underline{x}^T \underline{a}_1, \ldots, n_s = \underline{x}^T \underline{a}_s$ (i.e. $\underline{n}^T = (n_1 \ldots n_s)\big)$ normalized such that $\underline{a}_i^T \underline{a}_i = \delta ij$, such that all the variation in Y is due to only linear combinations of X plus stochastic error. This reduces the set of independent variables $x_1, x_2, \ldots, x_r$ to a new set of independent variables $n_1, \ldots, n_s$. Consequently, the rank of M will be less than or equal to s. Again consider the model (1.2). We now center the data by subtracting the column means from each variables of X and Y such that

$$\underline{1}^T X = \underline{O}^T, \underline{1}^T Y = \underline{O}^T. \tag{3.1}$$

Hence, the model (1.2) can be written as

$$Y = XM + E, \tag{3.2}$$

where $X(n \times r)$ and $Y(n \times t)$ are matrices whose n rows contain, respectively, independent and dependent data, and whose columns each sum to zero, E is the matrix of stochastic errors which are assumed to be uncorrelated row-wise. We have lost, (txc) parameters but there is a corresponding loss in the data since quantities

$$y_{ij} - \bar{y}_{.j}, \quad i = 1, 2, \ldots, n; j = 1, 2, \ldots, t$$

represents only $(n-1) \times t$ separate pieces of information to the fact that their sum to zero, whereas

$$y_{ij}, \quad i = 1, 2, \ldots, n; j = 1, 2, \ldots, t$$

represent $n \times t$ separate pieces of information. Effectively, the lost pieces of information have been used to enable the proper adjustments to be made to the model so that the C term can be removed. This transformation of the origin data to corrected data is consistent with least-square estimation of the vector of response constants $\underline{c}$. The problem now is to estimate the unknown matrix of regression coefficients $M(r \times t)$ subject to the rank constraint

$$\text{ran}(M) \leq s < \min(r, t).$$

The first step is to determine the unconstrained least-squares estimate of $\hat{M}$ by minimizing

$$\|Y - XM\|^2. \tag{3.3}$$

The unique least-squares solution can be written in generalized matrix form as

$$\hat{M} = X^+ Y, \tag{3.4}$$

when X is of full column rank, this is equivalent to the ordinary least-squares estimator,

$$\hat{M} = \left(X^T X\right)^{-1} X^T Y. \tag{3.5}$$

Hence, the corresponding unconstrained fitted of Y are obtained by

$$\hat{Y} = \hat{X}\hat{M}. \tag{3.6}$$

The next step is to consider the estimation of the matrix M when it is constrained to have rank at most s. First of.all, let us decompose (3.3) as

$$\|Y - XM\|^2 = \|Y - \hat{Y}\|^2 + \|\hat{Y} - XM\|^2. \tag{3.7}$$

The second term only, in the above decomposition, varies as M varies. We may choose M to satisfy $XM = (\hat{Y})_s$, where $(Y)_s$, is the partial sum to the s terms of the SVD of Y. By the Eckart-Young theorem M must minimize the second term in (3.7) and therefore must minimize the least-squares criterion (3.3). Finding $\hat{Y}$, perform the singular value decomposition of $\hat{Y}$ as follows:

$$\hat{Y} = Q \sum V^T \tag{3.8}$$

where $Q = (\underline{q}_1, ...\underline{q}_t); \Sigma = diag(\sigma_1, ..., \sigma_t); V^T = (\underline{v}_1, ...\underline{v}_t)^T$. Now, let M_s^* denote the optimal reduced rank regression coefficient matrix of rank s; then M_s^* can be computed by one of the following two procedures:

1. Evaluate the reduced-rank fitted values $Y_s^* = (\hat{Y})_s$ by taking the partial sum to the s terms of the SVD of Y, and then premultiplying it by X^+, the generalized inverse of X, we get

$$M_s^* = X^+ Y_s^* = X^T (\widehat{Y})_s. \tag{3.9}$$

2. We construct the $(t \times s)$ matrix V_s by taking the first s columns of the matrix V, i.e.

$$V_s = (\underline{v}_1, \ldots \underline{v}_s)$$

then evaluate M_s^* by

$$M_s^* = \hat{M} V_s V_s^T. \tag{3.10}$$

The corresponding reduced-rank fitted values of Y will be given by

$$Y_s^* = (\hat{Y})_s = X M_s^*. \tag{3.11}$$

Indeed both procedures are numerically equivalent. The residual sum of squares resulting from a rank s fit is then

$$\|Y - \hat{Y}\|^2 + \sigma_{s+1}^2 + \cdots + \sigma_t^2$$

i.e. the residual resulting from an unconstrained fit plus the contribution from the least significant singular values of $\hat{Y}$.

4 Determination of the Actual Rank of the Coefficient Matrix in the RRRM

If Y has full rank, i.e has rank t and E are independently and identically normally distributed N$(0, 1)$, then the least-squares estimation is equivalent to maximum likelihood estimation. Consider the model (3.1), then the likelihood function can be given by

$$L(M) = (2\pi)^{-\frac{nt}{2}} \exp\left\{-\frac{1}{2} \operatorname{tr} (Y - XM) (Y - XM)^T\right\} \tag{4.1}$$

and the log-likelihood function is

$$\log L(M) = \frac{-nt}{2} \log(2\pi) - \frac{1}{2} tr \left\{(Y - XM) \left(Y - XM^T\right)\right\} \tag{4.2}$$

$$\begin{aligned} \Lambda_t &= 2 \log L(M) = -nt - tr\left(YY^T\right) + tr\left(YY^T\right) \\ &= -nt - tr\left(YY^T\right) + \sum_{i=1}^{t} \sigma_i^2 \end{aligned} \tag{4.3}$$

where $\left\{\sigma_i^2 \mid i = 1, 2, \ldots\ldots, t\right\}$ are the eigenvalues of $\hat{Y}\hat{Y}^T$. Let $\boldsymbol{M_s^*}$ be the rank s maximum likelihood estimator of, then (4.4) becomes

$$Log L(M_s^*) = \frac{-nt}{2} log(2\pi) - \frac{1}{2} tr \left\{(Y - XM_s^*)(Y - XM_s^*)^T\right\}. \tag{4.4}$$

Letting ${Y_s}^* = XM_s^*$, (4.4) can be written as

$$\begin{aligned} \Lambda_s &= 2 log L(M_s^*) = -nt - tr(YY^T) + tr(Y_S^* Y_S^{*T}) \\ &= -nt - tr(YY^T) + \sum_{i=1}^{s} \sigma_i^2 \end{aligned} \tag{4.5}$$

where $\{\sigma_i^2 \,|i = 1, 2, ..., s\}$ are the eigenvalues of $Y_s^* Y_s^{*T}$. The log-likelihood ratio for testing a rank s model against a full rank model is given by the sum of the (t-s) smallest eigenvalues of $\hat{Y}^T \hat{Y}$. That is

$$\Lambda_t - \Lambda_s = \sigma_{s+1}^2 + ... + \sigma_t^2. \tag{4.6}$$

We reject the null hypothesis that M has rank s against the alternative that it is greater if the (t-s) smallest eigenvalues are not sufficiently small. For large samples, the asymptotic distribution is

$$n \sum_{i=s+1}^{t} \sigma_i^2 \sim {\chi_v}^2, \tag{4.7}$$

where $v = (t-s)(r-s)$ is the number of degrees of freedom, might be used to test s.

5 Numerical Examples (Applications)

In this section, we present same examples to verify the method . For each example, we estimate the reduced -rank regression coefficients matrix using Davies and Tso (1982) [2] procedure and then we apply the likelihood ratio test to determine the rank of the RRRM. A program was written to compute the singular values of the fitted values of Y, M_s^* and the residual sum of squares resulting from a rank s fit.

Example 1:

(a) The data

$$X = \begin{bmatrix} 2.1 & 1.5 & 2.2 \\ 2.3 & 1.1 & 2.5 \\ 1.2 & 2.1 & 2.0 \\ 2.2 & 3.3 & 2.4 \\ 2.3 & 2.0 & 2.4 \end{bmatrix} \quad M = \begin{bmatrix} 1 & 2 \\ 1.5 & 3 \\ 0.5 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1.5 & 1.2 \\ 1.5 & 1.2 \\ 1.5 & 1.2 \\ 1.5 & 1.2 \\ 1.5 & 1.2 \end{bmatrix}$$

There is only one linearly independent column in the matrix M as the second column is twice the first. Therefore the rank of M is 1.

(b) Estimation of coefficients matrix in RRRM

Singular Values of $\hat{Y}$

4.75119 1.46906

Coefficient Matrix $(\boldsymbol{M_s^*})$

$$M_1^* = \begin{bmatrix} -0.18052 & -0.39397 \\ 1.09826 & 2.39690 \\ 2.45988 & 5.3685 \end{bmatrix} \quad M_2^* = \begin{bmatrix} -0.31428 & -0.33268 \\ 1.39774 & 2.25968 \\ -0.28996 & 6.62854 \end{bmatrix}$$

Residual sum of squares

	s=1	s=2
$\Vert\boldsymbol{Y} - \boldsymbol{Y_S^*}\Vert^2$	3.61976	1.46161

(c) Testing of hypotheses and result.

We wish to test:

$$\boldsymbol{H_0 : s = 1}$$
$$\boldsymbol{H_1 : s > 1}.$$

The statistic is

$$\boldsymbol{\sigma_2^2 = 2.158137}$$

and the number of degrees of freedom is

$$(t-s)(r-s) = (2-1)(3-1) = 2.$$

For a level of significance of $\propto = \mathbf{0.01}$, $\chi^2_{\mathbf{2}}(\mathbf{0.01}) = \mathbf{9.21}$. Since $2.158137 < \chi^2_{\mathbf{2}}(\mathbf{0.01})$, we conclude that null hypothesis is true, that is the actual rank of M is 1.

Example 2:

(a) **The data**

$$X = \begin{bmatrix} 1 & 2 & 2 \\ 2.2 & 3 & 1.2 \\ 2.2 & 4 & 1.5 \\ 2.6 & 2.7 & 4 \\ 1.8 & 3.7 & 3 \end{bmatrix} \quad M = \begin{bmatrix} 1 & 3 & -4 \\ -1 & -3 & 4 \\ 2 & 6 & 8 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 4 & 2 \\ 3 & 4 & 2 \\ 3 & 4 & 2 \\ 3 & 4 & 2 \\ 3 & 4 & 2 \end{bmatrix}$$

It is clear that the second column is three the first column. Column three is equal to the negative value of the sum of the first and second columns. Thus, the rank M is 1.

(b) **Estimation of coefficients matrix in RRRM**

Singular Values of $\hat{Y}$

26.94091 1.24209 0.01113

Coefficient Matrix $(\boldsymbol{M^*_s})$

$$M^*_1 = \begin{bmatrix} 1.12757 & 3.66908 & -6.07972 \\ -0.83682 & -2.72299 & 4.39376 \\ 1.65281 & 5.37817 & -7.70734 \end{bmatrix} \quad M^*_2 = \begin{bmatrix} 1.03060 & 2.71183 & -6.07942 \\ -0.78199 & -2.18173 & 4.39376 \\ 1.68514 & 5.69734 & -7.70734 \end{bmatrix}$$

$$M^*_3 = \begin{bmatrix} 1.03377 & 2.71130 & -6.07942 \\ -0.77662 & -2.18263 & 4.39427 \\ 1.68550 & 5.69728 & -7.70731 \end{bmatrix}$$

Residual sum of squares

	s=1	s=2	s=3
$\Vert\boldsymbol{Y} - \boldsymbol{Y^*_S}\Vert^2$	7.99454	6.45175	6.45163

(c) **Testing of hypotheses and result.**

We wish to test:

$$\boldsymbol{H_0 : s = 1}$$

$$\boldsymbol{H_1 : s > 1}$$

The statistic is

$$\boldsymbol{\sigma^2_2 + \sigma^2_3 = 1.5429}.$$

The degrees of freedom is 4 and $\chi^2_{\mathbf{4}}(\mathbf{0.01}) = \mathbf{13.28}$, but

$$1.5429 < \chi^2_{\mathbf{4}}(\mathbf{0.01}).$$

So we accept the null hypothesis, and hence the actual rank of M is 1

Example 3:

(a) The data

$$X = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 2 & 1 & 4 & 2 & 1 \\ 3 & 2 & 1 & 1 & 3 \\ 2 & 1 & 3 & 2 & 1 \\ 3 & 2 & 1 & 4 & 5 \\ 1 & 3 & 4 & 2 & 5 \\ 1 & 2 & 5 & 3 & 4 \end{bmatrix} \quad M = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & -4 \\ 5 & 11 & -1 \\ -4 & -4 & 8 \\ 3 & 5 & -3 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 3 & 1 \\ 2 & 3 & 1 \\ 2 & 3 & 1 \\ 2 & 3 & 1 \\ 2 & 3 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$

Three times of column two is equal to the sum of seven times the first column and twice the third column. Hence, the rank of M is 2.

(b) Estimation of coefficients matrix in RRRM

Singular Values of $\hat{Y}$

42.93819 18.13227 1.44764

Coefficient Matrix $(\boldsymbol{M_s^*})$

$$M_1^* = \begin{bmatrix} 0.89210 & 1.45118 & -0.84816 \\ 2.04185 & 3.32149 & -1.94128 \\ 5.68908 & 9.25443 & -5.40887 \\ -4.56148 & -7.42017 & 4.33681 \\ 3.65144 & 5.93981 & -3.47160 \end{bmatrix} \quad M_2^* = \begin{bmatrix} 0.84736 & 2.61471 & 1.09555 \\ 2.08527 & 2.19241 & -3.82744 \\ 5.59805 & 11.62184 & -1.45404 \\ -4.65003 & -5.11705 & 8.18423 \\ 3.64779 & 6.03492 & -3.31272 \end{bmatrix}$$

$$M_3^* = \begin{bmatrix} 0.78223 & 2.64372 & 1.07668 \\ 2.06269 & 2.20247 & -3.83398 \\ 5.67385 & 11.58807 & -1.43208 \\ -4.65214 & -5.11611 & 8.18362 \\ 3.15592 & 6.25402 & -3.45520 \end{bmatrix}$$

Residual sum of squares

	s=1	s=2	s=3
$\lVert \boldsymbol{Y} - \boldsymbol{Y_S^*} \rVert^2$	336.18325	7.40402	5.30835

(c) Testing of hypotheses and result.

We examine the hypotheses:

$$\boldsymbol{H_0 : s = 1}$$

$$\boldsymbol{H_1 : s > 1}$$

The statistic is

$$\boldsymbol{\sigma_2^2 + \sigma_3^2 = 330.8748}.$$

The degrees of freedom is 8 and $\chi_8^2\,(\mathbf{0.01}) = \mathbf{20.09}$. Since $330.8748 > \chi_8^2\,(\mathbf{0.01})$.

Hence , $\boldsymbol{H_0}$ is rejected. Next, consider the hypotheses that

$$\boldsymbol{H_0 : s = 2}$$

$$\boldsymbol{H_1 : s > 2}.$$

The statistic is $\boldsymbol{\sigma_3^2 = 2.09366157}$. The degrees of freedom is 3 and $\boldsymbol{\chi_3^2\,(0.01) = 11.34}$.
We note that

$$2.09366157 < \chi_3^2\,(\mathbf{0.01})\,.$$

Therefore, the null hypothesis s=2 is accepted and the actual rank of M will be 2.

6 Conclusion

The numerical examples show that LRT is effective in determining the actual rank in the reduced rank regression model.

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