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A note on sharpening of Erdös-Lax and Turán-type inequalities for a constrained polynomial

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Abstract

The well-known Erdös-Lax and Turán-type inequalities, which relate the uniform norm of a univariate complex coefficient polynomial to its derivative on the unit circle in the plane, are discussed in this paper. We create some new inequalities here when there is a restriction on its zeros. The obtained results strengthen some recently proved Erdös-Lax and Turán-type inequalities for constrained polynomials and also produce various inequalities that are sharper than the previous ones known in a very rich literature on this subject.

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1 Introduction

In scientific investigations, experimental observations are converted into mathematical language, resulting in mathematical models. In order to solve these models, it may be necessary to calculate how large or small the maximum modulus of an algebraic polynomial derivative can be in terms of the polynomials maximum modulus. In practise, having bounds for these types of circumstances is critical. Because no closed formulae exist for exactly estimating these constraints, and the only information available in the literature is in the form of tables, approximations. These estimated boundaries are quite enough when computed efficiently suit the needs of scientists and investigators. As a result, there is a perpetual need to go forward to look for improved and better bounds than those provided in the literature. This need for more refined and updated bounds is what has motivated us to write this note. A fertile topic in analysis is the inequalities for polynomials and their derivatives, which generalise the classical inequalities for various norms and with varied constraints on utilising different approaches of geometric function theory. In the literature, for proving the inverse theorems in approximation theory, many inequalities in both directions relating the norm of the derivative and the polynomial itself play a significant role and, of course, have their own intrinsic appeal. Many research papers have been published on these inequalities for constrained polynomials, as evidenced by many recent studies (for example, see [10], [13], [17], [19]-[21]). We begin with the well-known Bernstein inequality [4] for the uniform norm on the unit disk in the plane: namely, if $P(z)$ is a polynomial of degree n, then

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}$$

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If we limit ourselves to polynomials with no zeros in $|z| < 1$, the above inequality (1.1) can then be emphasised. In fact, Erdös conjectured and later Lax [14] proved that, if $P(z) \neq 0$ in $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.2}$$

The inequality (1.2) is sharp and equality holds if $P(z)$ has all of its zeros on $|z| = 1$.
On the other hand, Turán's classical inequality [25] provides a lower bound estimate to the size of the derivative of a polynomial on the unit circle relative to the size of the polynomial itself when there is a restriction on its zeros. It states that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.3}$$

Inequality (1.3) was refined by Aziz and Dawood [2] in the form

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \left\{ \max_{|z|=1} |P(z)| + \min_{|z|=1} |P(z)| \right\}. \tag{1.4}$$

Equality in (1.3) and (1.4) holds for any polynomial which has all its zeros on $|z| = 1$.
Over the years, the inequalities (1.3) and (1.4) have been generalized and extended in several directions. For a polynomial $P(z)$ of degree n having all its zeros in $|z| \leq k,\ k \geq 1$, Govil [7], proved that

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k^n} \max_{|z|=1} |P(z)|. \tag{1.5}$$

As is easy to see that (1.5) becomes equality if $P(z) = z^n + k^n$, one would expect that if we exclude the class of polynomials having all zeros on $|z| = k$, then it may be possible to improve the bound in (1.5). In this direction as an improvement of (1.5) and a generalization to (1.4), it was shown by Govil [9] that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k,\ k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k^n} \left\{ \max_{|z|=1} |P(z)| + \min_{|z|=k} |P(z)| \right\}. \tag{1.6}$$

As an extension of (1.2), Malik [15] proved that, if $P(z) \neq 0$ in $|z| < k,\ k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{1.7}$$

The result is sharp and equality in (1.7) holds for $P(z) = (z+k)^n$.
On the other hand, if $P(z) \neq 0$ in $|z| < k,\ k \leq 1$, the precise estimate of maximum $|P'(z)|$ on $|z| = 1$ does not seem to be known in general, and this problem is still open. However, some special cases in this direction have been considered by many people where some partial extensions of (1.2) are established. In 1980, it was shown by Govil [8] that if $P(z)$ is a polynomial of degree n and $P(z) \neq 0$ in $|z| < k,\ k \leq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k^n} \max_{|z|=1} |P(z)|, \tag{1.8}$$

provided $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, where $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$. Under the same hypothesis as in (1.8), Aziz and Ahmad [1] established an improved inequality in the form

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k^n} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)| \right\}.$$

More generalised versions of Bernstein and Turán inequalities have developed in the literature, in which the underlying polynomial is substituted with more general classes of functions. Moving from the domain of ordinary derivatives of polynomials to the domain of polar derivatives is one such generalisation. Let us first introduce the concept of the polar derivative before moving on to further conclusions. For a polynomial $P(z)$ of degree n, we define

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z),$$

the polar derivative of $P(z)$ with respect to the point α. The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\left\{\frac{D_\alpha P(z)}{\alpha}\right\}=P'(z),$$

uniformly with respect to z for $|z|\le R,\ R>0$.
One might check the thorough literature on the polar derivative of polynomials from the comprehensive books of Marden [16], Milovanović et al. [18] and Rahman and Schmeisser [23]. In 1998, Aziz and Rather [3] established the polar derivative analogue of (1.5) by proving that if $P(z)$ is a polynomial of degree n having all its zeros in $|z|\le k,\ k\ge 1$, then for every $\alpha\in\mathbb{C}$ with $|\alpha|\ge k$,

$$\max_{|z|=1}|D_\alpha P(z)|\ge n\left(\frac{|\alpha|-k}{1+k^n}\right)\max_{|z|=1}|P(z)|. \tag{1.9}$$

The corresponding polar derivative analogue of (1.6) and a refinement of (1.9) was given by Govil and Mctume [11]. They proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z|\le k,\ k\ge 1$, then for any complex number α with $|\alpha|\ge 1+k+k^n$,

$$\max_{|z|=1}|D_\alpha P(z)|\ge n\left(\frac{|\alpha|-k}{1+k^n}\right)\max_{|z|=1}|P(z)|+n\left(\frac{|\alpha|-(1+k+k^n)}{1+k^n}\right)\min_{|z|=k}|P(z)|. \tag{1.10}$$

Very recently, Singh et al. [24] established the following refinement of (1.8) in the form of the following result:

Theorem 1.1. Let $P(z)=\sum_{v=0}^{n}a_vz^v$ be a polynomial of degree n which does not vanish in $|z|<k,\ k\le 1$, and $Q(z)=z^n\overline{P\left(\frac{1}{\bar z}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z|=1$, then

$$\max_{|z|=1}|P'(z)|\le\left(\frac{n}{1+k^n}-\frac{k^n(|a_0|-k^n|a_n|)}{(1+k^n)(|a_0|+k^n|a_n|)}\right)\max_{|z|=1}|P(z)|. \tag{1.11}$$

Equality in (1.11) holds for $P(z)=z^n+k^n$.

As a polar derivative analogue of Theorem 1.1, Singh et al. in the same paper proved the following result:

Theorem 1.2. Let $P(z)=\sum_{v=0}^{n}a_vz^v$ be a polynomial of degree n which does not vanish in $|z|<k,\ k\le 1$, and $Q(z)=z^n\overline{P\left(\frac{1}{\bar z}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z|=1$, then for any complex number α with $|\alpha|\ge 1$,

$$\max_{|z|=1}|D_\alpha P(z)|\le\left(\frac{n(|\alpha|+k^n)}{1+k^n}-\frac{(|\alpha|-1)k^n(|a_0|-k^n|a_n|)}{(1+k^n)(|a_0|+k^n|a_n|)}\right)\max_{|z|=1}|P(z)|. \tag{1.12}$$

As an improvement of (1.10), Singh et al. in the same paper proved the following result in this direction.

Theorem 1.3. If $P(z)=\sum_{v=0}^{n}a_vz^v$ is a polynomial of degree n having all its zeros in $|z|\le k,\ k\ge 1$, then for any complex number α with $|\alpha|\ge 1+k+k^n$,

$$\begin{aligned}\max_{|z|=1}|D_\alpha P(z)|&\ge\left(\frac{|\alpha|-k}{1+k^n}\right)\left\{n+\frac{k^n|a_n|-|a_0+e^{i\theta_0}m|}{k^n|a_n|+|a_0+e^{i\theta_0}m|}\right\}\max_{|z|=1}|P(z)|\\&+\left\{n\left(\frac{|\alpha|-(1+k+k^n)}{1+k^n}\right)+\frac{(|\alpha|-k)}{1+k^n}\left(\frac{k^n|a_n|-|a_0+e^{i\theta_0}m|}{k^n|a_n|+|a_0+e^{i\theta_0}m|}\right)\right\}m,\end{aligned} \tag{1.13}$$

where $m=\min_{|z|=k}|P(z)|$, $\theta_0=arg\big(P(e^{i\phi_0})\big)$ such that $|P(e^{i\phi_0})|=\max_{|z|=1}|P(z)|$.

In this paper, we continue our investigation of these types of results for a specific class of polynomials and establish some new inequalities for the polar derivative of a polynomial on the unit disk while accounting for the zeros and extremal coefficients of the polynomial.

2 Main Results

We begin now by presenting the following strengthening of (1.12).

Theorem 2.1. Let $P(z) = \sum_{v=0}^{n} a_v z^v$ be a polynomial of degree n which does not vanish in $|z| < k,\ k \leq 1$, and $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then for any complex number α with $|\alpha| \geq 1$,

$$\begin{aligned}\max_{|z|=1} |D_\alpha P(z)| \leq & \left\{ \frac{n(|\alpha| + k^n)}{1+k^n} - \frac{(|\alpha| - 1)k^n(|a_0| - k^n|a_n|)}{(1+k^n)(|a_0| + k^n|a_n|)} \right\} \max_{|z|=1} |P(z)| \\ & - \psi(k) \frac{(1-k)}{2} \left(\frac{(|\alpha| - 1)k^n(|a_0| - k^n|a_n|)}{(1+k^n)(|a_0| + k^n|a_n|)} \right) \max_{|z|=1} |P(z)|, \end{aligned} \tag{2.1}$$

where

$$\psi(k) = \frac{|a_0| - k^n|a_n|}{k|a_0| + k^n|a_n|}.$$

If we divide both sides of (2.1) by $|\alpha|$ and let $|\alpha| \to \infty$, we get the following refinement of (1.11).

Corollary 2.2. Let $P(z) = \sum_{v=0}^{n} a_v z^v$ be a polynomial of degree n which does not vanish in $|z| < k,\ k \leq 1$, and $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then

$$\begin{aligned}\max_{|z|=1} |P'(z)| \leq & \left\{ \frac{n}{1+k^n} - \frac{k^n(|a_0| - k^n|a_n|)}{(1+k^n)(|a_0| + k^n|a_n|)} \right\} \max_{|z|=1} |P(z)| \\ & - \psi(k) \frac{(1-k)}{2} \left(\frac{k^n(|a_0| - k^n|a_n|)}{(1+k^n)(|a_0| + k^n|a_n|)} \right) \max_{|z|=1} |P(z)|, \end{aligned} \tag{2.2}$$

where $\psi(k)$ is as defined in Theorem 2.1.

Remark 2.3. Recall that $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n which does not vanish in $|z| < k,\ k \leq 1$, and if $z_1, z_2, z_3, ..., z_n$ are the zeros of $P(z)$, then

$$\begin{aligned}\left| \frac{a_0}{a_n} \right| &= |z_1.z_2.z_3...z_n| \\ &= |z_1|.|z_2|.|z_3|...|z_n| \\ &\geq k^n, \end{aligned}$$

which implies

$$|a_n|k^n \leq |a_0|. \tag{2.3}$$

By using (2.3), it easily follows that

$$\frac{|a_0| - k^n|a_n|}{k|a_0| + k^n|a_n|} = \psi(k) \geq 0. \tag{2.4}$$

It may be remarked that, in general, for any polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$, of degree n having no zeros in $|z| < k,\ k \leq 1$, the inequalities (2.1) and (2.2) in view of (2.4), would give improvements over the bounds obtained from the inequalities (1.12) and (1.11) respectively.

In the sequel we prove the following refinement of (1.13), which in turn strengthens the bound in (1.10).

Theorem 2.4. If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having all its zeros in $|z| \leq k,\ k \geq 1$, then for any complex number α with $|\alpha| \geq 1 + k + k^n$,

$$\begin{aligned}\max_{|z|=1} |D_\alpha P(z)| \geq & \left(\frac{|\alpha| - k}{1+k^n} \right) \left\{ n + \frac{k^n|a_n| - |a_0 + e^{i\theta_0}m|}{k^n|a_n| + |a_0 + e^{i\theta_0}m|} \right\} \left(1 + \phi(k,m) \frac{k-1}{2} \right) \max_{|z|=1} |P(z)| \\ & + \left\{ n \left(\frac{|\alpha| - (1+k+k^n)}{1+k^n} \right) + \frac{(|\alpha| - k)}{1+k^n} \left(\frac{k^n|a_n| - |a_0 + e^{i\theta_0}m|}{k^n|a_n| + |a_0 + e^{i\theta_0}m|} \right) \right. \\ & \left. + \phi(k,m) \frac{k-1}{2} \left(\frac{k^n|a_n| - |a_0 + e^{i\theta_0}m|}{k^n|a_n| + |a_0 + e^{i\theta_0}m|} \right) \right\} m, \end{aligned} \tag{2.5}$$

where

$$\phi(k,m)=\frac{k^n|a_n|-|a_0+e^{i\theta_0}m|}{k^n|a_n|+k|a_0+e^{i\theta_0}m|},$$

$m=\min_{|z|=k}|P(z)|$, $\theta_0=arg\big(P(e^{i\phi_0})\big)$ such that $|P(e^{i\phi_0})|=\max_{|z|=1}|P(z)|$.

Remark 2.5. Recall that the polynomial $P(z)=\sum_{v=0}^{n}a_vz^v$ has all its zeros in $|z|\leq k,\ k\geq 1$, then for any complex number $|\lambda|e^{i\theta_0}$ with $|\lambda|<1$, it follows by Rouché's theorem that the polynomial $P(z)+|\lambda|e^{i\theta_0}m=(a_0+|\lambda|e^{i\theta_0}m)+a_1z+a_2z^2+...+a_nz^n$ has all its zeros in $|z|\leq k$, where $m=\min_{|z|=k}|P(z)|$, then

$$\left|\frac{a_0+|\lambda|e^{i\theta_0}m}{a_n}\right|=|z_1.z_2...z_n|\leq k^n,$$

implies by letting $|\lambda|\to 1$,

$$k^n|a_n|\geq|a_0+e^{i\theta_0}m|. \tag{2.6}$$

By using (2.6), it easily follows that

$$\frac{k^n|a_n|-|a_0+e^{i\theta_0}m|}{k^n|a_n|+k|a_0+e^{i\theta_0}m|}=\phi(k,m)\geq 0. \tag{2.7}$$

It may be noted that the inequality (2.5) in view of (2.7), would give improvement over the bound in (1.13), which in turn improves the bound in (1.10) excepting the case when all the zeros of $P(z)$ lie on $|z|=k$.

If we divide both sides of (2.5) by $|\alpha|$ and let $|\alpha|\to\infty$, we get the following result:

Corollary 2.6. If $P(z)=\sum_{v=0}^{n}a_vz^v$ is a polynomial of degree n having all its zeros in $|z|\leq k,\ k\geq 1$, then

$$\begin{aligned}\max_{|z|=1}|P'(z)|\geq{}&\frac{1}{1+k^n}\left\{n+\frac{k^n|a_n|-|a_0+e^{i\theta_0}m|}{k^n|a_n|+|a_0+e^{i\theta_0}m|}\right\}\left(1+\phi(k,m)\frac{k-1}{2}\right)\max_{|z|=1}|P(z)|\\&+\left\{\frac{n}{1+k^n}+\frac{1}{1+k^n}\left(\frac{k^n|a_n|-|a_0+e^{i\theta_0}m|}{k^n|a_n|+|a_0+e^{i\theta_0}m|}\right)\right\}m,\end{aligned} \tag{2.8}$$

where $m=\min_{|z|=k}|P(z)|$, $\theta_0=arg\big(P(e^{i\phi_0})\big)$ such that $|P(e^{i\phi_0})|=\max_{|z|=1}|P(z)|$ and $\phi(k,m)$ is as defined in Theorem 2.4.

As remarked above, in general, for any polynomial $P(z)=a_0+a_1z+a_2z^2+...+a_nz^n$, of degree n having all its zeros in $|z|\leq k,\ k\geq 1$, the inequality (2.8) in view of (2.7), would give improvement over the bound obtained from the inequality (1.6), excepting the case when all the zeros of $P(z)$ lie on $|z|=k$.

3 Auxiliary results

For the proofs of our main results, we shall make use of the following lemmas. The first lemma is a simple deduction from the Maximum Modulus Principle (see [22]).

Lemma 3.1. If $P(z)$ is a polynomial of degree n, then for $R\geq 1$,

$$\max_{|z|=R}|P(z)|\leq R^n\max_{|z|=1}|P(z)|.$$

The following lemma is due to Dubnin [6].

Lemma 3.2. If $P(z)=\sum_{v=0}^{n}a_vz^v$ is a polynomial of degree n which does not vanish in $|z|<1$, then for $R\geq 1$, we have

$$\max_{|z|=R}|P(z)|\leq\frac{(1+R^n)(|a_0|+R|a_n|)}{(1+R)(|a_0|+|a_n|)}\max_{|z|=1}|P(z)|. \tag{3.1}$$

Equality in (3.1) holds for $P(z)=\frac{a+bz^n}{2},\ |a|=|b|=1$.

Lemma 3.3. If $P(z)=\sum_{v=0}^{n} a_v z^v$, is a polynomial of degree n having all its zeros in $|z|\leq k,\ k\geq 1$, then

$$\max_{|z|=k}|P(z)|\geq \frac{2k^n}{1+k^n}\left(1+\frac{k-1}{2}\phi(k)\right)\max_{|z|=1}|P(z)|, \tag{3.2}$$

where

$$\phi(k)=\frac{k^n|a_n|-|a_0|}{k^n|a_n|+k|a_0|}.$$

Equality in (3.2) holds for $P(z)=z^n+k^n$.

Proof . Let $T(z)=P(kz)$. Since $P(z)$ has all its zeros in $|z|\leq k,\ k\geq 1$, the polynomial $T(z)$ has all its zeros in $|z|\leq 1$. Let $H(z)=z^nT\left(\frac{1}{z}\right)$ be the reciprocal polynomial of $T(z)$, then $H(z)$ has no zeros in $|z|<1$. Hence applying (3.1) of Lemma 3.2 to the polynomial $H(z)$, we get for $k\geq 1$,

$$\max_{|z|=k}|H(z)|\leq \frac{(1+k^n)(k^n|a_n|+k|a_0|)}{(1+k)(k^n|a_n|+|a_0|)}\max_{|z|=1}|H(z)|. \tag{3.3}$$

Since $|H(z)|=|T(z)|$ on $|z|=1$,

$$\max_{|z|=1}|H(z)|=\max_{|z|=1}|T(z)|=\max_{|z|=k}|P(z)|$$

and

$$\max_{|z|=k}|H(z)|=\max_{|z|=k}\left|z^nP\left(\frac{k}{z}\right)\right|=k^n\max_{|z|=1}|P(z)|.$$

The above when substituted in (3.3) gives

$$\max_{|z|=k}|P(z)|\geq\left(\frac{(1+k)(k^n|a_n|+|a_0|)}{(1+k^n)(k^n|a_n|+k|a_0|)}\right)k^n\max_{|z|=1}|P(z)|. \tag{3.4}$$

Using the fact that

$$\frac{(1+k)(k^n|a_n|+|a_0|)}{(1+k^n)(k^n|a_n|+k|a_0|)}=\frac{2}{1+k^n}+\frac{(k^n|a_n|-|a_0|)(k-1)}{(1+k^n)(k^n|a_n|+k|a_0|)},$$

in (3.4), we get

$$\max_{|z|=k}|P(z)|\geq \frac{2k^n}{1+k^n}\left(1+\frac{k-1}{2}\phi(k)\right)\max_{|z|=1}|P(z)|,$$

where

$$\phi(k)=\frac{k^n|a_n|-|a_0|}{k^n|a_n|+k|a_0|}.$$

This completes the proof of Lemma 3.3. $\square$

Lemma 3.4. If $P(z)$ is a polynomial of degree n and, $Q(z)=z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$, then on $|z|=1$,

$$|P'(z)|+|Q'(z)|\leq n\max_{|z|=1}|P(z)|.$$

The above lemma is due to Govil and Rahman [12].

Lemma 3.5. If $P(z)=\sum_{v=0}^{n} a_v z^v$, is a polynomial of degree n having all its zeros in $|z|\leq 1$, then for $|z|=1$ and $P(z)\neq 0$, we have

$$Re\left(\frac{zP'(z)}{P(z)}\right)\geq\frac{1}{2}\left(n+\frac{|a_n|-|a_0|}{|a_n|+|a_0|}\right).$$

The above lemma is due to Dubnin [5].

Lemma 3.6. If $P(z)=\sum_{v=0}^{n} a_v z^v$, is a polynomial of degree n having all its zeros in $|z|\le k,\ k\ge 1$, then for $|\alpha|\ge k$,

$$\max_{|z|=1}|D_\alpha P(z)|\ge\left(\frac{|\alpha|-k}{1+k^n}\right)\left(n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\left(1+\frac{k-1}{2}\phi(k)\right)\max_{|z|=1}|P(z)|,$$

where $\phi(k)$ is as defined in Lemma 3.3.

Proof . Since $P(z)$ is a polynomial of degree n having all its zeros in $|z|\le k,\ k\ge 1$, all the zeros of $T(z)=P(kz)$ lie in $|z|\le 1$. Therefore, applying Lemma 3.5 to the polynomial $T(z)$, we get for $|z|=1$ and $T(z)\ne 0$,

$$Re\left(\frac{zT'(z)}{T(z)}\right)\ge\frac{1}{2}\left(n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right),$$

which implies

$$|T'(z)|\ge\frac{1}{2}\left(n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)|T(z)|. \tag{3.5}$$

If $H(z)=z^n\overline{T\left(\frac{1}{\bar z}\right)}$, then $T(z)=z^n\overline{H\left(\frac{1}{\bar z}\right)}$. It is easy to verify that for $|z|=1$,

$$|H'(z)|=|nT(z)-zT'(z)|.$$

Since $T(z)$ has all its zeros in $|z|\le 1$, using the fact that $|T'(z)|\ge|H'(z)|$ on $|z|=1$, we have for $\frac{|\alpha|}{k}\ge 1$ and $|z|=1$,

$$\begin{aligned}\left|D_{\frac{\alpha}{k}}T(z)\right|&=\left|nT(z)+\left(\frac{\alpha}{k}-z\right)T'(z)\right|\\&\ge\left|\frac{\alpha}{k}\right|\left|T'(z)\right|-\left|nT(z)-zT'(z)\right|\\&\ge\left(\frac{|\alpha|}{k}-1\right)|T'(z)|.\end{aligned} \tag{3.6}$$

On combining (3.5) and (3.6), we get

$$\left|D_{\frac{\alpha}{k}}T(z)\right|\ge\left(\frac{|\alpha|-k}{2k}\right)\left(n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)|T(z)|. \tag{3.7}$$

The above inequality (3.7) is equivalent to

$$\max_{|z|=1}\left|nP(kz)+\left(\frac{\alpha}{k}-z\right)kP'(kz)\right|\ge\left(\frac{|\alpha|-k}{2k}\right)\left(n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)|P(kz)|.$$

The last inequality yields

$$\max_{|z|=k}|D_\alpha P(z)|\ge\left(\frac{|\alpha|-k}{2k}\right)\left(n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\max_{|z|=k}|P(z)|. \tag{3.8}$$

Since $D_\alpha P(z)$ is a polynomial of degree at most $n-1$, we have by Lemma 3.1 for $R=k\ge 1$,

$$\max_{|z|=k}|D_\alpha P(z)|\le k^{n-1}\max_{|z|=1}|D_\alpha P(z)|.$$

By using this and Lemma 3.3, the above inequality (3.8) clearly gives

$$\max_{|z|=1}|D_\alpha P(z)|\ge\left(\frac{|\alpha|-k}{1+k^n}\right)\left(n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\left(1+\frac{k-1}{2}\phi(k)\right)\max_{|z|=1}|P(z)|. \tag{3.9}$$

This completes the proof of Lemma 3.6. □
If we divide both sides of (3.9) by $|\alpha|$ and let $|\alpha|\to\infty$, we get the following:

$$\max_{|z|=1}|P'(z)|\ge\left(\frac{n}{1+k^n}+\frac{k^n|a_n|-|a_0|}{(1+k^n)(k^n|a_n|+|a_0|)}\right)\left(1+\frac{k-1}{2}\phi(k)\right)\max_{|z|=1}|P(z)|. \tag{3.10}$$

4 Proofs of the Theorems

Proof of Theorem 2.1. Let $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$. Since $P(z) = \sum\limits_{v=0}^{n} a_v z^v \neq 0$ in $|z| < k$, $k \leq 1$, the polynomial $Q(z)$ of degree n has all its zeros in $|z| \leq \frac{1}{k}$, $\frac{1}{k} \geq 1$. On applying inequality (3.10) to $Q(z)$ and using the fact that $\max_{|z|=1}|P(z)| = \max_{|z|=1}|Q(z)|$, we get

$$\begin{aligned}\max_{|z|=1}|Q'(z)| &\geq \left(\frac{n}{1+\frac{1}{k^n}} + \frac{(\frac{1}{k^n}|a_0| - |a_n|)}{(1+\frac{1}{k^n})(\frac{1}{k^n}|a_0| + |a_n|)}\right) \\ &\times \left(1 + \frac{\frac{1}{k}-1}{2}\left(\frac{\frac{1}{k^n}|a_0| - |a_n|}{\frac{1}{k^n}|a_0| + \frac{1}{k}|a_n|}\right)\right)\max_{|z|=1}|P(z)|,\end{aligned}$$

which implies

$$\max_{|z|=1}|Q'(z)| \geq \left(\frac{nk^n}{1+k^n} + \frac{k^n(|a_0| - k^n|a_n|)}{(1+k^n)(|a_0| + k^n|a_n|)}\right)\left(1 + \frac{1-k}{2}\psi(k)\right)\max_{|z|=1}|P(z)|, \tag{4.1}$$

where

$$\psi(k) = \frac{|a_0| - k^n|a_n|}{k|a_0| + k^n|a_n|}.$$

Since $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, we have

$$\max_{|z|=1}(|P'(z)| + |Q'(z)|) = \max_{|z|=1}|P'(z)| + \max_{|z|=1}|Q'(z)|. \tag{4.2}$$

On combining (4.1), (4.2) and Lemma 3.4, we get

$$\begin{aligned}&n\max_{|z|=1}|P(z)| \\ &\geq \max_{|z|=1}|P'(z)| + \left(\frac{nk^n}{1+k^n} + \frac{k^n(|a_0| - k^n|a_n|)}{(1+k^n)(|a_0| + k^n|a_n|)}\right)\left(1 + \frac{1-k}{2}\psi(k)\right)\max_{|z|=1}|P(z)|,\end{aligned}$$

which gives

$$\begin{aligned}\max_{|z|=1}|P'(z)| &\leq \left\{\frac{n}{1+k^n} - \frac{k^n(|a_0| - k^n|a_n|)}{(1+k^n)(|a_0| + k^n|a_n|)}\right\}\max_{|z|=1}|P(z)| \\ &\quad - \frac{(1-k)}{2}\psi(k)\left(\frac{k^n(|a_0| - k^n|a_n|)}{(1+k^n)(|a_0| + k^n|a_n|)}\right)\max_{|z|=1}|P(z)|.\end{aligned} \tag{4.3}$$

Also, it is easy to verify that for $|z| = 1$,

$$|Q'(z)| = |nP(z) - zP'(z)|. \tag{4.4}$$

Note that for any complex number α, and $|z| = 1$, we have

$$\begin{aligned}|D_\alpha P(z)| &= |nP(z) + (\alpha - z)P'(z)| \\ &\leq |nP(z) - zP'(z)| + |\alpha||P'(z)|,\end{aligned}$$

which gives by (4.4) and $|\alpha| \geq 1$, that

$$\begin{aligned}|D_\alpha P(z)| &\leq |Q'(z)| + |\alpha||P'(z)| \\ &= |Q'(z)| + |P'(z)| - |P'(z)| + |\alpha||P'(z)| \\ &\leq n\max_{|z|=1}|P(z)| + (|\alpha| - 1)|P'(z)| \quad \text{(by Lemma 3.4)} \\ &\leq n\max_{|z|=1}|P(z)| + (|\alpha| - 1)\max_{|z|=1}|P'(z)|.\end{aligned} \tag{4.5}$$

Combining (4.5) with (4.3) and rearranging the terms, we get

$$\max_{|z|=1}|D_\alpha P(z)| \le \left\{\frac{n(|\alpha|+k^n)}{1+k^n} - \frac{(|\alpha|-1)k^n(|a_0|-k^n|a_n|)}{(1+k^n)(|a_0|+k^n|a_n|)}\right\}\max_{|z|=1}|P(z)|$$
$$-\psi(k)\frac{(1-k)}{2}\left(\frac{(|\alpha|-1)k^n(|a_0|-k^n|a_n|)}{(1+k^n)(|a_0|+k^n|a_n|)}\right)\max_{|z|=1}|P(z)|.$$

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.4. If $P(z)$ is a polynomial of degree n having atleast one zero on $|z|=k$, then $m=\min_{|z|=k}|P(z)|=0$ and the result follows trivially from Lemma 3.6. So, without loss of generality, let us assume that $P(z)$ has all its zeros in $|z|<k,\ k\ge 1$, then it follows by Rouché's theorem that for any complex number λ with $|\lambda|<1$, the polynomial $P(z)+\lambda m=(a_0+\lambda m)+a_1z+a_2z^2+...+a_nz^n$ also has all its zeros in $|z|<k,\ k\ge 1$. Therefore, applying Lemma 3.6 to $P(z)+\lambda m$, we get for $|\alpha|\ge 1+k+k^n$,

$$\max_{|z|=1}|D_\alpha(P(z)+\lambda m)| \ge \left(\frac{|\alpha|-k}{1+k^n}\right)\left(n+\frac{k^n|a_n|-|a_0+\lambda m|}{k^n|a_n|+|a_0+\lambda m|}\right)$$
$$\times\left(1+\frac{k-1}{2}.\frac{k^n|a_n|-|a_0+\lambda m|}{k^n|a_n|+k|a_0+\lambda m|}\right)\max_{|z|=1}|P(z)+\lambda m|. \tag{4.6}$$

Let $0\le\phi_0<2\pi$, be such that $|P(e^{i\phi_0})|=\max_{|z|=1}|P(z)|$. Then inequality (4.6) gives

$$\max_{|z|=1}|D_\alpha P(z)+n\lambda m| \ge \left(\frac{|\alpha|-k}{1+k^n}\right)\left(n+\frac{k^n|a_n|-|a_0+\lambda m|}{k^n|a_n|+|a_0+\lambda m|}\right)$$
$$\times\left(1+\frac{k-1}{2}.\frac{k^n|a_n|-|a_0+\lambda m|}{k^n|a_n|+k|a_0+\lambda m|}\right)|P(e^{i\phi_0})+\lambda m|. \tag{4.7}$$

Now

$$|P(e^{i\phi_0})+\lambda m| = \Big||P(e^{i\phi_0})|e^{i\theta_0}+|\lambda|e^{i\phi_0}m\Big| = \Big||P(e^{i\phi_0})|+|\lambda|e^{i(\phi-\theta_0)}m\Big|.$$

Setting the argument of ϕ such that $\phi=\theta_0$, we get $|P(e^{i\phi_0})+\lambda m|=|P(e^{i\phi_0})|+|\lambda|m$, and then it follows from inequality (4.7) that

$$\max_{|z|=1}|D_\alpha P(z)|+n|\lambda|m \ge \left(\frac{|\alpha|-k}{1+k^n}\right)\left(n+\frac{k^n|a_n|-|a_0+|\lambda|e^{i\theta_0}m|}{k^n|a_n|+|a_0+|\lambda|e^{i\theta_0}m|}\right)$$
$$\times\left(1+\frac{k-1}{2}.\frac{k^n|a_n|-|a_0+|\lambda|e^{i\theta_0}m|}{k^n|a_n|+k|a_0+|\lambda|e^{i\theta_0}m|}\right)(|P(e^{i\phi_0})|+|\lambda|m),$$

which is equivalent to

$$\max_{|z|=1}|D_\alpha P(z)| \ge \left(\frac{|\alpha|-k}{1+k^n}\right)\left\{n+\frac{k^n|a_n|-|a_0+|\lambda|e^{i\theta_0}m|}{k^n|a_n|+|a_0+|\lambda|e^{i\theta_0}m|}\right\}$$
$$\times\left(1+\frac{k-1}{2}.\frac{k^n|a_n|-|a_0+|\lambda|e^{i\theta_0}m|}{k^n|a_n|+k|a_0+|\lambda|e^{i\theta_0}m|}\right)\max_{|z|=1}|P(z)|$$
$$+\left\{n\left(\frac{|\alpha|-(1+k+k^n)}{1+k^n}\right)+\frac{(|\alpha|-k)}{1+k^n}\left(\frac{k^n|a_n|-|a_0+|\lambda|e^{i\theta_0}m|}{k^n|a_n|+|a_0+|\lambda|e^{i\theta_0}m|}\right)\right.$$
$$\left.+\frac{k-1}{2}.\frac{k^n|a_n|-|a_0+|\lambda|e^{i\theta_0}m|}{k^n|a_n|+k|a_0+|\lambda|e^{i\theta_0}m|}\left(\frac{k^n|a_n|-|a_0+|\lambda|e^{i\theta_0}m|}{k^n|a_n|+|a_0+|\lambda|e^{i\theta_0}m|}\right)\right\}m. \tag{4.8}$$

Taking $|\lambda|\to 1$ in (4.8), the above inequality reduces to

$$\max_{|z|=1}|D_\alpha P(z)| \ge \left(\frac{|\alpha|-k}{1+k^n}\right)\left\{n+\frac{k^n|a_n|-|a_0+e^{i\theta_0}m|}{k^n|a_n|+|a_0+e^{i\theta_0}m|}\right\}\left(1+\phi(k,m)\frac{k-1}{2}\right)\max_{|z|=1}|P(z)|$$
$$+\left\{n\left(\frac{|\alpha|-(1+k+k^n)}{1+k^n}\right)+\frac{(|\alpha|-k)}{1+k^n}\left(\frac{k^n|a_n|-|a_0+e^{i\theta_0}m|}{k^n|a_n|+|a_0+e^{i\theta_0}m|}\right)\right.$$
$$\left.+\phi(k,m)\frac{k-1}{2}\left(\frac{k^n|a_n|-|a_0+e^{i\theta_0}m|}{k^n|a_n|+|a_0+e^{i\theta_0}m|}\right)\right\}m,$$

where

$$\phi(k,m) = \frac{k^n|a_n| - |a_0 + e^{i\theta_0}m|}{k^n|a_n| + k|a_0 + e^{i\theta_0}m|}.$$

This completes the proof of Theorem 2.4. □

5 Conclusions

A sequence of publications on various Erdös-lax and Turán-type inequalities has been published in recent years, and significant progress has been made. Both mathematics and practical fields are interested in inequalities of these types. In this work, we continue our investigation of inequalities of this nature by taking into consideration the location of all the zeros and extremal coefficients of the underlying polynomial.

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