Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 3213–3222
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.23176.2487

Hyers-Ulam-Rassias stability of orthogonality equation on restricted domains

Ginkyu Choi[a], Soon-Mo Jung[b,*]

[a]Department of Electronic and Electrical Engineering, College of Science and Technology, Hongik University, 30016 Sejong, Republic of Korea

[b]Mathematics Section, College of Science and Technology, Hongik University, 30016 Sejong, Republic of Korea

(Communicated by Reza Saadati)

Abstract

In this paper, we prove some theorems about the Hyers-Ulam-Rassias stability of linear isometries. In particular, this paper will address the stability of the orthogonality equation, $\langle f(x), f(y)\rangle = \langle x, y\rangle$, on the restricted domains.

Keywords: isometry, linear isometry, orthogonality equation, Hyers-Ulam-Rassias stability
2020 MSC: Primary 46C99, Secondary 39B82, 39B62, 46B04

1 Introduction

Throughout the paper, we will use $(E, \|\cdot\|)$ and $(F, \|\cdot\|)$ to denote normed spaces over $\mathbb{K}$, unless specifically mentioned about them, where $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. A mapping $f : E \to F$ is called an *isometry* if it satisfies

$$\|f(x) - f(y)\| = \|x - y\| \tag{1.1}$$

for all $x, y \in E$.

Taking into account the definition of Hyers and Ulam [18], for any fixed $\varepsilon \geq 0$, a function $f : E \to F$ is called an *ε-isometry* if f satisfies the inequality

$$\big|\|f(x) - f(y)\| - \|x - y\|\big| \leq \varepsilon \tag{1.2}$$

for all $x, y \in E$. If there exists a constant $K > 0$ such that for any ε-isometry $f : E \to F$, there exists an isometry $U : E \to F$ satisfying $\|f(x) - U(x)\| \leq K\varepsilon$ for all $x \in E$, then we say that the functional equation (1.1) has the *Hyers-Ulam stability* in the class of functions of E into F.

If we weaken the conditions associated with Hyers-Ulam stability, we will be able to study more diverse topics: Assume that $\varphi, \Phi : E \times E \to [0, \infty)$ satisfy some 'mild' conditions. Let $\mathcal{K}$ be a special class of functions of E into F. If for any function $f \in \mathcal{K}$ that satisfies the inequality

$$\big|\|f(x) - f(y)\| - \|x - y\|\big| \leq \varphi(x, y),$$

*Corresponding author
Email addresses: gkchoi@hongik.ac.kr (Ginkyu Choi), smjung@hongik.ac.kr (Soon-Mo Jung)

Received: March 2021 *Accepted:* May 2021

there exists a function $U \in \mathcal{K}$ such that $\|U(x) - U(y)\| = \|x - y\|$ and $\|f(x) - U(x)\| \le \Phi(x, x)$ for all $x, y \in E$, then we say that the functional equation (1.1) has the *Hyers-Ulam-Rassias stability* in the class $\mathcal{K}$. (The term *generalized Hyers-Ulam stability* has been sometimes used for the same concept.) These terminologies also apply to other types of functional equations.

As far as we know, Hyers and Ulam are recognized as the first mathematicians who studied the Hyers-Ulam stability of isometries (see [18]). They indeed used the properties of the inner product to prove the stability of isometries of a real Hilbert space E onto E: For any surjective ε-isometry $f : E \to E$ with $f(0) = 0$, there exists a surjective isometry $U : E \to E$ such that $\|f(x) - U(x)\| \le 10\varepsilon$ for all $x \in E$.

This result of Hyers and Ulam was generalized by Bourgin [2]: Assume that E is a real Banach space and that F belongs to the class of uniformly convex real Banach spaces which includes the spaces $L_p(0, 1)$ for $1 < p < \infty$. For any ε-isometry $f : E \to F$ with $f(0) = 0$, there exists a linear isometry $U : E \to F$ such that $\|f(x) - U(x)\| \le 12\varepsilon$ for every $x \in E$.

After that, Hyers and Ulam [19] studied the stability problem for the Banach spaces of continuous functions: Assume that S_1 and S_2 are compact metric spaces and $C(S_i)$ is the Banach space of real-valued continuous functions on S_i equipped with the supremum norm $\| \cdot \|_\infty$. If a homeomorphism $T : C(S_1) \to C(S_2)$ satisfies the inequality

$$\big| \|T(f) - T(g)\|_\infty - \|f - g\|_\infty \big| \le \varepsilon \tag{1.3}$$

for some $\varepsilon \ge 0$ and for all $f, g \in C(S_1)$, then there is an isometry $U : C(S_1) \to C(S_2)$ such that $\|T(f) - U(f)\|_\infty \le 21\varepsilon$ for any $f \in C(S_1)$.

This very result of Hyers and Ulam was once again widely generalized by Bourgin (see [4]): Suppose that S_1 and S_2 are completely regular Hausdorff spaces and $T : C(S_1) \to C(S_2)$ is a surjective function that satisfies the inequality (1.3) for some $\varepsilon \ge 0$ and for all $f, g \in C(S_1)$. Then there is a linear isometry $U : C(S_1) \to C(S_2)$ with $\|T(f) - U(f)\|_\infty \le 10\varepsilon$ for any $f \in C(S_1)$. After that, Bourgin [5] continued to study the stability of isometries in finite-dimensional Banach space.

In 1978, Gruber [16] proved the following elaborate theorem: Let E and F be real normed spaces. Assume that $f : E \to F$ is a surjective ε-isometry. Moreover, suppose $U : E \to F$ is an isometry with $f(p) = U(p)$ for a $p \in E$. If $\|f(x) - U(x)\| = o(\|x\|)$ as $\|x\| \to \infty$ uniformly, then U is a surjective linear isometry and $\|f(x) - U(x)\| \le 5\varepsilon$ for all $x \in E$. Furthermore, if f is continuous, then $\|f(x) - U(x)\| \le 3\varepsilon$ for any $x \in E$.

Years later, Gevirtz [15] studied the stability of isometries defined between real Banach spaces: Let E and F be real Banach spaces. For each surjective ε-isometry $f : E \to F$, there exists a surjective isometry $U : E \to F$ that satisfies $\|f(x) - U(x)\| \le 5\varepsilon$ for any $x \in E$. After that, the upper bound 5ε was improved to a sharper 2ε by Omladič and Šemrl [24].

Due to [13, 29], we say that a function $f : E \to F$ is an (ε, p)*-isometry* when f satisfies

$$\big| \|f(x) - f(y)\| - \|x - y\| \big| \le \varepsilon \|x - y\|^p$$

for some constants $p \ge 0$ and $\varepsilon \ge 0$ and for all $x, y \in E$. Dolinar [13] demonstrated the *superstability* of (ε, p)-isometries: If p is larger than 1, then every surjective (ε, p)-isometry $f : E \to F$ between finite-dimensional real Banach spaces is an isometry.

There are many other papers related to the stability of isometries, but it is unfortunate that we cannot quote them all in this paper due to page limitations. Nevertheless, see [1, 3, 14, 20, 21, 23, 26, 27, 28, 30, 31, 32, 33] for more general information on the stability of isometries and related topics.

The *orthogonality equation*

$$\langle f(x), f(y) \rangle = \langle x, y \rangle \tag{1.4}$$

is well known to characterize linear isometries that map a Hilbert space into another one. It would be interesting and meaningful to study the stability of the orthogonality equation in order to shed new light on the characteristics of linear isometries as well as to study the stability of the equation (1.1).

In this paper, the Hyers-Ulam-Rassias stability problems for linear isometries are studied in several different perspectives. More precisely, we prove some theorems about the Hyers-Ulam-Rassias stability of the orthogonality equation on the restricted domains.

2 Preliminaries

From now on, we use E and F to denote Hilbert spaces over $\mathbb{K}$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. A function $f : E \to F$ is said to be *inner product preserving* if and only if f is a solution to the orthogonality equation (1.4) for any $x, y \in E$. It is to be noted that a function $f : E \to F$ is a solution to the orthogonality equation (1.4) if and only if it is a linear isometry.

On the other hand, a function $f : E \to F$ is a solution to the functional equation

$$\langle f(x), f(y)\rangle = \langle y, x\rangle$$

for any $x, y \in E$ if and only if it is a *conjugate-linear isometry*, *i.e.*, f is an isometry that satisfies $f(\lambda x + \mu y) = \overline{\lambda} f(x) + \overline{\mu} f(y)$ for all $x, y \in E$ and $\lambda, \mu \in \mathbb{K}$.

We define $\mathcal{S} = \{z \in \mathbb{K} : |z| = 1\}$. Any two functions $f, g : E \to F$ are said to be *phase-equivalent* if and only if there exists a function $\sigma : E \to \mathcal{S}$ such that $g(x) = \sigma(x)f(x)$ for any $x \in E$.

The functional equation

$$|\langle f(x), f(y)\rangle| = |\langle x, y\rangle| \tag{2.1}$$

is called the *Wigner equation* or the *generalized orthogonality equation*. The following theorem was introduced in [34].

Theorem 2.1. If a function $f : E \to F$ satisfies the Wigner equation (2.1) for all $x, y \in E$, then f is phase-equivalent to a linear isometry or a conjugate-linear isometry.

Let $(E, \langle\cdot,\cdot\rangle)$ be a Hilbert space with the associated norm defined as $\|x\| = \sqrt{|\langle x, x\rangle|}$. For any fixed $d \ge 0$, we define either $D = \{x \in E : \|x\| \ge d\}$ or $D = \{x \in E : \|x\| \le d\}$. (In the latter case, we will exclude the trivial case of $D = \{0\}$). Now we choose an appropriate constant c that satisfies the following condition:

$$\begin{cases} 0 < c < 1 & (\text{for } D = \{x \in E : \|x\| \ge d\}), \\ c > 1 & (\text{for } D = \{x \in E : \|x\| \le d\}). \end{cases} \tag{2.2}$$

In practice, we will either choose an arbitrary constant c that satisfies the conditions (2.2) and (2.3), or an arbitrary constant c that satisfies the conditions (2.2) and (2.5) as we see in several theorems and corollaries below. The following theorem was proved in [11, Theorem 1] (or see [12, Theorem 3]).

Theorem 2.2. Let E and F be Hilbert spaces over $\mathbb{K}$. Assume that a function $\varphi : E \times E \to [0, \infty)$ satisfies the following condition

$$\lim_{m+n\to\infty} c^{m+n} \varphi\left(\frac{1}{c^m}x, \frac{1}{c^n}y\right) = 0 \tag{2.3}$$

for all $x, y \in D$. If a function $f : E \to F$ satisfies the inequality

$$\big||\langle f(x), f(y)\rangle| - |\langle x, y\rangle|\big| \le \varphi(x, y) \tag{2.4}$$

for all $x, y \in D$, then there exists a unique function $I : E \to F$ (up to a phase-equivalent function) that satisfies the Wigner equation (2.1) for all $x, y \in E$ as well as

$$\|f(x) - I(x)\| \le \sqrt{\varphi(x, x)}$$

for all $x \in D$.

It follows from Theorem 2.1 that if E and F are real Hilbert spaces, then the function $I : E \to F$ in Theorem 2.2 has to be phase-equivalent to a linear isometry as we see in the following corollary.

Corollary 2.3. Let E and F be real Hilbert spaces. Assume that a function $\varphi : E\times E \to [0,\infty)$ satisfies the condition (2.3) for all $x, y \in D$. If a function $f : E \to F$ satisfies the inequality (2.4) for all $x, y \in D$, then there exist a function $\sigma : E \to \{-1, 1\}$ and a linear isometry $U : E \to F$ that satisfy $\|f(x) - \sigma(x)U(x)\| \le \sqrt{\varphi(x,x)}$ for all $x \in D$.

Proof . According to Theorem 2.2, there exists a unique function $I : E \to F$ (up to a phase-equivalent function) that satisfies the Wigner equation (2.1) for all $x, y \in E$ as well as

$$\|f(x) - I(x)\| \le \sqrt{\varphi(x,x)}$$

for all $x \in D$. Due to Theorem 2.1, the function $I : E \to F$ is phase-equivalent to a linear isometry, since E and F are real Hilbert spaces. That is, there exist a function $\sigma : E \to \{-1, 1\}$ and a linear isometry $U : E \to F$ such that $I(x) = \sigma(x)U(x)$ for any $x \in E$. □

The Hyers-Ulam-Rassias stability of the orthogonality equation was proved in [11, Theorem 2] using a similar method used to prove Theorem 2.2. The result mentioned above is introduced in the following theorem, omitting the proof.

Theorem 2.4. Let E and F be Hilbert spaces over $\mathbb{K}$. Assume that a function $\varphi : E \times E \to [0,\infty)$ satisfies the condition (2.3) for all $x, y \in D$. If a function $f : E \to F$ satisfies the inequality

$$\big|\langle f(x), f(y)\rangle - \langle x, y\rangle\big| \le \varphi(x,y)$$

for all $x, y \in D$, then there exists a unique linear isometry $U : E \to F$ that satisfies

$$\|f(x) - U(x)\| \le \sqrt{\varphi(x,x)}$$

for all $x \in D$.

In the following theorem, the upper bound of the inequality (2.7) may be larger than the upper bound given in Theorem 2.4. This is an obvious drawback of the following theorem. However, it is an advantage of the following theorem to weaken the condition (2.3) so that the next theorem has a more broad application than Theorem 2.4. Indeed, the new condition (2.5) may not look much different from the old condition (2.3), but the various theorems introduced in the next section will show that the new condition has significantly weakened the old one. That is, Theorem 2.5 is an improved version of Theorem 2.4 and the former will play a central role in this paper. We will see in the next section that the limit values included in the inequality (2.7) below become nonnegative constants in the usual application environment.

The proof of the following theorem is based strongly on the well known properties of the inner product of Hilbert spaces and the direct method that was first conceived by Hyers [17]. Since the following theorem has been proved in detail in [12, Theorem 5], the proof will be omitted here.

Theorem 2.5. Let E and F be Hilbert spaces over $\mathbb{K}$ and assume that a function $\varphi : E \times E \to [0,\infty)$ satisfies the condition

$$\lim_{m,n\to\infty} c^{m+n}\varphi\left(\frac{1}{c^m}x, \frac{1}{c^n}y\right) = 0 \tag{2.5}$$

for all $x, y \in D$. If a function $f : E \to F$ satisfies the inequality

$$\big|\langle f(x), f(y)\rangle - \langle x, y\rangle\big| \le \varphi(x,y) \tag{2.6}$$

for all $x, y \in D$, then there exists a unique linear isometry $U : E \to F$ that satisfies

$$\|f(x) - U(x)\| \le \sqrt{\varphi(x,x) + \lim_{n\to\infty} c^n\varphi\left(\frac{1}{c^n}x, x\right) + \lim_{n\to\infty} c^n\varphi\left(x, \frac{1}{c^n}x\right)} \tag{2.7}$$

for all $x \in D$.

3 Stability of linear isometries

In this section, we will use E and F to denote some Hilbert spaces over $\mathbb{K}$, unless specifically mentioned about them, where $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.

When the control function has the form of $\varphi(x, y) = \theta\|x\|^p\|y\|^q$ which was first suggested by Rassias [25] for approximate additive functions, it would be quite interesting to study the stability problem of the orthogonality equation, depending on the form of D.

Theorem 3.1. Assume that d, p, q, and θ are constants with $d \geq 0$, $p \geq 1$, $q \geq 1$ (excepting the case of $p = q = 1$), and $\theta > 0$. Furthermore, let $D = \{x \in E : \|x\| \leq d\}$. If a function $f : E \to F$ satisfies the inequality

$$\big|\langle f(x), f(y)\rangle - \langle x, y\rangle\big| \leq \theta\|x\|^p\|y\|^q \tag{3.1}$$

for all $x, y \in D$, then there exists a unique linear isometry $U : E \to F$ that satisfies

$$\|f(x) - U(x)\| \leq \begin{cases} \sqrt{\theta\|x\|^{p+q}} & (\text{for } p > 1 \text{ and } q > 1), \\ \sqrt{2\theta\|x\|^{p+q}} & (\text{for either } p = 1 \text{ or } q = 1) \end{cases}$$

for all $x \in D$.

Proof . Due to the condition (2.2), we select a constant c satisfying $c > 1$. Let us define a function $\varphi : E\times E \to [0, \infty)$ by $\varphi(x, y) = \theta\|x\|^p\|y\|^q$ for all $x, y \in E$. Since one of p and q has to be larger that 1, we show that

$$\lim_{m,n\to\infty} c^{m+n}\varphi\left(\frac{1}{c^m}x, \frac{1}{c^n}y\right) = \lim_{m,n\to\infty} \theta\frac{c^{m+n}}{c^{pm+qn}}\|x\|^p\|y\|^q = 0$$

for all $x, y \in D$, which states that φ satisfies the condition (2.5) for all $x, y \in D$.

We know that

$$\lim_{n\to\infty} c^n\varphi\left(\frac{1}{c^n}x, x\right) = \lim_{n\to\infty} \theta\frac{c^n}{c^{pn}}\|x\|^{p+q} = \begin{cases} \theta\|x\|^{p+q} & (\text{for } p = 1), \\ 0 & (\text{for } p > 1) \end{cases}$$

and

$$\lim_{n\to\infty} c^n\varphi\left(x, \frac{1}{c^n}x\right) = \lim_{n\to\infty} \theta\frac{c^n}{c^{qn}}\|x\|^{p+q} = \begin{cases} \theta\|x\|^{p+q} & (\text{for } q = 1), \\ 0 & (\text{for } q > 1). \end{cases}$$

Since either $p > 1$ and $q > 1$, or $p = 1$ and $q > 1$, or $p > 1$ and $q = 1$, according to Theorem 2.5, there exists a unique linear isometry $U : E \to F$ that satisfies the given inequality for all $x \in D$. □

Theorem 3.2. Assume that d, p, q, and θ are constants with $d > 0$, $p \leq 1$, $q \leq 1$ (excepting the case of $p = q = 1$), and $\theta > 0$. Moreover, let $D = \{x \in E : \|x\| \geq d\}$. If a function $f : E \to F$ satisfies the inequality (3.1) for all $x, y \in D$, then there exists a unique linear isometry $U : E \to F$ such that

$$\|f(x) - U(x)\| \leq \begin{cases} \sqrt{\theta\|x\|^{p+q}} & (\text{for } p < 1 \text{ and } q < 1), \\ \sqrt{2\theta\|x\|^{p+q}} & (\text{for either } p = 1 \text{ or } q = 1) \end{cases}$$

for all $x \in D$.

Proof . Considering the condition (2.2), we fix a constant c with $0 < c < 1$. We define a function $\varphi : E \times E \to [0, \infty)$ by $\varphi(x, y) = \theta\|x\|^p\|y\|^q$ for all $x, y \in E$. Since $0 < c < 1$ and one of p and q is smaller than 1, we have

$$\lim_{m,n\to\infty} c^{m+n}\varphi\left(\frac{1}{c^m}x, \frac{1}{c^n}y\right) = \lim_{m,n\to\infty} \theta\frac{c^{m+n}}{c^{pm+qn}}\|x\|^p\|y\|^q = 0$$

for all $x, y \in D$. Thus, φ satisfies the condition (2.5) for any $x, y \in D$.

Moreover, it holds that

$$\lim_{n\to\infty} c^n \varphi\left(\frac{1}{c^n}x, x\right) = \lim_{n\to\infty} \theta \frac{c^n}{c^{pn}} \|x\|^{p+q} = \begin{cases} 0 & (\text{for } p < 1), \\ \theta\|x\|^{p+q} & (\text{for } p = 1) \end{cases}$$

and

$$\lim_{n\to\infty} c^n \varphi\left(x, \frac{1}{c^n}x\right) = \lim_{n\to\infty} \theta \frac{c^n}{c^{qn}} \|x\|^{p+q} = \begin{cases} 0 & (\text{for } q < 1), \\ \theta\|x\|^{p+q} & (\text{for } q = 1). \end{cases}$$

Since either $p < 1$ and $q < 1$, or $p = 1$ and $q < 1$, or $p < 1$ and $q = 1$, due to Theorem 2.5, there exists a unique linear isometry $U : E \to F$ that satisfies the given inequality for all $x \in D$. □

If $0 \le p \le 1$ and $0 \le q \le 1$ but not $p = q = 1$, then we can assume that $d \ge 0$ in Theorem 3.2. It would be interesting to compare our Theorem 3.1 and Theorem 3.2 with [9, Theorem 2]. We note that [9, Theorem 2] is the result on the case when the function f satisfies inequality (3.1) for all $x, y \in E$, at most excluding the origin, and $p = q \neq 1$, while the function f satisfies inequality (3.1) for all $x, y \in D$ in our theorems. In particular, for the case when E is finite-dimensional and $p = q = 1$, Chmielinski [10, Theorem 2] proved a similar theorem as [9, Theorem 2].

When $D = \{x \in E : \|x\| \ge d\}$, we can prove the Hyers-Ulam stability of the orthogonality equation on the restricted domain D.

Theorem 3.3. Assume that d and ε are constants satisfying $d \ge 0$ and $\varepsilon > 0$, and let $D = \{x \in E : \|x\| \ge d\}$. If a function $f : E \to F$ satisfies the following inequality

$$\big|\langle f(x), f(y)\rangle - \langle x, y\rangle\big| \le \varepsilon \tag{3.2}$$

for all $x, y \in D$, then there exists a unique linear isometry $U : E \to F$ that satisfies

$$\|f(x) - U(x)\| \le \sqrt{\varepsilon}$$

for all $x \in D$.

Proof . First, we consider the condition (2.2) and select a constant c satisfying $0 < c < 1$. In addition, we set $\varphi(x, y) = \varepsilon$ for all $x, y \in E$. Then, we easily verify that

$$\lim_{m,n\to\infty} c^{m+n} \varphi\left(\frac{1}{c^m}x, \frac{1}{c^n}y\right) = \lim_{m,n\to\infty} \varepsilon c^{m+n} = 0$$

for all $x, y \in D$, *i.e.*, φ satisfies the condition (2.5) for all $x, y \in D$.

Since

$$\lim_{n\to\infty} c^n \varphi\left(\frac{1}{c^n}x, x\right) = \lim_{n\to\infty} \varepsilon c^n = 0$$

and

$$\lim_{n\to\infty} c^n \varphi\left(x, \frac{1}{c^n}x\right) = \lim_{n\to\infty} \varepsilon c^n = 0,$$

by Theorem 2.5, there exists a unique linear isometry $U : E \to F$ satisfying the given inequality for all $x \in D$. □

In the case of $D = \{x \in E : \|x\| \le d\}$, we are not able to prove Theorem 3.3 by applying Theorem 2.5 because we have to select a constant c larger than 1 and $\varphi(x, y) = \varepsilon$ in this case, and hence the condition (2.5) is not satisfied unfortunately.

We remark that Chmielinski [6] proved the superstability of the orthogonality equation for the class of functions $f : \mathbb{R}^n \to \mathbb{R}^n$, where n is an integer larger than 1. More precisely, if a function $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the inequality (3.2) for all $x, y \in \mathbb{R}^n$, then f is a solution to the orthogonality equation, *i.e.*, f is a linear isometry.

It would be interesting to study Theorem 2.5 when the control function has the form of $\varphi(x, y) = \theta(\|x\|^p + \|y\|^q)$. We would like to point out that we cannot prove the following theorem by using Theorem 2.4. From this fact we can be sure that Theorem 2.5 is more useful than Theorem 2.4.

Theorem 3.4. Assume that d, p, q, and θ are constants with $d > 0$, $p \le 1$, $q \le 1$, and $\theta > 0$. Moreover, let $D = \{x \in E : \|x\| \ge d\}$. If a function $f : E \to F$ satisfies the inequality

$$\big|\langle f(x), f(y)\rangle - \langle x, y\rangle\big| \le \theta\big(\|x\|^p + \|y\|^q\big) \tag{3.3}$$

for all $x, y \in D$, then there exists a unique linear isometry $U : E \to F$ such that

$$\|f(x) - U(x)\| \le \begin{cases} \sqrt{\theta(\|x\|^p + \|x\|^q)} & (\text{for } p < 1 \text{ and } q < 1), \\ \sqrt{\theta(2\|x\| + \|x\|^q)} & (\text{for } p = 1 \text{ and } q < 1), \\ \sqrt{\theta(\|x\|^p + 2\|x\|)} & (\text{for } p < 1 \text{ and } q = 1), \\ 2\sqrt{\theta\|x\|} & (\text{for } p = 1 \text{ and } q = 1) \end{cases}$$

for all $x \in D$.

Proof . Due to (2.2), we choose a constant c with $0 < c < 1$ and let $\varphi : E \times E \to [0, \infty)$ be a function defined by $\varphi(x, y) = \theta\big(\|x\|^p + \|y\|^q\big)$ for all $x, y \in E$. Since $p \le 1$ and $q \le 1$, we have

$$\lim_{m,n\to\infty} c^{m+n}\varphi\left(\frac{1}{c^m}x, \frac{1}{c^n}y\right) = \lim_{m,n\to\infty} \theta\left(\frac{c^{m+n}}{c^{pm}}\|x\|^p + \frac{c^{m+n}}{c^{qn}}\|y\|^q\right) = 0$$

for all $x, y \in D$, which implies that φ satisfies the condition (2.5) for all $x, y \in D$.

Since

$$\lim_{n\to\infty} c^n\varphi\left(\frac{1}{c^n}x, x\right) = \lim_{n\to\infty} \theta\left(\frac{c^n}{c^{pn}}\|x\|^p + c^n\|x\|^q\right) = \begin{cases} 0 & (\text{for } p < 1), \\ \theta\|x\| & (\text{for } p = 1) \end{cases}$$

and

$$\lim_{n\to\infty} c^n\varphi\left(x, \frac{1}{c^n}x\right) = \lim_{n\to\infty} \theta\left(c^n\|x\|^p + \frac{c^n}{c^{qn}}\|x\|^q\right) = \begin{cases} 0 & (\text{for } q < 1), \\ \theta\|x\| & (\text{for } q = 1), \end{cases}$$

Theorem 2.5 implies that there exists a unique linear isometry $U : E \to F$ that satisfies the given inequality for all $x \in D$. □

If $0 \le p \le 1$ and $0 \le q \le 1$, then we can assume that $d \ge 0$ in Theorem 3.4. When $D = \{x \in E : \|x\| \le d\}$, we are not able to prove Theorem 3.4 by using Theorem 2.5 because we should select some $c > 1$ and the condition (2.5) does not seem to be satisfied under any selection of the values of p and q.

It would also be interesting to introduce the following theorem dealing with the case of $\varphi(x, y) = \varepsilon\|x - y\|^p$ in connection with (ε, p)-isometries.

Theorem 3.5. Assume that d, p, and ε are constants with $d \ge 0$, $0 < p \le 1$, and $\varepsilon > 0$. Moreover, define $D = \{x \in E : \|x\| \ge d\}$. If a function $f : E \to F$ satisfies the inequality

$$\big|\langle f(x), f(y)\rangle - \langle x, y\rangle\big| \le \varepsilon\|x - y\|^p$$

for all $x, y \in D$, then there exists a unique linear isometry $U : E \to F$ that satisfies

$$\|f(x) - U(x)\| \le \begin{cases} 0 & (\text{for } 0 < p < 1), \\ \sqrt{2\varepsilon\|x\|} & (\text{for } p = 1) \end{cases}$$

for all $x \in D$.

Proof . First, we refer to the condition (2.2) and select an appropriate constant c satisfying $0 < c < 1$. Assume that a function $\varphi : E \times E \to [0, \infty)$ is defined by $\varphi(x, y) = \varepsilon \|x - y\|^p$ for all $x, y \in E$. Since $0 < c < 1$ and $0 < p \le 1$, we have

$$\lim_{m,n\to\infty} c^{m+n}\varphi\left(\frac{1}{c^m}x, \frac{1}{c^n}y\right) = \lim_{m,n\to\infty} \varepsilon c^{(1-p)(m+n)}\|c^n x - c^m y\|^p = 0$$

for all $x, y \in D$, which implies that φ satisfies the condition (2.5) for all $x, y \in D$.

We know that

$$\lim_{n\to\infty} c^n\varphi\left(\frac{1}{c^n}x, x\right) = \lim_{n\to\infty} \varepsilon\frac{c^n}{c^{pn}}\|x - c^n x\|^p = \begin{cases} 0 & (\text{for } 0 < p < 1), \\ \varepsilon\|x\| & (\text{for } p = 1) \end{cases}$$

and

$$\lim_{n\to\infty} c^n\varphi\left(x, \frac{1}{c^n}x\right) = \lim_{n\to\infty} \varepsilon\frac{c^n}{c^{pn}}\|c^n x - x\|^p = \begin{cases} 0 & (\text{for } 0 < p < 1), \\ \varepsilon\|x\| & (\text{for } p = 1). \end{cases}$$

Hence, our assertion immediately follows from Theorem 2.5. □

Unfortunately, we do not yet know if Theorem 3.5 holds for the case where $D = \{x \in E : \|x\| \le d\}$ and $p > 1$.

We may compare Theorem 3.5 with [22, Theorem 5], in which the stability of the equation (1.1) instead of (1.4) was proved under the assumptions that $D = \{x \in E : \|x\| \le d\}$ and $p > 1$.

4 Discussion

It is widely known that the orthogonality equation is an equation that determines the linear isometries between the Hilbert spaces over $\mathbb{K}$. Therefore, it would be very interesting and meaningful to study the stability of the orthogonality equation in order to shed new light on the characteristics of linear isometric as well as to study the stability of the equation (1.1).

As far as we know, Chmielinski was the first mathematician to study the stability of the orthogonality equation. Indeed, he mainly studied the Hyers-Ulam stability of the orthogonality equation on the whole space, not the restricted domain (see [7, 8]).

It is an important difference from Chmielinski's papers that the present paper deals with the stability problems of the orthogonality equation by specially focusing on the stability on the restricted domains.

By using the fact that the function $y = \sqrt{x}$ is concave on $x \ge 0$, we can easily prove the following inequality. Hence, we omit the proof.

Lemma 4.1. For all real numbers $a, b \ge 0$, it holds that

(i) $\sqrt{a + b} \le \sqrt{a} + \sqrt{b}$;

(ii) $\left|\sqrt{a} - \sqrt{b}\right| \le \sqrt{|a - b|}$;

(iii) $|a - b| \le 2\max\left\{\sqrt{a}, \sqrt{b}\right\}\left|\sqrt{a} - \sqrt{b}\right|$.

Using [22, Theorem 3], we can prove the Hyers-Ulam stability of the orthogonality equation on a restricted domain.

Theorem 4.2. Let E be a real Hilbert space, $d \ge 1$, and $0 < \varepsilon \le \frac{1}{20}d$ be fixed, and define $D_0^\circ = \{x \in E : \|x\| > d\} \cup \{0\}$. If a function $f : E \to E$ satisfies the inequality

$$\left|\langle f(x), f(y)\rangle - \langle x, y\rangle\right| \le \varepsilon$$

for all $x, y \in D_0^\circ$, then there exists a unique linear isometry $U : E \to E$ such that

$$\|f(x) - U(x) - f(0)\| \le 5\big(2 + \sqrt{2}\big)\sqrt{\varepsilon}$$

for all $x \in D_0^\circ$.

Proof . If we set $a = \langle f(x) - f(y), f(x) - f(y)\rangle$ and $b = \langle x - y, x - y\rangle$, then it follows from Lemma 4.1 (i) and (ii) and (3.2) that

$$\begin{aligned}
&\big|\|f(x) - f(y)\| - \|x - y\|\big| \\
&\quad = \left|\sqrt{\langle f(x) - f(y), f(x) - f(y)\rangle} - \sqrt{\langle x - y, x - y\rangle}\,\right| \\
&\quad \le \sqrt{|\langle f(x) - f(y), f(x) - f(y)\rangle - \langle x - y, x - y\rangle|} \\
&= \sqrt{|(\langle f(x), f(x)\rangle - \langle x, x\rangle) - 2(\langle f(x), f(y)\rangle - \langle x, y\rangle) + (\langle f(y), f(y)\rangle - \langle y, y\rangle)|} \\
&\le \sqrt{|\langle f(x), f(x)\rangle - \langle x, x\rangle|} + \sqrt{2\,|\langle f(x), f(y)\rangle - \langle x, y\rangle|} + \sqrt{|\langle f(y), f(y)\rangle - \langle y, y\rangle|} \\
&\le \big(2 + \sqrt{2}\big)\sqrt{\varepsilon}
\end{aligned}$$

for all $x, y \in D_0^\circ$. According to [22, Theorem 3], there exists a unique linear isometry $U : E \to E$ such that

$$\|f(x) - U(x) - f(0)\| \le 5\big(2 + \sqrt{2}\big)\sqrt{\varepsilon}$$

for all $x \in D_0^\circ$. □

As we can see, the result of Theorem 4.2 is not better than that of Theorem 3.3. Taking this into account, we can see that Theorem 2.5 is very useful.

Acknowledgments

This work was supported by 2022 Hongik University Research Fund. This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2020R1F1A1A01049560).

References

[1] J.A. Baker, *Isometries in normed spaces*, Amer. Math. Month. **78** (1971), no. 6, 655–658.

[2] D.G. Bourgin, *Approximate isometries*, Bull. Amer. Math. Soc. **52** (1946), 704–714.

[3] R. Bhatia and P. Šemrl, *Approximate isometries on Euclidean spaces*, Amer. Math. Monthly **104** (1997), 497–504.

[4] D.G. Bourgin, *Approximately isometric and multiplicative transformations on continuous function rings*, Duke Math. J. **16** (1949), 385–397.

[5] R.D. Bourgin, *Approximate isometries on finite dimensional Banach spaces*, Trans. Amer. Math. Soc. **207** (1975), 309–328.

[6] J. Chmieliński, *On the superstability of the generalized orthogonality equation in Euclidean spaces*, Ann. Math. Sil. **8** (1994), 127–140.

[7] J. Chmieliński, *On the Hyers-Ulam stability of the generalized orthogonality equation in real Hilbert spaces*, Stability of Mappings of Hyers-Ulam Type (Th. M. Rassias and J. Tabor, ed.), Hadronic Press, Palm Harbor, 1994, pp. 31–41.

[8] J. Chmieliński, *The stability of the Wigner equation in complex Hilbert spaces*, Wyż. Szkoła Ped. Kraków Rocznik Nauk.-Dydakt. Prace Matematyczne **15** (1998), 49–55.

[9] J. Chmieliński, *On a singular case in the Hyers-Ulam-Rassias stability of the Wigner equation*, J. Math. Anal. Appl. **289** (2004), 571–583.

[10] J. Chmieliński, *Stability of the orthogonality preserving property in finite-dimensional inner product spaces*, J. Math. Anal. Appl. **318** (2006), 433–443.

[11] J. Chmielinski and S.-M. Jung, *The stability of the Wigner equation on a restricted domain*, J. Math. Anal. Appl. **254** (2001), 309–320.

[12] G. Choi and S.-M. Jung, *The stability of isometries on restricted domains*, Symmetry-Basel **13** (2021), Issue 2, 282, 11 pages.

[13] G. Dolinar, *Generalized stability of isometries*, J. Math. Anal. Appl. **242** (2000), 39–56.

[14] J.W. Fickett, *Approximate isometries on bounded sets with an application to measure theory*, Studia Math. **72** (1981), 37–46.

[15] J. Gevirtz, *Stability of isometries on Banach spaces*, Proc. Amer. Math. Soc. **89** (1983), 633–636.

[16] P.M. Gruber, *Stability of isometries*, Trans. Amer. Math. Soc. **245** (1978), 263–277.

[17] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.

[18] D.H. Hyers and S.M. Ulam, *On approximate isometries*, Bull. Amer. Math. Soc. **51** (1945), 288–292.

[19] D.H. Hyers and S.M. Ulam, *Approximate isometries of the space of continuous functions*, Ann. Math. **48** (1947), 285–289.

[20] S.-M. Jung, *Hyers-Ulam-Rassias stability of isometries on restricted domains*, Nonlinear Stud. **8** (2001), 125–134.

[21] S.-M. Jung, *Asymptotic properties of isometries*, J. Math. Anal. Appl. **276** (2002), 642–653.

[22] S.-M. Jung and B. Kim, *Stability of isometries on restricted domains*, J. Korean Math. Soc. **37** (2000), no. 1, 125–137.

[23] J. Lindenstrauss and A. Szankowski, *Non linear perturbations of isometries*, Astérisque **131** (1985), 357–371.

[24] M. Omladič and P. Šemrl, *On non linear perturbations of isometries*, Math. Ann. **303** (1995), 617–628.

[25] J.M. Rassias, *On a new approximation of approximately linear mappings by linear mappings*, Discuss. Math. **7** (1985), 193–196.

[26] Th.M. Rassias, *Properties of isometric mappings*, J. Math. Anal. Appl. **235** (1999), 108–121.

[27] Th.M. Rassias, *Isometries and approximate isometries*, Internat. J. Math. Math. Sci. **25** (2001), 73–91.

[28] Th.M. Rassias and C. S. Sharma, *Properties of isometries*, J. Natural Geom. **3** (1993), 1–38.

[29] P. Šemrl, *Hyers-Ulam stability of isometries on Banach spaces*, Aequationes Math. **58** (1999), 157–162.

[30] F. Skof, *Sulle δ-isometrie negli spazi normati*, Rendiconti Di Mat. Ser. VII, Roma **10** (1990), 853–866.

[31] F. Skof, *On asymptotically isometric operators in normed spaces*, Ist. Lomb. Sc. Lett. A **131** (1997), 117–129.

[32] R.L. Swain, *Approximate isometries in bounded spaces*, Proc. Amer. Math. Soc. **2** (1951), 727–729.

[33] J. Väisälä, *Isometric approximation property of unbounded sets*, Result. Math. **43** (2003), 359–372.

[34] E.P. Wigner, *Gruppentheorie und ihre Anwendungen auf die Quantenmechanik der Atomspektren*, Friedr. Vieweg und Sohn, Braunschweig, 1931.