Int. J. Nonlinear Anal. Appl. 14 (2023) 1, 1311–1329
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.26761.3406

On the solutions of some nonlinear quadratic integral equations on an unbounded interval

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(Communicated by Haydar Akca)

Abstract

In this paper, we study the existence of the solutions of a class of functional integral equations which contain a number of classical nonlinear integral equations as special cases. The investigations are placed in the Banach space of real functions defined, continuous on the real half-axis and vanishing at infinity. The method used in our considerations depends on suitable conjunction of the technique of measures of noncompactness with the classical Schauder fixed point theorem. The results obtained in this paper extend and improve essentially some known results in the recent literature. Besides, three examples showing the generality and applicability of our results are presented.

Keywords: Nonlinear integral equation, measure of noncompactness, Schauder fixed-point theorem
2020 MSC: Primary 45G10, 47H08; Secondary 47H10, 45M99

1 Introduction

It is well known that the theory of nonlinear integral equations of various types appears in many applications that arise in the fields of mathematical analysis, nonlinear functional analysis, mathematical physics and engineering (see [8, 9, 22]). A lot of real world problems in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory can be described and analysed with help of integral equations. Especially, the so-called quadratic integral equation of Chandrasekhar's type can be very often encountered in many applications, [11, 12, 13, 14, 16, 21]. Many authors have studied the existence of solutions for several classes of nonlinear quadratic integral equations.

The main subject of this paper is to investigate the following the nonlinear functional integral equation given by:

$$x(t) = f\left(t, (T_1x)(t), (T_2x)(t)\int_0^\infty u\left(t, s, x(s)\right)ds\right),\ t \in \mathbb{R}_+, \tag{1.1}$$

where the functions f, u and the operators T_i $(i = 1, 2)$ are known, while $x = x(t)$ is an unknown function. It is worthwhile also mentioning that Eq.(1.1) contains as particular cases a lot of integral equations. For example, the equation

$$x(t) = a(t) + x(t)\int_0^\infty \frac{t}{t+s}\phi(s)x(s)ds,\ t \in \mathbb{R}_+ \tag{1.2}$$

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Received: April 2022 *Accepted:* August 2022

is closely related to the famous quadratic integral equation of Chandrasekhar type:

$$x(t) = 1 + x(t) \int_0^1 \frac{t}{t+s} \phi(s) x(s) ds, \ t \in [0,1]. \tag{1.3}$$

The equation (1.3) is considered in many papers and monographs, [2, 4, 13, 18].

The sufficient conditions on the existence of the solutions of the equation

$$x(t) = \int_0^\infty k(t,s) f\left(s, x(s)\right) ds, \ t \in \mathbb{R}_+ \tag{1.4}$$

are given in [1].

The existence of the solution of nonlinear integral equations

$$x(t) = a(t) + x(t) \int_0^\infty k(t,s) h\left(s, x(s)\right) ds, \ t \in \mathbb{R}_+ \tag{1.5}$$

and

$$x(t) = H(t, x(t)) + x(t) \int_0^\infty k(t,s) \varphi(s) (f(x(s)) + g(x(s))) ds, \ t \in \mathbb{R}_+ \tag{1.6}$$

is researched in [20] and [30], respectively.

The existence and asymptotic stability of the solutions of the nonlinear integral equation

$$x(t) = a(t) + g(t, x(t)) \int_0^\infty K(t,s) h\left(s, x(s)\right) ds, \ t \in \mathbb{R}_+ \tag{1.7}$$

are studied in [6].

The monotonic solutions of the nonlinear integral equation

$$x(t) = g(t) + \int_0^\infty u\left(t, s, x(s)\right) ds, \ t \in \mathbb{R}_+ \tag{1.8}$$

and the existence of the solutions of the Urysohn integral equation

$$x(t) = a(t) + f(t, x(t)) \int_0^\infty u\left(t, s, x(s)\right) ds, \ t \in \mathbb{R}_+ \tag{1.9}$$

are investigated in [5] and [7, 10, 15, 20, 26, 27, 28], respectively.

Besides, the equations

$$x(t) = (T_1 x)(t) + (T_2 x)(t) \int_0^\infty u\left(t, s, x(s)\right) ds, \ t \in \mathbb{R}_+ \tag{1.10}$$

and

$$x(t) = F\left(t, x(t), \int_0^\infty u\left(t, s, x(s)\right) ds\right), \ t \in \mathbb{R}_+ \tag{1.11}$$

are examined in [19, 29] and [26], respectively.

The equations (1.2)-(1.11) can be easily obtained from the equation (1.1) by special selection of the functions and the operators.

Many equations studied so far are the special form of the equation (1.1) (see, [1, 5, 6, 7, 10, 15, 17, 20, 23, 24, 25, 27, 28, 30]). Using the technique of a suitable measure of noncompactness and Schauder fixed point theorem, we prove an existence theorem for Eq.(1.1). We give three nontrivial examples that explain the generalizations and applications of our results. So our work improves and completes some results mentioned before.

The integral equations (1.1) and (1.11) are quite similar to each other. But the equation (1.1) is more general than the equation (1.11). **Some equations can be written in both (1.1) and (1.11) form. In this case, is there any difference between the existence theorem in the presented paper and Theorem 1 given in [26]? It will be seen in Remark 4.4 and Remark 4.6 that the results obtained in our paper are a generalization and extension of both the results in [26] and in some other articles.**

2 Preliminaries

In this section, we give a collection of auxiliary facts which will be needed further on. Assume that $(E, \|.\|)$ is a real Banach space with zero element θ . Let $B(x, r)$ denote the closed ball centered at x and with radius r. The symbol B_r stands for the ball $B(\theta, r)$. If X is a subset of E, then $\overline{X}$ and Conv$X$ denote the closure and convex closure of X, respectively. With the symbols λX and $X + Y$, we denote the standard algebraic operations on sets. Moreover, we denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E and $\mathfrak{N}_E$ its subfamily consisting of all relatively compact subsets. The definition of the concept of a measure of noncompactness presented below comes from [3].

Definition 2.1. A function $\mu : \mathfrak{M}_E \to \mathbb{R}_+ = [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies following conditions:

(1) The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.

(2) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.

(3) $\mu(\overline{X}) = \mu(\text{Conv}X) = \mu(X)$.

(4) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$, for $\lambda \in [0, 1]$.

(5) If $\{X_n\}$ is a sequence of nonempty, bounded and closed subsets of the set E such that $X_{n+1} \subset X_n$, $(n = 1, 2, ...)$ and $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \cap_{n=1}^{\infty} X_n$ is nonempty.

Notice that the intersection set X_∞ from (5) belongs to $\ker \mu$. In fact, from the inequality $\mu(X_\infty) \leq \mu(X_n)$ for any $n = 1, 2, ...$ we have that $\mu(X_\infty) = 0$. This property of the set X_∞ will be crucial later.

In the sequel, we will work in the Banach space $C_0(\mathbb{R}_+)$ consisting of real functions defined, continuous on $\mathbb{R}_+$ and vanishing at infinity. The space $C_0(\mathbb{R}_+)$ is furnished with the standard norm $\|x\| = \sup\{|x(t)| : t \in \mathbb{R}_+\}$.

Further we recall the definition of the measure of noncompactness in the space $C_0(\mathbb{R}_+)$ which will be used in our considerations. In order to define it, let us fix a nonempty and bounded subset X of $C_0(\mathbb{R}_+)$. For $x \in X$, $\varepsilon \geq 0$ and $L > 0$ denoted by $\omega^L(x, \varepsilon)$ the modulus of continuity of function x, i.e.,

$$\omega^L(x, \varepsilon) = \sup\{|x(s) - x(t)| : t, s \in [0, L] \text{ and } |t - s| \leq \varepsilon\}.$$

Further, let us put:

$$\omega^L(X, \varepsilon) = \sup\{\omega^L(x, \varepsilon) : x \in X\},\ \omega_0^L(X) = \lim_{\varepsilon\to 0} \omega^L(X, \varepsilon)$$

and

$$\omega_0(X) = \lim_{L\to\infty} \omega_0^L(X). \tag{2.1}$$

Next define:

$$\beta^L(x) = \sup\{|x(t)| : t \geq L\},\ \beta^L(X) = \sup\left\{\beta^L(x) : x \in X\right\}$$

and

$$\beta(X) = \lim_{L\to\infty} \beta^L(X). \tag{2.2}$$

Finally, let us consider the function μ defined on the family $\mathfrak{M}_{C_0(\mathbb{R}_+)}$ by the equality:

$$\mu(X) = \omega_0(X) + \beta(X). \tag{2.3}$$

It may be found in [3] that the function μ is a measure of noncompactness in the space $C_0(\mathbb{R}_+)$. Moreover, the kernel $\ker \mu$ contains nonempty and bounded sets X such that functions from X are locally equicontinuous on $\mathbb{R}_+$ and tend to zero at infinity uniformly with respect to the set X, i.e. for each $\varepsilon \geq 0$ there exists $L > 0$ with the property that

$$|x(t)| \leq \varepsilon \text{ for all } x \in X \text{ and } t \geq L.$$

This property of $\ker \mu$ will be important in our further study.

3 The Main Result

We will consider Eq.(1.1) under the following assumptions $(H1) - (H8)$:

(H1) $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous such that $f(t, 0, 0) \to 0$ as $t \to \infty$.

Remark 3.1. The assumption $(H1)$ implies that there exists a constant F satisfying

$$\sup \{|f(t, 0, 0)| : t \geq 0\} \leq F.$$

(H2) The function f satisfies the Lipschitz condition with the nonnegative constants l_1 and l_2 with respect to second and third variables, i.e.

$$|f(t, x_1, y) - f(t, x_2, y)| \leq l_1|x_1 - x_2|$$

for all $t \in \mathbb{R}_+$ and $x_1, x_2, y \in \mathbb{R}$ and

$$|f(t, x, y_1) - f(t, x, y_2)| \leq l_2|y_1 - y_2|$$

for all $t \in \mathbb{R}_+$ and $x, y_1, y_2 \in \mathbb{R}$.

(H3) The operators $T_i : C_0(\mathbb{R}_+) \to C_0(\mathbb{R}_+)$ are continuous and there exist nondecreasing functions $d_i : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|T_i x\| \leq d_i(\|x\|), \ (i = 1, 2)$$

for all $x \in C_0(\mathbb{R}_+)$.

(H4) $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exist a continuous function $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and a continuous nondecreasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $h(0) = 0$ and

$$|u(t, s, x)| \leq g(t, s)h(|x|)$$

for all $t, s \in \mathbb{R}_+$ and $x \in \mathbb{R}$.

(H5) For every $t \geq 0$ the function $s \to g(t, s)$ is integrable on $\mathbb{R}_+$ and the function $t \to \int_0^\infty g(t, s)ds$ is bounded on $\mathbb{R}_+$. That is, there exists a constant G satisfying

$$\sup \left\{ \int_0^\infty g(t, s)ds : t \geq 0 \right\} \leq G.$$

(H6) The inequality

$$l_1 d_1(r) + l_2 d_2(r)h(r)G + F \leq r$$

has a positive solution r_0.

(H7) There exist the nonnegative constants m_{i,r_0} and b_{r_0} for r_0 such that the inequalities

$$\mu(T_i X) \leq m_{i,r_0}\, \mu(X), \ (i = 1, 2) \text{ and } \beta\,(h_X) \leq b_{r_0}\, \beta(X)$$

hold for all nonempty and bounded subset X of the ball B_{r_0}, where β and μ are defined by (2.2) and (2.3), h is the function given in the assumption $(H4)$, $h_X = \{h_x : x \in X\}$ and the function $h_x : \mathbb{R}_+ \to \mathbb{R}_+$ for each $x \in X$ is defined by $h_x(t) = h\,(|x(t)|)$.

Remark 3.2. The set h_X is nonempty and bounded subset of $C_0(\mathbb{R}_+)$, since the subset X of B_{r_0} is nonempty and the estimate $|h_x(t)| \leq h(\|x\|) \leq h(r_0)$ holds for all $x \in X$ and $t \in \mathbb{R}_+$.

(H8) $l_1 m_{1,r_0} + l_2 m_{2,r_0} h(r_0) G + 2 l_2 d_2(r_0) b_{r_0} G < 1$.

Lemma 3.3. If the assumptions $(H4)$ and $(H5)$ are satisfied, then the inequality

$$\left| \int_0^\infty u\,(t, s, x(s))\, ds \right| \leq h\,(\|x\|)\, G \tag{3.1}$$

holds for all $t \in \mathbb{R}_+$ and $x \in C_0(\mathbb{R}_+)$. Also, the function $t \to \int_0^\infty u\,(t, s, x(s))\, ds$ is continuous on $\mathbb{R}_+$ for an arbitrarily fixed $x \in C_0(\mathbb{R}_+)$.

Proof . Since

$$|u\,(t, s, x(s))| \leq g(t, s) h\,(|x(s)|) \leq g(t, s) h\,(\|x\|)$$

for all $t, s \in \mathbb{R}_+$ and $x \in C_0(\mathbb{R}_+)$, the function $t \to \int_0^\infty u\,(t, s, x(s))\, ds$ is well defined on $\mathbb{R}_+$ for an arbitrarily fixed $x \in C_0(\mathbb{R}_+)$ and the inequality (3.1) holds for all $t \in \mathbb{R}_+$ and $x \in C_0(\mathbb{R}_+)$.

Let us fix arbitrarily $L > 0$ and $\varepsilon \geq 0$ and take arbitrary numbers $t_1, t_2 \in [0, L]$ with $|t_1 - t_2| \leq \varepsilon$. Then,

$$\begin{aligned} & \left| \int_0^\infty u\,(t_1, s, x(s))\, ds - \int_0^\infty u\,(t_2, s, x(s))\, ds \right| \\ \leq\ & \int_0^L |u\,(t_1, s, x(s)) - u\,(t_2, s, x(s))|\, ds \\ & + \int_L^\infty |u\,(t_1, s, x(s)) - u\,(t_2, s, x(s))|\, ds \\ \leq\ & L\omega^L_{\|x\|}(u, \varepsilon) + [\sup\{h\,(|x(s)|) : s \geq L\}] \int_L^\infty [g(t_1, s) + g(t_2, s)]\, ds \\ \leq\ & L\omega^L_{\|x\|}(u, \varepsilon) + 2Gh\big(\sup\{|x(s)| : s \geq L\}\big), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} \omega^L_{\|x\|}(u, \varepsilon) \quad = \quad & \sup\{|u(t_1, s, y) - u(t_2, s, y)| : t_1, t_2, s \in [0, L], \\ & y \in [-\|x\|, \|x\|\,] \text{ and } |t_1 - t_2| \leq \varepsilon\}\,. \end{aligned}$$

Further, from the uniform continuity of the function u on the set $[0, L] \times [0, L] \times [-\|x\|, \|x\|\,]$ we derive that $\omega^L_{\|x\|}(u, \varepsilon) \to 0$ as $\varepsilon \to 0$. Besides, since $x \in C_0(\mathbb{R}_+)$, taking into account the continuity of the function h together with $h(0) = 0$, we can choose a number L so big that the last term of the inequality (3.2) is sufficiently small. Thus we deduce that the function $t \to \int_0^\infty u\,(t, s, x(s))\, ds$ is continuous on the interval $[0, L]$ for any $L > 0$ big enough and consequently it is continuous on the whole interval $\mathbb{R}_+$. □

Lemma 3.4. If the assumptions $(H1) - (H5)$ are satisfied, then the operator T defined by

$$(Tx)(t) = f\left(t, (T_1 x)(t), (T_2 x)(t) \int_0^\infty u\,(t, s, x(s))\, ds\right) \tag{3.3}$$

transforms the space $C_0(\mathbb{R}_+)$ into itself. Besides, the inequality

$$\|Tx\| \leq l_1 d_1(\|x\|) + l_2 d_2(\|x\|) h(\|x\|) G + F \tag{3.4}$$

holds for all $x \in C_0(\mathbb{R}_+)$.

Proof . It is obvious from $(H1)$, $(H3)$ and Lemma 3.3 that the function Tx given by (3.3) is continuous on $\mathbb{R}_+$.

Next we show that $(Tx)(t) \to 0$ as $t \to \infty$. For each $t \in \mathbb{R}_+$ we have the estimate:

$$\begin{aligned}
|(Tx)(t)| \leq & \left| f\left(t, (T_1x)(t), (T_2x)(t) \int_0^\infty u\left(t, s, x(s)\right) ds\right) \right. \\
& \left. - f\left(t, 0, (T_2x)(t) \int_0^\infty u\left(t, s, x(s)\right) ds\right) \right| \\
& + \left| f\left(t, 0, (T_2x)(t) \int_0^\infty u\left(t, s, x(s)\right) ds\right) - f\left(t, 0, 0\right) \right| \\
& + |f\left(t, 0, 0\right)|.
\end{aligned} \tag{3.5}$$

By using of $(H2)$ and Lemma 3.3 we get from (3.5) that

$$\begin{aligned}
|(Tx)(t)| & \leq l_1 \left|(T_1x)(t)\right| + l_2 \left|(T_2x)(t) \int_0^\infty u\left(t, s, x(s)\right) ds\right| + \left|f\left(t, 0, 0\right)\right| \\
& \leq l_1 \left|(T_1x)(t)\right| + l_2 \left|(T_2x)(t)\right| h\left(\|x\|\right) G + \left|f\left(t, 0, 0\right)\right|
\end{aligned} \tag{3.6}$$

for all $t \in \mathbb{R}_+$. The estimate (3.6), together with $(H1)$ and $(H3)$, gives that $(Tx)(t) \to 0$ as $t \to \infty$. Thus the operator T transforms the space $C_0(\mathbb{R}_+)$ into itself. Additionally, by (3.6) and hypotheses, we obtain that

$$|(Tx)(t)| \leq l_1\|T_1x\| + l_2\|T_2x\|h(\|x\|)G + F \tag{3.7}$$

for all $t \in \mathbb{R}_+$. So we have (3.4) from (3.7) and $(H3)$. □

Remark 3.5. Let's assume that the assumptions $(H1) - (H6)$ hold. Then, by considering Lemma 3.4, we infer that $T : B_{r_0} \to B_{r_0}$ from (3.3), (3.4) and $(H6)$.

Theorem 3.6. If the assumptions $(H1) - (H7)$ hold, then the estimate

$$\mu(TX) \leq k\mu(X) \tag{3.8}$$

holds for any nonempty subset X of the ball B_{r_0}. Here, T is the operator defined by (3.3) and k is the constant given as $k = l_1 m_{1,r_0} + l_2 m_{2,r_0} h(r_0) G + 2l_2 d_2(r_0) b_{r_0} G$.

Proof . Let us take a nonempty subset X of the ball B_{r_0}. It is clear by Remark 3.5 that $T : B_{r_0} \to B_{r_0}$. Fix $\varepsilon \geq 0$, $L > 0$, $t_1, t_2 \in [0, L]$ with $|t_1 - t_2| \leq \varepsilon$ and take an arbitrary function $x \in X$. Then, we get that:

$$\begin{aligned}
& |(Tx)(t_1) - (Tx)(t_2)| \\
\leq & \left| f\left(t_1, (T_1x)(t_1), (T_2x)(t_1) \int_0^\infty u\left(t_1, s, x(s)\right) ds\right) - f\left(t_2, (T_1x)(t_1), (T_2x)(t_1) \int_0^\infty u\left(t_1, s, x(s)\right) ds\right) \right| \\
& + \left| f\left(t_2, (T_1x)(t_1), (T_2x)(t_1) \int_0^\infty u\left(t_1, s, x(s)\right) ds\right) - f\left(t_2, (T_1x)(t_2), (T_2x)(t_1) \int_0^\infty u\left(t_1, s, x(s)\right) ds\right) \right| \\
& + \left| f\left(t_2, (T_1x)(t_2), (T_2x)(t_1) \int_0^\infty u\left(t_1, s, x(s)\right) ds\right) - f\left(t_2, (T_1x)(t_2), (T_2x)(t_2) \int_0^\infty u\left(t_2, s, x(s)\right) ds\right) \right|.
\end{aligned} \tag{3.9}$$

From (3.9) and $(H2)$ we get that

$$\begin{aligned}
|(Tx)(t_1) - (Tx)(t_2)| \leq & \ \omega_{r_0}^L(f, \varepsilon) + l_1 \left|(T_1x)(t_1) - (T_1x)(t_2)\right| \\
& + l_2 \left|(T_2x)(t_1) \int_0^\infty u(t_1, s, x(s)) ds - (T_2x)(t_2) \int_0^\infty u(t_2, s, x(s)) ds\right| \\
= & \ \omega_{r_0}^L(f, \varepsilon) + l_1 \left|(T_1x)(t_1) - (T_1x)(t_2)\right| \\
& + l_2 \left| \left[(T_2x)(t_1) - (T_2x)(t_2)\right] \int_0^\infty u\left(t_1, s, x(s)\right) ds \right. \\
& \left. + (T_2x)(t_2) \int_0^\infty \left[u\left(t_1, s, x(s)\right) - u\left(t_2, s, x(s)\right)\right] ds \right|,
\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}\omega_{r_0}^L(f,\varepsilon) &= \sup\left\{|f(t_1,x_1,y)-f(t_2,x_1,y)| : t_1,t_2\in[0,L], x_1\in[-d_1(r_0),\, d_1(r_0)\,],\right.\\ &\quad \left. y\in[-d_2(r_0)h(r_0)G,\, d_2(r_0)h(r_0)G\,] \text{ and } |t_1-t_2|\le\varepsilon\right\}.\end{aligned}$$

So we have by (3.10) and assumptions that:

$$\begin{aligned}
|(Tx)(t_1)-(Tx)(t_2)| &\le \omega_{r_0}^L(f,\varepsilon)+l_1\,|(T_1x)(t_1)-(T_1x)(t_2)|\\
&\quad +l_2\,|(T_2x)(t_1)-(T_2x)(t_2)|\int_0^\infty |u(t_1,s,x(s))|ds\\
&\quad +l_2\,|(T_2x)(t_2)|\int_0^\infty |u(t_1,s,x(s))-u(t_2,s,x(s))|ds\\
&\le \omega_{r_0}^L(f,\varepsilon)+l_1\omega^L(T_1x,\varepsilon)+l_2\omega^L(T_2x,\varepsilon)h(\|x\|)\int_0^\infty g(t_1,s)ds\\
&\quad +l_2d_2(\|x\|)\int_0^\infty |u(t_1,s,x(s))-u(t_2,s,x(s))|\,ds\\
&\le \omega_{r_0}^L(f,\varepsilon)+l_1\omega^L(T_1x,\varepsilon)+l_2h(r_0)G\omega^L(T_2x,\varepsilon)\\
&\quad +l_2d_2(r_0)\left\{\int_0^L |u\,(t_1,s,x(s))-u\,(t_2,s,x(s))|\,ds\right.\\
&\quad \left.+\int_L^\infty |u\,(t_1,s,x(s))-u\,(t_2,s,x(s))|\,ds\right\}\\
&\le \omega_{r_0}^L(f,\varepsilon)+l_1\omega^L(T_1x,\varepsilon)+l_2h(r_0)G\omega^L(T_2x,\varepsilon)+l_2d_2(r_0)L\omega_{r_0}^L(u,\varepsilon)\\
&\quad +l_2d_2(r_0)\,[\sup\{h\,(|x(s)|):s\ge L\}]\int_L^\infty [g(t_1,s)+g(t_2,s)]\,ds\\
&\le \omega_{r_0}^L(f,\varepsilon)+l_1\omega^L(T_1x,\varepsilon)+l_2h(r_0)G\omega^L(T_2x,\varepsilon)\\
&\quad +l_2d_2(r_0)\left[L\omega_{r_0}^L(u,\varepsilon)+2G\sup\{h\,(|x(s)|):s\ge L\}\right]
\end{aligned}$$

which yields that:

$$\omega^L(Tx,\varepsilon) \;\le\; \omega_{r_0}^L(f,\varepsilon)+l_1\omega^L(T_1x,\varepsilon)+l_2h(r_0)G\omega^L(T_2x,\varepsilon)+l_2d_2(r_0)\left[L\omega_{r_0}^L(u,\varepsilon)+2G\beta^L(h_x)\right], \tag{3.11}$$

where

$$\omega^L(T_ix,\varepsilon)=\sup\{|(T_ix)(t_1)-(T_ix)(t_2)|:t_1,t_2\in[0,L] \text{ and } |t_1-t_2|\le\varepsilon\},\ \ (i=1,2)$$

and

$$\begin{aligned}\omega_{r_0}^L(u,\varepsilon) &= \sup\left\{|u(t_1,s,y)-u(t_2,s,y)| : t_1,t_2,s\in[0,L],\right.\\ &\quad \left. y\in[-r_0,r_0\,] \text{ and } |t_1-t_2|\le\varepsilon\right\}.\end{aligned}$$

Hence, we get by (3.11) that:

$$\omega^L(TX,\varepsilon) \;\le\; \omega_{r_0}^L(f,\varepsilon)+l_1\omega^L(T_1X,\varepsilon)+l_2h(r_0)G\omega^L(T_2X,\varepsilon)+l_2d_2(r_0)\left[L\omega_{r_0}^L(u,\varepsilon)+2G\beta^L(h_X)\right]. \tag{3.12}$$

Since the functions f and u are uniformly continuous on $[0,L]\times[-d_1(r_0),\, d_1(r_0)\,]\times[-d_2(r_0)h(r_0)G,\, d_2(r_0)h(r_0)G\,]$ and $[0,L]\times[0,L]\times[-r_0,r_0\,]$, respectively, we get that $\omega_{r_0}^L(f,\varepsilon)\to 0$ and $\omega_{r_0}^L(u,\varepsilon)\to 0$ as $\varepsilon\to 0$. Now taking into account the properties of the component involved in the estimate (3.12), we have:

$$\omega_0^L(TX)\le l_1\omega_0^L(T_1X)+l_2h(r_0)G\omega_0^L(T_2X)+2l_2d_2(r_0)G\beta^L(h_X)$$

which yields that:

$$\omega_0(TX)\le l_1\omega_0(T_1X)+l_2h(r_0)G\omega_0(T_2X)+2l_2d_2(r_0)G\beta(h_X). \tag{3.13}$$

Further taking $x \in X$ and choosing arbitrarily $L > 0$, in view of the estimate (3.5) we obtain that:

$$\begin{aligned}
\sup\{|(Tx)(t)| : t \geq L\} &\leq l_1 \sup\{|(T_1x)(t)| : t \geq L\} + \sup\{|f(t,0,0)| : t \geq L\} \\
&\quad + l_2 \sup\left\{|(T_2x)(t)| \int_0^\infty |u(t,s,x(s))|ds : t \geq L\right\} \\
&\leq l_1 \sup\{|(T_1x)(t)| : t \geq L\} + \sup\{|f(t,0,0)| : t \geq L\} \\
&\quad + l_2 \sup\{|(T_2x)(t)| : t \geq L\}\, h(\|x\|) \sup\left\{\int_0^\infty g(t,s)ds : t \geq L\right\} \\
&\leq l_1 \sup\{|(T_1x)(t)| : t \geq L\} + \sup\{|f(t,0,0)| : t \geq L\} \\
&\quad + l_2 h(r_0) G \sup\{|(T_2x)(t)| : t \geq L\}. \qquad (3.14)
\end{aligned}$$

Hence, by considering hypotheses, we get by (3.14) that:

$$\beta(TX) \leq l_1\beta(T_1X) + l_2h(r_0)G\beta(T_2X). \qquad (3.15)$$

Now, linking (2.3), (3.13) and (3.15) and by using the assumption $(H7)$ and the inequality $\beta(X) \leq \mu(X)$ we derive that:

$$\begin{aligned}
\mu(TX) &\leq l_1\mu(T_1X) + l_2h(r_0)G\mu(T_2X) + 2l_2d_2(r_0)G\beta(h_X) \\
&\leq l_1m_{1,r_0}\mu(X) + l_2m_{2,r_0}h(r_0)G\mu(X) + 2l_2d_2(r_0)b_{r_0}G\beta(X) \\
&\leq k\mu(X)
\end{aligned}$$

which completes the proof.□

We can now give the following theorem about the existence of a solution of the integral equation (1.1):

Theorem 3.7. Under the assumptions $(H1) - (H8)$, there exists at least one solution $x = x(t)$ of Eq. (1.1) in the space $C_0(\mathbb{R}_+)$.

Proof . We define operator T on $C_0(\mathbb{R}_+)$ in the following way:

$$(Tx)(t) = f\left(t, (T_1x)(t), (T_2x)(t)\int_0^\infty u(t,s,x(s))\,ds\right).$$

Consider the sequence of sets $(B_{r_0}^n)$, where $B_{r_0}^1 = \operatorname{Conv} T(B_{r_0})$, $B_{r_0}^2 = \operatorname{Conv} T(B_{r_0}^1)$ and so on. Observe that all sets of this sequence are nonempty, bounded, closed and convex. Moreover, $B_{r_0}^{n+1} \subset B_{r_0}^n$ for all $n \in \{1, 2, \ldots\}$. Further, keeping in mind Theorem 3.6, we get from (3.8) that:

$$\mu(B_{r_0}^n) \leq k^n\mu(B_{r_0}). \qquad (3.16)$$

Obviously in view of assumption $(H8)$ we have that $k < 1$. Hence, from condition (5) of Definition 2.1 we infer that the set $Y = \bigcap_{n=1}^\infty B_{r_0}^n$ is nonempty, bounded, closed and convex. In fact, since $\mu(Y) \leq \mu(B_{r_0}^n)$ for any $n \geq 1$, we deduce by (3.16) that $\mu(Y) = 0$ and thus $Y \in \ker\mu$. It should be also noted that the operator T maps the set Y into itself.

Now we show that T is continuous on the set Y. To do this fix $\varepsilon \geq 0$, $x_0 \in Y$ and take functions $x \in Y$ such that $\|x - x_0\| \leq \varepsilon$. Taking into account the fact that $Y \in \ker\mu$ and the description of sets belonging to $\ker\mu$ we can find a number $L > 0$ such that for each $z \in Y$ and $t \geq L$ the inequality $|z(t)| \leq \varepsilon$ is satisfied. Since $T : Y \to Y$, we obtain that

$$|(Tx)(t) - (Tx_0)(t)| \leq |(Tx)(t)| + |(Tx_0)(t)| \leq 2\varepsilon \qquad (3.17)$$

for all $x \in Y$ and $t \geq L$. On the other hand, for $t \in [0, L]$ we get that:

$$\begin{aligned}
&|(Tx)(t) - (Tx_0)(t)| \\
\leq &\left|f\left(t, (T_1x)(t), (T_2x)(t)\int_0^\infty u(t,s,x(s))\,ds\right) - f\left(t, (T_1x_0)(t), (T_2x)(t)\int_0^\infty u(t,s,x(s))\,ds\right)\right| \\
&+ \left|f\left(t, (T_1x_0)(t), (T_2x)(t)\int_0^\infty u(t,s,x(s))\,ds\right) - f\left(t, (T_1x_0)(t), (T_2x_0)(t)\int_0^\infty u(t,s,x_0(s))\,ds\right)\right|. \qquad (3.18)
\end{aligned}$$

From (3.18) and $(H2)$ we have that:

$$\begin{aligned}
&|(Tx)(t)-(Tx_0)(t)| \\
\leq\ & l_1\left|(T_1x)(t)-(T_1x_0)(t)\right| + l_2\left|(T_2x)(t)\int_0^\infty u\left(t,s,x(s)\right)ds-(T_2x_0)(t)\int_0^\infty u\left(t,s,x_0(s)\right)ds\right| \\
=\ & l_1\left|(T_1x)(t)-(T_1x_0)(t)\right| \\
& + l_2\left|\left[(T_2x)(t)-(T_2x_0)(t)\right]\int_0^\infty u\left(t,s,x(s)\right)ds+(T_2x_0)(t)\int_0^\infty \left[u\left(t,s,x(s)\right)-u\left(t,s,x_0(s)\right)\right]ds\right|. \qquad (3.19)
\end{aligned}$$

Thus, we can write by (3.19) that

$$\begin{aligned}
|(Tx)(t)-(Tx_0)(t)| \leq\ & l_1\left|(T_1x)(t)-(T_1x_0)(t)\right| + l_2\left|(T_2x)(t)-(T_2x_0)(t)\right|\int_0^\infty \left|u\left(t,s,x(s)\right)\right|ds \\
& + l_2\left|(T_2x_0)(t)\right|\int_0^\infty \left|u\left(t,s,x(s)\right)-u\left(t,s,x_0(s)\right)\right|ds
\end{aligned}$$

and so by assumptions

$$\begin{aligned}
|(Tx)(t)-(Tx_0)(t)| \leq\ & l_1\|T_1x-T_1x_0\| + l_2\|T_2x-T_2x_0\|\, h(\|x\|)\int_0^\infty g(t,s)ds \\
& + l_2d_2(\|x_0\|)\left(\int_0^L \left|u(t,s,x(s))-u(t,s,x_0(s))\right|ds\right. \\
& \left. + \int_L^\infty \left|u(t,s,x(s))-u(t,s,x_0(s))\right|ds\right)
\end{aligned}$$

which yields that

$$\begin{aligned}
|(Tx)(t)-(Tx_0)(t)| \leq\ & l_1\|T_1x-T_1x_0\| + l_2\|T_2x-T_2x_0\|h(r_0)G + l_2d_2(r_0)L\bar{\omega}^L_{r_0}(u,\varepsilon) \\
& + l_2d_2(r_0)\left[\sup\left\{h\left(|x(s)|\right): s\geq L\right\}\right]\int_L^\infty g(t,s)ds \\
& + l_2d_2(r_0)\left[\sup\left\{h\left(|x_0(s)|\right): s\geq L\right\}\right]\int_L^\infty g(t,s)ds \\
\leq\ & l_1\|T_1x-T_1x_0\| + l_2\|T_2x-T_2x_0\|h(r_0)G + l_2d_2(r_0)L\bar{\omega}^L_{r_0}(u,\varepsilon) \\
& + l_2d_2(r_0)G\left[h\left(\sup\left\{|x(s)|: s\geq L\right\}\right)+h\left(\sup\left\{|x_0(s)|: s\geq L\right\}\right)\right] \qquad (3.20)
\end{aligned}$$

for all $t\in[0,L]$, where

$$\bar{\omega}^L_{r_0}(u,\varepsilon)=\sup\{|u(t,s,x)-u(t,s,y)| : t,s\in[0,L];\ x,y\in[-r_0,r_0] \text{ and } |x-y|\leq\varepsilon\}.$$

Observe that $\bar{\omega}^L_{r_0}(u,\varepsilon)\to 0$ as $\varepsilon\to 0$. Moreover, taking into account $Y\in \ker\mu$ and the continuity of h together with $h(0)=0$, we can choose L in such a way that the term

$$h\left(\sup\left\{|x(s)|: s\geq L\right\}\right)+h\left(\sup\left\{|x_0(s)|: s\geq L\right\}\right)$$

in estimate (3.20) is small enough for all $x\in Y$. By the above facts and continuity of the operators T_1 and T_2, we conclude from (3.17) and (3.20) that the operator T is continuous at the arbitrary fixed point x_0. Thus, T is continuous on the set Y. Finally, linking all above established properties of the set Y and the operator $T: Y\to Y$ and using the Schauder fixed point principle we infer that the operator T has at least one fixed point x in the set Y. This means that there exists at least one solution $x=x(t)$ of Eq. (1.1) in the space $C_0(\mathbb{R}_+)$. $\square$

4 Examples

Example 4.1. Consider the following nonlinear integral equation of the form:

$$x(t)=\frac{1}{1+t}+x(t)\int_0^\infty \frac{t}{t+s}\phi(s)x(s)ds, \qquad (4.1)$$

where $t \in \mathbb{R}_+$. Notice that Eq. (4.1) is a similar form of the famous quadratic integral equation of Chandrasekhar type:

$$x(t) = 1 + x(t) \int_0^1 \frac{t}{t+s} \phi(s) x(s) ds, \; t \in [0,1].$$

Also, Eq.(4.1) is a special case of (1.1) if we put:

$$f(t,x,y) = x + y, \; (T_1 x)(t) = \frac{1}{1+t}, \; (T_2 x)(t) = x(t)$$

and

$$u(t,s,x) = \begin{cases} 0, & s = 0, \; t \geq 0, \; x \in \mathbb{R} \\ \frac{t}{t+s} \phi(s) x, & s \neq 0, \; t \geq 0, \; x \in \mathbb{R}. \end{cases} \tag{4.2}$$

Here, the characteristic function $\phi : \mathbb{R}_+ \to \mathbb{R}$ is continuous and satisfies $\phi(0) = 0$.

The function f is continuous on $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$ and the equality $\lim_{t\to\infty} f(t,0,0) = 0$ holds. Besides, f satisfies the Lipschitz condition with the nonnegative constants $l_1 = 1$ and $l_2 = 1$ with respect to second and third variables. Therefore the assumptions $(H1)$ and $(H2)$ of Theorem 3.7 are valid and the constant F can be taken as $F = \sup \{|f(t,0,0)| : t \geq 0\} = 0$.

Since

$$|(T_1 x)(t) - (T_1 y)(t)| = 0 \text{ and } |(T_2 x)(t) - (T_2 y)(t)| = |x(t) - y(t)|$$

for all $t \in \mathbb{R}_+$ and $x, y \in C_0(\mathbb{R}_+)$, we derive that:

$$\|T_1 x - T_1 y\| = 0 \text{ and } \|T_2 x - T_2 y\| = \|x - y\|$$

which yield that the operators T_1 and T_2 are continuous on the space $C_0(\mathbb{R}_+)$. Further for all $t \in \mathbb{R}_+$ and $x \in C_0(\mathbb{R}_+)$, the equalities

$$|(T_1 x)(t)| = \frac{1}{1+t} \text{ and } |(T_2 x)(t)| = |x(t)|$$

hold. Therefore, $\|T_1 x\| = 1$ and $\|T_2 x\| = \|x\|$ for all $x \in C_0(\mathbb{R}_+)$. Hence, the assumption $(H3)$ is satisfied with $d_1(t) = 1$ and $d_2(t) = t$.

The function $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ defined by (4.2) is continuous, *[11]*. If the function ϕ is chosen as $\phi(s) = \frac{s\,e^{-s}}{6}$, then it is obtained that the inequality

$$|u(t,s,x)| \leq g(t,s) h(|x|) \tag{4.3}$$

holds for all $t, s \in \mathbb{R}_+$ and $x \in \mathbb{R}$, where the functions $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and $h : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (4.3) are defined by

$$g(t,s) = \begin{cases} 0, & s = 0, \; t \geq 0 \\ \frac{tse^{-s}}{6(t+s)}, & s \neq 0, \; t \geq 0 \end{cases}$$

and $h(t) = t$. The continuity of g and h is clear. Also, $h(0) = 0$ and h is nondecreasing. Additionally,

$$\int_0^\infty g(t,s) ds = \int_0^\infty \frac{tse^{-s}}{6(t+s)} ds \leq \frac{1}{6} \int_0^\infty se^{-s} ds = \frac{1}{6}$$

for all $t \geq 0$. So the assumptions $(H4)$ and $(H5)$ hold and we can chose the constant G as $G = \frac{1}{6}$.

The inequality in the assumption $(H6)$ is equivalent to:

$$1 + \frac{r^2}{6} \leq r. \tag{4.4}$$

The number r_0 chosen as $3 - \sqrt{3} \leq r_0 < 2$ satisfies (4.4).

Apart from this, fixing a nonempty and bounded subset X of the ball B_{r_0}. Let $x \in X$, $\varepsilon \geq 0$, $L > 0$ and $t_1, t_2 \in [0, L]$ such that $|t_1 - t_2| \leq \varepsilon$. Then,

$$|(T_1x)(t_1) - (T_1x)(t_2)| = \left|\frac{1}{1+t_1} - \frac{1}{1+t_2}\right|$$

and

$$|(T_2x)(t_1) - (T_2x)(t_2)| = |x(t_1) - x(t_2)|\,.$$

The function $t \to \frac{1}{1+t}$ is uniformly continuous on the set $[0, L]$. This implies that $\omega^L(T_1x, \varepsilon) \to 0$ as $\varepsilon \to 0$. Besides, $\omega^L(T_2x, \varepsilon) = \omega^L(x, \varepsilon)$. So, it is clear that $\omega_0(T_1X) = 0$ and $\omega_0(T_2X) = \omega_0(X)$. By the equalities

$$\beta^L(T_1x) = \sup\{|(T_1x)(t)| : t \geq L\} = \sup\left\{\frac{1}{1+t} : t \geq L\right\} = \frac{1}{1+L},$$

$$\beta^L(T_2x) = \sup\{|(T_2x)(t)| : t \geq L\} = \sup\{|x(t)| : t \geq L\} = \beta^L(x)$$

and

$$\beta^L(h_x) = \sup\{|h_x(t)| : t \geq L\} = \sup\{h\left(|x(t)|\right) : t \geq L\} = \sup\{|x(t)| : t \geq L\} = \beta^L(x),$$

we derive that $\beta(T_1X) = 0$, $\beta(T_2X) = \beta(X)$ and $\beta\left(h_X\right) = \beta(X)$. Thus, we have by (2.3) that $\mu(T_1X) = 0$ and $\mu(T_2X) = \mu(X)$. As a result, it is understood that the assumption $(H7)$ holds with the constants $m_{1,r_0} = 0$, $m_{2,r_0} = 1$ and $b_{r_0} = 1$.

The inequality in $(H8)$ has the form:

$$\frac{r_0}{2} < 1. \tag{4.5}$$

Since $3 - \sqrt{3} \leq r_0 < 2$, (4.5) is satisfied.

Since all of the assumptions of Theorem 3.7 are fullfilled, we deduce that the integral equation (4.1) has at least one solution $x = x(t)$ belonging to the space $C_0(\mathbb{R}_+)$.

Remark 4.2. Let us consider the nonlinear integral equation:

$$x(t) = \frac{1}{1+t} + x(t)\int_0^\infty \frac{t\lambda\phi(s)}{t+s}\log\left(1 + |x(s)|\right)ds,\ t \in \mathbb{R}_+ \tag{4.6}$$

which is a similar version of the quadratic integral equation of the generalized Chandrasekhar's type:

$$x(t) = 1 + x(t)\int_0^1 \frac{t\lambda\phi(s)}{t+s}\log\left(1 + |x(s)|\right)ds,\ t \in [0,1]. \tag{4.7}$$

The equation (4.7) was considered in many papers (see, *[11, 13]*) and it arose originally in connection with scattering through a homogeneous semi-infinite plane atmosphere, *[13]*. If we take

$$f(t,x,y) = x + y,\ (T_1x)(t) = \frac{1}{1+t},\ (T_2x)(t) = x(t),$$

$$u(t,s,x) = \begin{cases} 0, & s = 0,\ t \geq 0,\ x \in \mathbb{R} \\ \frac{t\lambda}{t+s}\phi(s)\log\left(1+|x|\right), & s \neq 0,\ t \geq 0,\ x \in \mathbb{R} \end{cases}$$

and $\phi(s) = \frac{s\,e^{-s}}{6}$, then (4.6) is a special form of (1.1). Since $\log\left(1+|x|\right) \leq |x|$ for all $x \in \mathbb{R}$, the inequality $|u(t,s,x)| \leq g(t,s)|x|$ holds for all $t, s \in \mathbb{R}_+$ and $x \in \mathbb{R}$, where

$$g(t,s) = \begin{cases} 0, & s = 0,\ t \geq 0 \\ \frac{t|\lambda|}{6(t+s)}se^{-s}, & s \neq 0,\ t \geq 0. \end{cases}$$

Also, by taking $h(t)=t$, we can easily see by Example 4.1 that the assumptions $(H1)-(H5)$ of Theorem 3.7 hold. Besides, it is clear that the assumption $(H6)$ is equivalent to the inequality:

$$1+\frac{r^2|\lambda|}{6}\leq r. \tag{4.8}$$

If the real parameter λ is chosen as $0<|\lambda|<\frac{4}{3}$, then (4.8) holds for the number r_0 with $\frac{3-3\sqrt{1-\frac{2}{3}|\lambda|}}{|\lambda|}\leq r_0<\frac{2}{|\lambda|}$ and the validity of the assumption $(H7)$ can be shown as in Example 4.1.

Finally, the inequality in $(H8)$ is equivalent to:

$$\frac{r_0|\lambda|}{2}<1 \tag{4.9}$$

and (4.9) is satisfied for $\frac{3-3\sqrt{1-\frac{2}{3}|\lambda|}}{|\lambda|}\leq r_0<\frac{2}{|\lambda|}$.

Since the conditions of Theorem 3.7 are satisfied, the equation (4.6) has at least one solution in the space $C_0(\mathbb{R}_+)$.

Example 4.3. Let us consider the following nonlinear integral equation:

$$\begin{aligned} x(t) &= \frac{t}{1+t^2}+\frac{4\sin x(t)}{25\left(1+\sin^2 x(t)\right)} \\ &+\frac{2x^2(t)\int_0^\infty \frac{\cos t}{(1+s)^2}x^3(s)ds}{50\left[1+\left(x^2(t)\int_0^\infty \frac{\cos t}{(1+s)^2}x^3(s)ds\right)^2\right]},\ t\in\mathbb{R}_+. \end{aligned} \tag{4.10}$$

If we put

$$f(t,x,y)=\frac{t}{1+t^2}+\frac{4x}{25\,(1+x^2)}+\frac{2y}{50\,(1+y^2)},$$

$$(T_1x)(t)=\sin x(t),\ (T_2x)(t)=x^2(t)\text{ and }u(t,s,x)=\frac{\cos t}{(1+s)^2}x^3,$$

then Eq. (4.10) is a special case of Eq. (1.1).

It is easily verified that the assumptions of Theorem 3.7 are satisfied.

Indeed, the function f is continuous on the set $\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}$ and f satisfies $\lim_{t\to\infty}f(t,0,0)=\lim_{t\to\infty}\frac{t}{1+t^2}=0$.

Since

$$|f(t,x_1,y)-f(t,x_2,y)|\leq\frac{4}{25}\left(\frac{|x_1-x_2|}{(1+x_1^2)\,(1+x_2^2)}+\frac{|x_1|\,|x_2|}{(1+x_1^2)\,(1+x_2^2)}|x_1-x_2|\right),$$

$$|f(t,x,y_1)-f(t,x,y_2)|\leq\frac{2}{50}\left(\frac{|y_1-y_2|}{(1+y_1^2)\,(1+y_2^2)}+\frac{|y_1|\,|y_2|}{(1+y_1^2)\,(1+y_2^2)}|y_1-y_2|\right)$$

for all $t\in\mathbb{R}_+$ and $x,x_1,x_2,y,y_1,y_2\in\mathbb{R}$, we can write the inequalities:

$$|f(t,x_1,y)-f(t,x_2,y)|\leq\frac{4}{25}\left(|x_1-x_2|+\frac{1}{2}\frac{1}{2}|x_1-x_2|\right)=\frac{1}{5}|x_1-x_2|,$$

$$|f(t,x,y_1)-f(t,x,y_2)|\leq\frac{2}{50}\left(|y_1-y_2|+\frac{1}{2}\frac{1}{2}|y_1-y_2|\right)=\frac{1}{20}|y_1-y_2|.$$

Thus, l_1 and l_2 can be taken as $l_1=\frac{1}{5}$ and $l_2=\frac{1}{20}$.

T_1 and T_2 are the continuous operators on the space $C_0(\mathbb{R}_+)$. Further for all $x\in C_0(\mathbb{R}_+)$ and $t\in\mathbb{R}_+$, we have that:

$$|(T_1x)(t)|\leq\|x\|\text{ and }|(T_2x)(t)|\leq\|x\|^2.$$

Hence the assumption $(H3)$ is satisfied with $d_1(t)=t$ and $d_2(t)=t^2$.

Now notice that the function u is continuous on the set $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$. Moreover, since

$$|u(t,s,x)| = \left|\frac{\cos t}{(1+s)^2}x^3\right| = \frac{|\cos t|}{(1+s)^2}|x|^3$$

for all $t, s \in \mathbb{R}_+$ and $x \in \mathbb{R}$, if we choose $g(t,s) = \frac{|\cos t|}{(1+s)^2}$ and $h(t) = t^3$, we see that the assumption $(H4)$ is satisfied.

To check that the assumption $(H5)$ is satisfied let us observe that $s \to \frac{|\cos t|}{(1+s)^2}$ is integrable on $\mathbb{R}_+$ and $t \to \int_0^\infty \frac{|\cos t|}{(1+s)^2}ds$ is bounded on $\mathbb{R}_+$. Thus it is easily seen that:

$$G = \sup\left\{\int_0^\infty \frac{|\cos t|}{(1+s)^2}ds : t \in \mathbb{R}^+\right\} = \sup\left\{|\cos t| : t \in \mathbb{R}^+\right\} = 1.$$

Besides, $F = \sup\{|f(t,0,0)| : t \geq 0\} = \frac{1}{2}$ and the inequality in the assumption $(H6)$ has the form:

$$\frac{r}{5} + \frac{r^5}{20} + \frac{1}{2} \leq r. \tag{4.11}$$

It can be easily verified that if $0.631265 \leq r_0 < \sqrt{2}$, then r_0 is the solution of (4.11). Apart from this for $\varepsilon \geq 0$, $L > 0$, $\|x\| \leq r_0$ and $t, s \in [0, L]$ such that $|t-s| \leq \varepsilon$, we have:

$$\begin{aligned} |(T_1x)(t) - (T_1x)(s)| &= |\sin x(t) - \sin x(s)| \\ &\leq |x(t) - x(s)|. \end{aligned} \tag{4.12}$$

Further, it can be seen that:

$$\begin{aligned} |(T_2x)(t) - (T_2x)(s)| &= \left|x^2(t) - x^2(s)\right| \\ &= |x(t) - x(s)|\,|x(t) + x(s)| \\ &\leq 2r_0|x(t) - x(s)|. \end{aligned} \tag{4.13}$$

From the estimates (4.12) and (4.13) in view of the (2.1) we have that

$$\omega_0(T_1X) \leq \omega_0(X) \text{ and } \omega_0(T_2X) \leq 2r_0\omega_0(X),$$

respectively. Besides, we get that:

$$\begin{aligned} \sup\{|(T_1x)(t)| : t \geq L\} &= \sup\{|\sin x(t)| : t \geq L\} \\ &\leq \sup\{|x(t)| : t \geq L\}. \end{aligned} \tag{4.14}$$

Thus from the estimate (4.14) we have that:

$$\beta(T_1X) \leq \beta(X).$$

Moreover, we derive that:

$$\begin{aligned} \sup\{|(T_2x)(t)| : t \geq L\} &= \sup\left\{\left|x^2(t)\right| : t \geq L\right\} \\ &\leq r_0 \sup\{|x(t)| : t \geq L\} \\ &\leq 2r_0 \sup\{|x(t)| : t \geq L\} \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} \sup\{|h_x(t)| : t \geq L\} &= \sup\{h(|x(t)|) : t \geq L\} \\ &= \sup\left\{|x(t)|^3 : t \geq L\right\} \\ &\leq r_0^2 \sup\{|x(t)| : t \geq L\}. \end{aligned} \tag{4.16}$$

Thus from the estimates (4.15) and (4.16) we have that:

$$\beta(T_2X) \leq 2r_0\beta(X) \text{ and } \beta\left(h_X\right) \leq r_0^2\beta(X).$$

Therefore, taking into account the above estimates and (2.3), we get m_{1,r_0}, m_{2,r_0} and b_{r_0} are equal to 1, $2r_0$ and r_0^2, respectively.

Besides, it is obtained that

$$l_1m_{1,r_0} + l_2m_{2,r_0}h(r_0)G + 2l_2d_2(r_0)b_{r_0}G = \frac{1}{5} + \frac{r_0^4}{5} < 1$$

for $0.631265 \leq r_0 < \sqrt{2}$. Hence the assumption $(H8)$ is satisfied.

Thus we showed that all assumptions of Theorem 3.7 are fulfilled. This yields that the Eq.(4.10) has at least one solution $x = x(t)$ in the space $C_0(\mathbb{R}_+)$.

Remark 4.4. Notice that the integral equation (4.10) can't be derived from any of the integral equations handled in *[1, 5, 6, 7, 10, 15, 17, 19, 20, 23, 24, 25, 27, 28, 29, 30]*.

Besides, (4.10) can be obtained from (1.11) by choosing

$$F(t,x,y) = \frac{t}{1+t^2} + \frac{4\sin x}{25\left(1+\sin^2 x\right)} + \frac{2x^2y}{50\left(1+x^4y^2\right)}$$

and $u(t,s,x) = \frac{\cos t}{(1+s)^2}x^3$ for $t,s \in \mathbb{R}_+$ and $x,y \in \mathbb{R}$. But, there is no function $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the inequality

$$|u(t,s,x)| \leq g(t,s)|x|$$

for all $t,s \in \mathbb{R}_+$ and $x \in \mathbb{R}$. **Therefore, the assumption (H_3) in *[26]* doesn't hold. Consequently, the existence theorem given in *[26]* cannot be applied to the integral equation (4.10).**

Example 4.5. Let us consider the following integral equation:

$$\begin{aligned} x(t) &= \frac{1}{4}\exp(-t) + \frac{1}{3}\ln\left(1+\frac{|x(t)|}{1+t}\right) \\ &\quad + \frac{1}{10}\arctan\left(x(t)\sqrt{x^2(t)+1}\int_0^\infty t\exp\left(-ts-s\right)x(s)ds\right),\ t\in\mathbb{R}_+. \end{aligned} \tag{4.17}$$

Observe that this equation has the form of Eq.(1.1) if we put:

$$f(t,x,y) = \frac{1}{4}\exp(-t) + \frac{1}{3}\ln\left(1+|x|\right) + \frac{1}{10}\arctan y,$$

$$(T_1x)(t) = \frac{x(t)}{1+t},\ (T_2x)(t) = x(t)\sqrt{x^2(t)+1} \text{ and } u(t,s,x) = t\exp\left(-ts-s\right)x.$$

It is easily shown that the function $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and $f(t,0,0) = \frac{1}{4}\exp(-t) \to 0$ as $t \to \infty$.

Without loss of generality we can suppose that $|x_1| < |x_2|$. So, there exists $\eta \in (|x_1|, |x_2|)$ satisfying the inequality

$$\begin{aligned} |f(t,x_1,y) - f(t,x_2,y)| &= \frac{1}{3}\left|\ln\left(1+|x_1|\right) - \ln(1+|x_2|)\right| \\ &\leq \frac{1}{3(1+\eta)}\left|x_1 - x_2\right| \end{aligned} \tag{4.18}$$

for all $t \in \mathbb{R}_+$ and $y \in \mathbb{R}$. Taking into account (4.18), we have

$$|f(t,x_1,y) - f(t,x_2,y)| \leq \frac{1}{3}\left|x_1 - x_2\right|$$

for all $t \in \mathbb{R}_+$ and $x_1, x_2, y \in \mathbb{R}$.

We assume that $y_1 < y_2$ without loss of generality. Thus, for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$, we get

$$\begin{aligned} |f(t,x,y_1) - f(t,x,y_2)| &= \frac{1}{10} |\arctan y_1 - \arctan y_2| \\ &\leq \frac{1}{10(1+\rho^2)} |y_1 - y_2|, \end{aligned} \tag{4.19}$$

where $\rho \in (y_1, y_2)$. By considering (4.19), we have

$$|f(t,x,y_1) - f(t,x,y_2)| \leq \frac{1}{10} |y_1 - y_2|$$

for all $t \in \mathbb{R}_+$ and $x, y_1, y_2 \in \mathbb{R}$.

Consequently, l_1 and l_2 satisfying the condition $(H2)$ of Theorem 3.7 can be chosen as $l_1 = \frac{1}{3}$ and $l_2 = \frac{1}{10}$.

T_1 and T_2 are continuous operators on the space $C_0(\mathbb{R}_+)$. Additionally for all $x \in C_0(\mathbb{R}_+)$ and $t \in \mathbb{R}_+$ we have that:

$$|(T_1x)(t)| = \frac{|x(t)|}{1+t} \leq \frac{\|x\|}{1+t} \leq \|x\|$$

and

$$|(T_2x)(t)| = \left| x(t)\sqrt{x^2(t)+1} \right| \leq \|x\| \sqrt{\|x\|^2 + 1}.$$

Hence the assumption $(H3)$ is satisfied with $d_1(t) = t$ and $d_2(t) = t\sqrt{t^2+1}$.

The function $u(t,s,x)$ is continuous on the set $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$. Further we get

$$|u(t,s,x)| = |t \exp(-ts - s)\, x| = t \exp(-ts - s)\, |x|$$

for all $t, s \in \mathbb{R}_+$ and $x \in \mathbb{R}$. Thus the functions appearing in assumption $(H4)$ have the form $g(t,s) = t \exp(-ts - s)$ and $h(t) = t$. Obviously $s \to t \exp(-ts - s)$ is integrable on $\mathbb{R}_+$ and $t \to \int_0^\infty t \exp(-ts - s)\, ds$ is bounded on $\mathbb{R}_+$. Moreover, we have that:

$$G = \sup \left\{ \int_0^\infty t \exp(-ts - s)\, ds : t \in \mathbb{R}^+ \right\} = \sup \left\{ \frac{t}{t+1} : t \in \mathbb{R}^+ \right\} = 1.$$

Now notice that the inequality

$$l_1 d_1(r) + l_2 d_2(r) h(r) G + F \leq r$$

in the assumption $(H6)$ is equivalent to:

$$\frac{r}{3} + \frac{r^2}{10}\sqrt{r^2+1} + \frac{1}{4} \leq r. \tag{4.20}$$

It can be easily verified that if $0.400985 \leq r_0 \leq 1.159091$, then r_0 is the solution of (4.20).

Let us take $\varepsilon \geq 0$, $L > 0$, $\|x\| \leq r_0$ and $t, s \in [0, L]$ such that $|t - s| \leq \varepsilon$. So we get that:

$$\begin{aligned} &|(T_1x)(t) - (T_1x)(s)| \\ = &\left| \frac{x(t)}{1+t} - \frac{x(s)}{1+s} \right| \\ = &\left| \left(\frac{1}{1+t} - \frac{1}{1+s} \right) x(t) + \frac{1}{1+s} (x(t) - x(s)) \right| \\ \leq &\frac{|s-t|}{(1+t)(1+s)} |x(t)| + \frac{1}{1+s} |x(t) - x(s)| \\ \leq &|s-t|\, \|x\| + |x(t) - x(s)| \\ \leq &\varepsilon\, r_0 + |x(t) - x(s)|. \end{aligned} \tag{4.21}$$

Further, we assume that $x(t) < x(s)$ without loss of generality, then it can be easily seen that there exists $\eta \in (x(t), x(s))$ satisfying the inequality

$$\begin{aligned}
& |(T_2x)(t) - (T_2x)(s)| \\
= & \left| x(t)\sqrt{x^2(t)+1} - x(s)\sqrt{x^2(s)+1} \right| \\
= & \left| [x(t) - x(s)]\sqrt{x^2(t)+1} + x(s)\left[\sqrt{x^2(t)+1} - \sqrt{x^2(s)+1}\right] \right| \\
\leq & |x(t) - x(s)| \left|\sqrt{x^2(t)+1}\right| + |x(s)| \left|\sqrt{x^2(t)+1} - \sqrt{x^2(s)+1}\right| \\
\leq & \sqrt{r_0^2+1}|x(t) - x(s)| + r_0 \frac{2|\eta|}{2\sqrt{\eta^2+1}} |x(t) - x(s)| . \qquad (4.22)
\end{aligned}$$

By considering (4.22), we obtain that

$$|(T_2x)(t) - (T_2x)(s)| \leq \left(\sqrt{r_0^2+1} + r_0\right) |x(t) - x(s)| \qquad (4.23)$$

for all $t, s \in [0, L]$ with $|t - s| \leq \varepsilon$.

From estimates (4.21) and (4.23) in view of the (2.1), we have that:

$$\omega_0(T_1X) \leq \omega_0(X) \text{ and } \omega_0(T_2X) \leq \left(\sqrt{r_0^2+1} + r_0\right) \omega_0(X).$$

Further, we get that:

$$\begin{aligned}
\sup\{|(T_1x)(t)| : t \geq L\} &= \sup\left\{\left|\frac{x(t)}{1+t}\right| : t \geq L\right\} \\
&\leq \|x\| \sup\left\{\frac{1}{1+t} : t \geq L\right\} \\
&= \frac{\|x\|}{1+L}. \qquad (4.24)
\end{aligned}$$

Moreover, it is clear that:

$$\begin{aligned}
\sup\{|(T_2x)(t)| : t \geq L\} &= \sup\left\{\left|x(t)\sqrt{x^2(t)+1}\right| : t \geq L\right\} \\
&= \sqrt{r_0^2+1} \sup\{|x(t)| : t \geq L\} \\
&\leq \left(\sqrt{r_0^2+1} + r_0\right) \sup\{|x(t)| : t \geq L\} . \qquad (4.25)
\end{aligned}$$

Besides,

$$\begin{aligned}
\sup\{|h_x(t)| : t \geq L\} &= \sup\{h(|x(t)|) : t \geq L\} \\
&= \sup\{|x(t)| : t \geq L\} . \qquad (4.26)
\end{aligned}$$

Thus from estimates (4.24), (4.25) and (4.26) in view of (2.2) we have that:

$$\beta(T_1X) = 0,\ \beta(T_2X) \leq \left(\sqrt{r_0^2+1} + r_0\right) \beta(X) \text{ and } \beta(h_X) = \beta(X).$$

So,

$$\mu(T_1X) = \omega_0(T_1X) + \beta(T_1X) \leq \omega_0(X) \leq \mu(X),$$

$$\begin{aligned}\mu(T_2X) &= \omega_0(T_2X)+\beta(T_2X)\\ &\le \left(\sqrt{r_0^2+1}+r_0\right)(\omega_0(X)+\beta(X))\\ &= \left(\sqrt{r_0^2+1}+r_0\right)\mu(X).\end{aligned}$$

Therefore we get that $m_{1,r_0}=1$, $m_{2,r_0}=\sqrt{r_0^2+1}+r_0$ and $b_{r_0}=1$. Keeping in mind the above obtained constants, we get that

$$l_1m_{1,r_0}+l_2m_{2,r_0}h(r_0)G+2l_2d_2(r_0)b_{r_0}G=\frac{1}{3}+\frac{\left(\sqrt{r_0^2+1}+r_0\right)r_0}{10}+\frac{r_0\sqrt{r_0^2+1}}{5}<1$$

for $0.400985\le r_0\le 1.159091$ and thus the assumption $(H8)$ is satisfied.

Finally, we conclude that the assumptions of Theorem 3.7 hold. This implies that the considered integral equation (4.17) has a solution $x=x(t)$ belonging to the space $C_0(\mathbb{R}_+)$.

Remark 4.6. Notice that the integral equation (4.17) can't be derived from any of the integral equations handled in *[1, 5, 6, 7, 10, 15, 17, 19, 20, 23, 24, 25, 27, 28, 29, 30].*

On the other hand, (4.17) can be derived from the equation (1.11) by taking

$$F(t,x,y)=\frac{1}{4}\exp(-t)+\frac{1}{3}\ln\left(1+\frac{|x|}{1+t}\right)+\frac{1}{10}\arctan\left(x\sqrt{x^2+1}\,y\right)$$

and $u(t,s,x)=t\exp(-ts-s)x$ for $t,s\in\mathbb{R}_+$ and $x,y\in\mathbb{R}$. But, since

$$\begin{aligned}&\lim_{T\to\infty}\left\{\sup\left\{\int_0^T g(t,s)ds : t\in[T,\infty)\right\}\right\}\\ =\ &\lim_{T\to\infty}\left\{\sup\left\{\int_0^T t\exp(-ts-s)ds : t\in[T,\infty)\right\}\right\}\\ =\ &\lim_{T\to\infty}\left\{\sup\left\{\frac{t}{1+t}\left(1-\frac{1}{\exp(Tt+T)}\right) : t\in[T,\infty)\right\}\right\}\\ =\ &1,\end{aligned}$$

the assumption (H_5) **in***[26]* **doesn't hold. Hence, the existence theorem given in** *[26]* **cannot be applied to** (4.17)**.**

5 Concluding Remarks

This paper contains a result on the existence of solutions of the nonlinear integral equation (1.1). The equation (1.1) is more general than many integral equations examined up to now. The investigations of the paper are placed in the Banach space of real functions defined, continuous on the real half-axis and vanishing at infinity. The main tools used in our considerations are the concept of a measure of noncompactness and the classical Schauder fixed point theorem. The result obtained in this paper generalizes and extends several ones obtained earlier for some integral equations. We give some examples illustrating the different of this paper than the other studies in [1, 5, 6, 7, 10, 15, 17, 19, 20, 23, 24, 25, 26, 27, 28, 29, 30].

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