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High performance of the Remez algorithm in finding polynomial approximations for the solution of integral equations

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Abstract

In this paper, the solution of the general integral equation of the second kind is approximated using polynomials. These polynomials are obtained based on the Remez algorithm and the minimization of residual function. The nature of the use of the Remez algorithm in the proposed method will lead to the conversion of the integral equation to a system of algebraic equations and obtaining the best polynomial approximation for the solution of an integral equation. Also, the convergence analysis of the approach is discussed. Finally, some numerical examples and comparisons with previous results confirm the efficiency and high accuracy of the presented method.

Keywords: Nonlinear integral equations, Remez algorithm, Best polynomial approximation, Residual function, Minimization, Infinity norm
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1 Introduction

One of the fundamental classes of equations are the integral ones. Integral equations are utilized in various fields of science, such as physics, biology, economics, and engineering, etc. For this reason, these equations have received more attention in recent decades. General approaches for solving integral equations are classified in two types: analytical and numerical methods. Analytical methods are usually not available to solve integral equations. Therefore, much effort has been made for producing numerical methods for solving various types of integral equations. In fact, there are several numerical methods for solving variety of Fredholm and Voltera integral equations. Especially, some numerical methods, such as projection and collocation methods, have been used to solve the problem. Most of these methods, appropriate linear combinations of basic functions, such as Chebyshev polynomials [1, 2, 3, 4, 5, 6], Legendre polynomials [7, 8, 9], Bernoulli polynomials [10], Bernstein polynomials [11, 12, 13], monomials[14, 15], Tau-collocation method[16], wavelets [17, 18, 19] have been used. The coefficients of linear combinations of basic functions are obtained through these methods. And most of the these methods are performable only for Fredholm or Volterra (linear or nonlinear) integral equations. Therefore, it is necessary to present a generic method that can be applied to all integral equations of the second kind which includes Fredholm, Volterra, and mixed Fredholm-Volterra (linear and nonlinear) and as a result, best coefficients of linear combinations are yielded.

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The problem of best polynomial approximation for a continuous function on a closed interval is one of the most important and applicable subjects of the approximation theory. The Remez algorithm is an iterative algorithm and effective technique for obtaining the best polynomial approximation with very high accuracy. The basis of the Remez algorithm is to find a certain number of points on the desired interval and to solve the system of their corresponding equations. By updating the points and repeating the process, it is possible to achieve the desired accuracy in obtaining the best polynomial approximation. Concerning this issue, first, the basic idea of [15] and the Remez algorithm is developed and then it is applied to integral equations of the second kind. The current study attempts to propose a new approach in order to find the best polynomial approximations for a given integral equation of the second kind. In this approach, the solution of a given integral equation is represented as a polynomial (linear combination of monomials). Furthermore, the calculation method of the corresponding coefficients of these monomials is discussed based on the Remez algorithm. Moreover, the convergence and the error analysis of the proposed scheme are studied. This study shows that the obtained approximations are very close to the best polynomial approximation for the solution of an integral equation, which is the main advantage of the proposed method. The accuracy and convergence rate of this method are compared with the other existing ones, which indicates improvements in the results. Accordingly, this method is appropriate and practical for solving all integral equations of the second kind, integro- differential equations, multi-dimensional integral equations, ordinary and partial differential equations. The structure of the paper is as follows:
In Section 2, the Remez algorithm is introduced. In the third section, the method of solving the integral equations of the second kind by using the Remez algorithm is presented. Sections 4 and 5 are devoted to the convergence analysis of the presented method. Finally, to observe the efficiency of the proposed approach, it is used to examine some numerical examples and obtained results are compared with those of previous methods.

2 The Remez Algorithm

With the assumption that n is a nonnegative integer, $\mathbf{P}_n$ denotes the set of all (real valued) polynomials of degree $\leq n$; such that if $p_n \in \mathbf{P}_n$ then

$$p_n(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0. \tag{2.1}$$

Definition 2.1. For given $f \in C[a,b]$ and $n \geq 0$, fixed, $p_n^* \in P_n$ is called a polynomial of best approximation of degree n to the function f in the ∞-norm whenever:

$$\|f - p_n^*\|_\infty = \min_{p_n \in \mathbf{P}_n} \|f - p_n\|_\infty, \tag{2.2}$$

where

$$\|f - p_n\|_\infty = \max_{a \leq x \leq b} |f(x) - p_n(x)|.$$

Chebyshev proved that such polynomial exists and is unique. So, it is often referred to as the minimax polynomial.

Theorem 2.2. (The Oscillation Theorem) Suppose that $f \in C[a,b]$, a polynomial $p_n \in \mathbf{P}_n$ is a minimax polynomial for f on $[a,b]$ if and only if, there exists a sequence of $n+2$ points x_i, $i = 0,1,...,n+1$, such that $a \leq x_0 < ... < x_{n+1} \leq b$,

$$f(x_i) - p_n(x_i) = (-1)^{(i)} e, \quad i = 0,1,...,n+1, \tag{2.3}$$

and

$$e = \max_{a \leq x \leq b} |f(x) - p_n(x)|. \tag{2.4}$$

By using the Oscillation Theorem, minimax polynomial for the function f on the interval $[a,b]$ can be obtained. For this objective , the system of $n+2$ linear equations (2.3) in the $n+2$ unknowns $\{a_0, a_1, ..., a_n, e\}$ is solved(for x_i's selected). The great challenge is that e does not always satisfy in (2.4). To solve this problem, Remez has been presented an iterative algorithm known to the Remez algorithm as the following:
Step I: Choose arbitrarily $n+2$ points $x_i, \quad 0 \leq i \leq n+1$ in the given interval $[a,b]$.
Step II: By solving the following system of linear equations, $a_0, a_1, ..., a_n$ are found.

$$f(x_i) - p_n(x_i) = (-1)^i e, \;\; i = 0,1,...,n+1, \tag{2.5}$$

where p_n is as (2.1).
Step III: Obtaining p_n from step II yields the root of $|f - p_n|$, z_i, on interval $[x_i, x_{i+1}]$, $\quad i = 0, 1, ..., n+1$.
Step IV: First we divide the interval $[a, b]$ into the $(n+2)$ subintervals $[a, z_0]$, $[z_0, z_1]$, ..., $[z_{n-1}, z_n]$, $[z_n, b]$. Then in each of these subintervals, the point at which $|f - p_n|$ attains its maximum value is computed and is denoted by x_i^*, $i = 0, 1, ..., n+1$.
Step V: It stops and p_n obtained from step II is chosen as the best polynomial approximation; otherwise it is set $x_i := x_i^*$ and goes to step II.
It has been proved that the system of equations of Eq. (2.5) have unique solution, therefore other steps of the algorithm are applicable.

3 Explaining the Method

The general form of a Hammerstein integral equation is given as follows:

$$y(x) = \lambda\text{ş} \int_0^1 k_1(x,t)F_1(t, y(t))dt + \gamma\text{ş} \int_0^x k_2(x,t)F_2(t, y(t))dt + g(x), \tag{3.1}$$

where the parameters λ, γ and $k_1(x,t)$, $k_2(x,t)$, F_1, F_2 and $g(x)$ are known functions, while $y(x)$ is an unknown function which must be determined.
Suppose that the approximate solution of Eq. (3.1) is presented as Eq. (2.1), therefore

$$y(x) - p_n(x) = R[p_n](x) + \epsilon[p_n](x), \tag{3.2}$$

where:

$$R[p_n](x) = -p_n(x) + \lambda\text{ş} \int_0^1 k_1(x,t)F_1(t, p_n(t))dt + \gamma\text{ş} \int_0^x k_2(x,t)F_2(t, p_n(t))dt + g(x), \tag{3.3}$$

$$\epsilon[p_n](x) = \lambda\text{ş} \int_0^1 k_1(x,t)[F_1(t, y(t)) - F_1(t, p_n(t))]dt + \gamma\text{ş} \int_0^x k_2(x,t)[F_2(t, y(t)) - F_2(t, p_n(t))]dt. \tag{3.4}$$

So, Eq. (3.2) can be written as follows:

$$(y(x) - \epsilon[p_n](x)) - p_n(x) = R[p_n](x). \tag{3.5}$$

In the next sections, it will be shown that there exists a polynomial $p_n^* \in P_n$ such that

$$\|R[p_n^*]\|_\infty = \min_{p_n \in P_n} \|R[p_n]\|_\infty. \tag{3.6}$$

Also, it will be proved that the sequence $\{p_n^*\}$ is convergent to the exact solution of Eq. (3.1).
According to Eqs. (3.5) and (3.6), it is concluded that:

$$\min_{p_n \in P_n} \|y - \epsilon[p_n] - p_n\|_\infty = \|y - \epsilon[p_n^*] - p_n^*\|_\infty = \|R[p_n^*]\|_\infty. \tag{3.7}$$

Eq. (3.4) and the convergence of the sequence $\{p_n^*\}$ result in

$$\lim_{n \longrightarrow \infty} \|\epsilon[p_n^*]\|_\infty = 0. \tag{3.8}$$

Now, it is shown that $\{p_n^*\}$ is best polynomial approximation for $y - \epsilon[p_n^*]$ (where $y(x)$ is the exact solution of Eq. (3.1)). According to Eq. (3.5), it is clear that

$$y(x) - \epsilon[p_n^*](x) - p_n(x) = R[p_n^*](x) + p_n^*(x) - p_n(x). \tag{3.9}$$

Due to Eq. (3.9), there exist a polynomial $\bar{p}_n$ such that:

$$\|y - \epsilon[p_n^*] - \bar{p}_n\|_\infty = \min_{p_n \in P_n} \|y - \epsilon[p_n^*] - p_n\|_\infty = \min_{p_n \in P_n} \|R[p_n^*] + p_n^* - p_n\|_\infty = \|R[p_n^*] + p_n^* - \bar{p}_n\|_\infty. \tag{3.10}$$

Moreover, p_n^* and $\bar{p}_n$ are almost the same, since according to Eq. (3.10) we have

$$\|p_n^* - \bar{p}_n\|_\infty \leq 2\|R[p_n^*]\|_\infty. \tag{3.11}$$

On the other hand, due to the definition of $R[p_n^*]$ it can be written

$$\|R[p_n^*]\|_\infty \leq \|R[p_n^{**}]\|_\infty \leq L\|p_n^{**} - y\|_\infty, \quad \forall n \in \mathbb{N} \tag{3.12}$$

where L is a constant and p_n^{**} is the best polynomial approximation for the solution of Eq. (3.1).

Theorem 2.4.6 of [21] for interval $[-1,1]$ gives

$$E_n^{[-1,1]}(y) = \|p_n^{**} - y\|_\infty = \frac{1}{2^n}\frac{|y^{(n+1)}(\xi)|}{(n+1)!}, \quad \xi \in (-1,1). \tag{3.13}$$

Assuming $g(x) = y(\frac{x+1}{2})$, $-1 \leq x \leq 1$ we get

$$E_n^{[0,1]}(y) = E_n^{[-1,1]}(g). \tag{3.14}$$

From Eqs. (3.13) and (3.14) we have

$$E_n^{[0,1]}(y) = E_n^{[-1,1]}(g) = \frac{1}{2^n}\frac{|g^{(n+1)}(\xi)|}{(n+1)!}. \tag{3.15}$$

Based on the definition of g and Eq. (3.15), it is deduced that

$$\|p_n^{**} - y\|_\infty = \frac{1}{2^{2n+1}}\frac{|y^{(n+1)}(\xi)|}{(n+1)!}, \quad \xi \in [0,1]. \tag{3.16}$$

With a little ignorance, Eqs. (3.11), (3.12) and (3.16), especially for n sufficiently large, are shown

$$\bar{p}_n = p_n^*. \tag{3.17}$$

Since $\bar{p}_n$ is best approximation for $y - \epsilon[p_n^*]$ then, to obtain $\bar{p}_n$ (or same p_n^*) of Eq. (3.10) can be achieved based on the Remez algorithm which is based on the solution of the following equations system :

$$(y(x_i) - \epsilon[p_n^*](x_i)) - \bar{p}_n(x_i) = (-1)^i e, \quad i = 0,1,...,n+1, \tag{3.18}$$

where x_i, $\quad 0 \leq i \leq n+1$ are chosen in the interval $[0,1]$.
Eqs. (3.5), (3.17) and (3.18) yield

$$R[p_n^*](x_i) = (-1)^i e, \quad i = 0,1,...,n+1. \tag{3.19}$$

Therefore, despite of the unknown $y(x)$, by using of Eqs. (3.5), (3.17) and (3.18), steps of the Remez algorithm for obtaining p_n^* are executable.
Conclusion: In fact, to finding the best polynomial approximation p_n for $y - \epsilon[p_n^*]$ is equivalent with minimization $\|R[p_n]\|_\infty$ for $p_n \in P_n$ and it can be implemented by using the Remez algorithm based on solving the system (3.19).
Main conclusion I: Minimization of $\|R[p_n]\|_\infty$ leads to generation of the sequence $\{p_n^*\}$ which:
1- Converges to solution of integral equation.
2- $\epsilon[p_n^*]$ is convergent to zero according to (3.4)**.**
Since p_n^* is the best polynomial approximation for $y - \epsilon[p_n^*]$ and $\epsilon[p_n^*] \longrightarrow 0$ hence
3- By increasing n, p_n^* is best polynomial approximation for solution of integral equation.

4 The convergence analysis of linear integral equations

In this section, it is assumed that in Eq. (3.1), $F_1(t, y(t)) = y(t)$ and $F_2(t, y(t)) = y(t)$, which yields the linear form of integral equation. It is supposed that the form of approximate solutions of integral equation is as Eq. (2.1). Substituting Eq. (2.1) into linear integral equation causes that the residual function $R[p_n]$ turns into

$$R\,[p_n] = a_0\phi_0 + a_1\phi_1 + ... + a_n\phi_n + g, \tag{4.1}$$

where

$$\phi_i(x) = -x^i + \lambda \int_0^1 k_1(x,t)t^i dt + \gamma \int_0^x k_2(x,t)t^i dt, \qquad 0 \le i \le n.$$

Let minimization $R\,[p_n]$ occur for $a_i^*,\ i = 0,1,2,...,n$, therefore,

$$\|a_0^*\phi_0 + a_1^*\phi_1 + ... + a_n^*\phi_n + g\|_\infty = \min_{p_n \in P_n} \|R[p_n]\|_\infty. \tag{4.2}$$

Now, we show that the sequence $\{p_n^*\}$ is convergent to the solution of linear integral equation where:

$$p_n^*(x) = a_n^* x^n + a_{n-1}^* x^{n-1} + ... + a_1^* x + a_0^*. \tag{4.3}$$

As a result, items 2, 3 of **Main conclusion** for linear integral equation are concluded.

Theorem 4.1. There exists a p_n^* as Eq. (4.3) such that $\|R[p_n^*]\|_\infty = \min\limits_{p_n \in P_n} \|R[p_n]\|_\infty$.

Proof . The set $V = \{a_0\phi_0 + a_1\phi_1 + ... + a_n\phi_n | (a_0, a_1, ..., a_n) \in \mathbb{R}^{n+1}\}$ is a finite dimensional subspace of normed space $C[0,1]$. Hence there is a best approximation function g with respect to V [20]. □

Theorem 4.2. If $k_2\,(x,t)$ is continuous on $0 \le t \le x \le 1$ and $f(x)$ is continuous on $0 \le x \le 1$, then the integral equation $y(x) - \gamma \int_0^x k_2\,(x,t)\, y(t)\, d\,t \;=\; g(x)$ possesses a unique continuous solution for $0 \le x \le 1$.

Proof . See [20]. □

Theorem 4.3. The sequence $\{p_n^*\}$ of presented method is convergent to the exact solution of Eq. (3.1) whenever in Eq. (3.1) $\lambda = 0$. (linear Volterra integral equation)

Proof . At first, corresponding to the integral equation

$$y(x) - \gamma \int_0^x k_2\,(x,t)\, y(t)\, d\,t \;=\; g(x),$$

we define the linear transformation $T : C\,[0,1] \to C\,[0,1]$ by

$$T\,[v]\,(x) = \; v(x) - \gamma \int_0^x k_2(x,t)v(t)dt. \tag{4.4}$$

Obviously the operator T is bounded, because $k_2(x,t)$ is bounded on the set

$$\Big\{(x,t)\big|0 \le t \le x \le 1\Big\}.$$

Also T is one-to-one, since Eq. (3.1) has a unique solution. Moreover, according to Theorem 4.2, T is onto. Thus there exist $0 < \alpha$ and $0 < \beta$, such that (see [16]) $\alpha\,\|v\|_\infty \le \|Tv\|_\infty \le \beta\,\|v\|_\infty$ for all $v \in V$ where

$$\alpha = \frac{1}{\|T^{-1}\|_\infty} \text{ and } \beta = \|T\|_\infty\,. \tag{4.5}$$

In particular for $v = p_n - y$, we have:

$$\alpha\,\|p_n - y\|_\infty \le \|R\,[p_n]\,\|_\infty \le \beta\,\|p_n - y\|_\infty\,, \tag{4.6}$$

where y is the exact solution Eq. (3.1). Since $span\,\{\varphi_0, \varphi_1, ..., \varphi_n, \varphi_{n+1}, ...\}$ is dense in $C\,[0,1]$ and $y \in C\,[0,1]$, for given $\varepsilon > 0$, there exists some p_N such that $\|R\,[p_N]\,\|_\infty < \alpha\varepsilon$. According to definitions of p_n^* and $R\,[p_n^*]$ for any $n \ge N$ we have:

$$\|R\,[p_n^*]\,\|_\infty \le \|R\,[p_N^*]\|_\infty \le \|R\,[p_N]\|_\infty \le \alpha\varepsilon. \tag{4.7}$$

By (4.6) and (4.7) the proof is completed. □

Theorem 4.4. (Fredholm alternative) Let V be a Banach space and let $K : V \to V$ be a compact operator where $K\left[v\right](x) = -\int_0^1 k_1(x,t)\,v(t)\,dt$. Then the equation

$$v(x) - \lambda \int_0^1 k_1\,(x,t)\,v(t)\,d\,t \;=\; g(x),$$

has a unique solution $v \in V$ for any $f \in V$, if and only if the homogeneous equation

$$v(x) - \lambda \int_0^1 k_1\,(x,t)\,v(t)\,d\,t \;=\; 0,$$

has only the trivial solution $v = 0$. In this case, the operator $I + K$ has a bounded inverse $(I + K)^{-1}$.

Proof . See [17]. □

Theorem 4.5. The sequence of approximate solutions $\{p_n^*\}$ of the proposed method is convergent to the exact solution of Eq. (3.1), whenever in Eq. (3.1) $\gamma = 0$ (Fredholm integral equation).

Proof . The proof is the same as of Theorem (3). □

Theorem 4.6. The sequence of approximate solutions $\{p_n^*\}$ of the proposed method is convergent to the exact solution of Eq. (3.1) whenever in Eq. (3.1), $\gamma, \lambda \neq 0$ (mixed Volterra − Fredholm integral equations).

Proof . By defining

$$T\left[v\right](x) = \; v(x) - \gamma \int_0^x k_2(x,t)v(t)dt, \tag{4.8}$$

and

$$S\left[v\right](x) = -\,\lambda \int_0^1 k_1(x,t)v(t)dt, \tag{4.9}$$

Eq. (3.1) becomes

$$T\left[v\right](x) + S\left[v\right](x) = f(x). \tag{4.10}$$

Obviously T^{-1} is bounded and S is compact. Hence, the linear operator $T^{-1}S$ is compact too. Due to [17], the linear operator $I{+}T^{-1}S$ has a bounded inverse, as Eq. (3.1) has a unique solution. Therefore similar to the proof of Theorem 4.3, the result can be obtained. □

Conclusion II: With the aid of Theorems 4.3, 4.4, 4.5 and 4.6 it is concluded that:

$$\alpha_A\;\; \|R\left[p_n\right]\|_\infty \leq \|p_n - y\|_\infty \leq \beta_A\; \|R\left[p_n\right]\|_\infty\,; \qquad \forall n, \tag{4.11}$$

where $\alpha_A = \frac{1}{\|A\|_\infty}$, $\beta_A = \left\|A^{-1}\right\|_\infty$, and

$$Ay(x) = y(x) - \lambda \int_0^1 k_1(x,t)y(t)dt - \gamma \int_0^x k_2(x,t)y(t)dt. \tag{4.12}$$

Using (4.4) gives

$$\alpha_A \left\|R\left[p_n^*\right]\right\|_\infty \;\leq \|p_n^* - y\|_\infty \leq \beta_A \left\|R\left[p_n^*\right]\right\|_\infty. \tag{4.13}$$

Conclusion III: Assuming p_n^{**} is the best approximation for the solution of linear integral equation (2.1), according to definition $\|R[p_n^*]\|_\infty$ and inequalities (4.11), (4.13) we can write

$$\|p_n^* - y\|_\infty \leq \kappa(A)\|p_n^{**} - y\|_\infty \tag{4.14}$$

where $\kappa(A) = \|A\|\|A^{-1}\| \geq 1$. Consequently p_n^* can be very close to p_n^{**}.

Also by definition of $\epsilon[p_n](x)$ and the fact that k_1 and k_2 are continuous, we have

$$\|\epsilon[p_n^*]\|_\infty \leq k\|y - p_n^*\|_\infty. \tag{4.15}$$

5 The convergence analysis of nonlinear integral equations

In this section, it is assumed that $F_1(x, y(x))$ and $F_2(x, y(x))$ with respect to variable $y(x)$ are nonlinear. Even though the presented method is practical for all nonlinear integral equations of the second kind, the existence of the nonlinear terms in the integrand of these equations will increase the computation time and complexity. In order to solve this problem, first we linearize the integrand term by changing variables $Z_1(x) = F_1(x, y(x))$ and $Z_2(x) = F_2(x, y(x))$. Therefore, Eq. (3.1) is converted into following system

$$\begin{cases} Z_1(x) = F_1(x, L[Z_1, Z_2](x)), \\ Z_2(x) = F_2(x, L[Z_1, Z_2](x)), \end{cases} \tag{5.1}$$

where $L[Z_1, Z_2](x) = \lambda \text{ș} \int_0^1 k_1(x,t) Z_1(t)dt + \gamma \text{ș} \int_0^x k_2(x,t) Z_2(t)dt + g(x)$.
Then the system (5.1) is solved by the Remez algorithm. It is clear that $y^*(x) = \lambda \text{ș} \int_0^1 k_1(x,t) Z_1^*(t)dt + \gamma \text{ș} \int_0^x k_2(x,t) Z_2^*(t)dt + g(x)$ is the solution of Eq. (3.1) where (Z_1^*, Z_2^*) is the solution of system (5.1). It is assumed that p_n, q_n are polynomials of degree n the same as (2.1). So from (5.1) we have

$$\begin{cases} (Z_1(x) - \epsilon_1[p_n, q_n](x)) - p_n(x) = R_1[p_n, q_n](x) \\ (Z_2(x) - \epsilon_2[p_n, q_n](x)) - q_n(x) = R_2[p_n, q_n](x) \end{cases} \tag{5.2}$$

where

$$\begin{cases} \epsilon_1[p_n, q_n](x) = F_1(x, L[Z_1, Z_2](x)) - F_1(x, L[p_n, q_n](x)), \\ \epsilon_2[p_n, q_n](x) = F_2(x, L[Z_1, Z_2](x)) - F_2(x, L[p_n, q_n](x)), \\ R_1[p_n, q_n](x) = -p_n(x) + F_1(x, L[p_n, q_n](x)), \\ R_2[p_n, q_n](x) = -q_n(x) + F_2(x, L[p_n, q_n](x)). \end{cases} \tag{5.3}$$

Now, in a similar way as in section 3, we look for p_n^* and q_n^* which are the best polynomial approximations for $Z_1 - \epsilon_1$ and $Z_2 - \epsilon_2$ in (5.3) such that by increasing n, ϵ_1 and ϵ_2 tend to zero.
Remark: Corresponding to $\lambda = 0$ or $\gamma = 0$ have $R_2[p_n, q_n](x) = R_2[q_n](x)$ or $R_1[p_n, q_n](x) = R_1[p_n](x)$, respectively, that for obtaining p_n^* or q_n^* proceed the same as previous sections.
Corresponding to $\lambda \neq 0$ and $\gamma \neq 0$, at first is defined:

$$R[p_n, q_n] := \max\{\|R_1[p_n, q_n]\|_\infty,\ \|R_2[p_n, q_n]\|_\infty\}. \tag{5.4}$$

Then, we put $R[p_n^*, q_n^*] = \min\limits_{p_n, q_n \in P_n} R[p_n, q_n]$.
Finally, to find p_n^*, q_n^*, the follwing system of equations will be solved .

$$\begin{cases} R_1[p_n, q_n](x_i) = (-1)^i e_1, \ \ i = 0, 1, ..., n+1, \\ R_2[p_n, q_n](x_j) = (-1)^j e_2. \ \ j = 0, 1, ..., n+1. \end{cases} \tag{5.5}$$

Theorem 5.1. Suppose that $\tilde{F}_1 = F_1(x, g(x))$ and $\tilde{F}_2 = F_2(x, g(x))$, if $|\lim\limits_{s\to\infty}(s - F_1(x, js))| > 1 + \|\tilde{F}_1\|$ and $|\lim\limits_{s\to\infty}(s - F_2(x, js))| > 1 + \|\tilde{F}_2\|$ (j is a constant). Then there exists p_n^* and q_n^* where $\|R_1[p_n^*, q_n^*]\|_\infty = \min\limits_{p_n, q_n \in P_n} \|R_1[p_n, q_n]\|_\infty$ and $\|R_2[p_n^*, q_n^*]\|_\infty = \min\limits_{p_n, q_n \in P_n} \|R_2[p_n, q_n]\|_\infty$.

Proof . First let $\lambda = 0$ (Volterra), therefore

$$Z_2(x) = F_2(x, L[Z_2](x)), \tag{5.6}$$

$$R_2[q_n](x) = -q_n(x) + F_2(x, L[q_n](x)), \tag{5.7}$$

where

$$L[Z](x) = \gamma \text{ș} \int_0^x k_2(x,t) Z(t)dt + g(x). \tag{5.8}$$

The set Ω_n is defined as follows:

$$\Omega_n = \{q_n \in P_n |\ \|R_2[q_n]\|_\infty \leq 1 + \|\tilde{F}_2\|_\infty\}. \tag{5.9}$$

Clearly $0 \in \Omega_n$, so Ω_n is nonempty and based on definition of Ω_n, it follows that $\|R_2[q_n]\|_\infty$ attains its minimum over the set Ω_n and this set is uniformly bounded, otherwise there exists sequence $\{q_m\}$ of set Ω_n such that

$$|q_m(x_m)| = \|q_m\|_\infty \to \infty. \tag{5.10}$$

Case 1: F_2 satisfies similar to Lipshitz condition with constant r, in other words

$$|F_2(x, q_m(x)) - F_2(x, Z_2(x))| \leq r|q_m(x) - Z_2(x)|. \tag{5.11}$$

Hence, subtracting (5.6) from (5.7) and using Lipschitz condition gives

$$|q_m(x_m) - Z_2(x_m)| \leq |R_2[q_m](x_m)| + r \mid \int_0^{x_m} k_2(x_m, t)(q_m(t) - Z_2(t))dt|. \tag{5.12}$$

It is clear that $Z_2(x_m)$ and $|R_2[q_m](x_m)|$ are finite, therefore, due to (5.10) and dividing the two sides of Eq. (5.12) by $|q_m(x_m)|$ and then taking the limit on both sides of Eq. (5.12), it can be deduced that

$$0 < \frac{1}{r} \leq \lim_{m\to\infty} \frac{|g(x_m) + \int_0^{x_m} k_2(x_m, t)q_m(t)dt - g(x_m) - \int_0^{x_m} k_2(x_m, t)Z_2(t))dt|}{|q_m(x_m)|} \leq k_2. \tag{5.13}$$

where $k_2 = \max\limits_{0\leq x,t\leq 1} |k_2(x, t)|$. Thus, there exist r_1 and r_2 such that for sufficiently large m value, we have:

$$r_1 q_m(x_m) \leq g(x_m) + \int_0^{x_m} k_2(x_m, t)q_m(t)dt \leq r_2 q_m(x_m). \tag{5.14}$$

From (5.7) and (5.14) it is obtained

$$R_2[q_m](x_m) = -q_m(x_m) + F_2(x_m, jq_m(x_m)). \tag{5.15}$$

Due to the $q_m \in \Omega_n$, itis in contradiction with the assumption $|\lim\limits_{s\to\infty}(s - F_2(x, js))| > 1 + \|\tilde{F}_2\|$.

Case 2: Case 1 does not satisfy. Then for any $r \in \mathbb{N}$, there exists $q_r \in \Omega_n$ such that

$$r|q_r(x) - Z_2(x)| < |F_2(x, q_r(x)) - F_2(x, Z_2(x))|. \tag{5.16}$$

Also, subtraction (5.6) , (5.7) and then applying absolute value is yields

$$|F_2(x, q_r(x)) - F_2(x, Z_2(x))| - |q_r(x) - Z_2(x)| \leq |R_2[q_r](x)|. \tag{5.17}$$

Since r can take large values and $q_r \in \Omega_n$, then (5.16) and (5.17) contradict bounded $R_2[q_r]$. Therefore, it is proven that Ω_n is bounded. So, the function $\Gamma(q_n) := \|R_2[q_n]\|_\infty$ is continuous, therefore, Ω_n is a closed set. Hence the continuous function Γ attains its minimum over the set Ω_n(compact). According to the definition Ω_n we have

$$\min_{q_n\in P_n} \|R_2[q_n]\|_\infty = \min_{q_n\in\Omega_n} \|R_2[q_n]\|_\infty, \tag{5.18}$$

which completes the proof. Fredholm and Fredholm-Volerra are proved similar to Volterra. □

Theorem 5.2. If $\lim\limits_{s\to\infty}(s - F_1(x, js)) \neq 0$ and $\lim\limits_{s\to\infty}(s - F_2(x, js)) \neq 0$, then sequences $\{q_n^*\}$ and $\{p_n^*\}$ of presented method are uniformly bounded.

Proof . It can be proven in analogous way as Theorem 5.1 . □

Theorem 5.3. The sequence $\{q_n^*\}$ of presented method is convergent to the exact solution of Eq. (3.1) whenever in Eq. (3.1), $\lambda = 0$. (nonlinear Volterra integral equation)

Proof . The Theorem 5.2 states that the sequence $\{q_n^*\}$ is uniformly bounded, hence F_2 over $\{q_n^*\} \cup \{Z_2\}$ satisfies in the Lipshitz condition. The subtracting (5.6) from (5.7) and applying of Lipschitz condition for F_2 leads to:

$$\|R_2[q_n]\|_\infty \leq (1 + rk_2)\|Z_2 - q_n\|_\infty, \tag{5.19}$$

and

$$|Z_2(x) - q_n(x)| \leq |R_2[q_n](x)| + r\int_0^x k_2(x,t)|Z_2(t) - q_n(t)|dt, \tag{5.20}$$

where r is the Lipschitz constant of function F_2 and $k_2 = \max_{0\leq x\leq t\leq 1} |k_2(x,t)|$.
Using Grönwall's inequality for (5.20), especially when $q_n = q_n^*$, it follows that

$$\|Z_2 - q_n^*\|_\infty \leq \|R_2[q_n^*]\|_\infty e^{rk_2}. \tag{5.21}$$

Based on the definition $R_2[q_n^*]$, obviously

$$\|R_2[q_n^*]\|_\infty \longrightarrow 0. \tag{5.22}$$

Inequality (5.21) and (5.22) prove the convergence $\{q_n^*\}$ to Z_2(exact solution). □

Theorem 5.4. The sequence $\{p_n^*\}$ of presented method is convergent to the exact solution of Eq. (3.1) whenever in Eq. (3.1), $\gamma = 0$. (nonlinear Fredholm integral equation)

Proof . In this case we have

$$R_1[p_n](x) = -p_n(x) + F_1(x, L[p_n](x)). \tag{5.23}$$

where

$$L[Z](x) = \lambda ș \int_0^1 k_1(x,t)Z(t)dt + g(x), \tag{5.24}$$

Eq. (5.23) gives

$$|p_n^*(x) - p_n^*(z)| \leq |R_1[p_n^*](x)| + |R_1[p_n^*](z)| + r_1|L[p_n^*](x) - L[p_n^*](z)| + r_2|x - z|, \tag{5.25}$$

where r_1 and r_2 are the Lipschitz constants of function F_1. Since $\|R_1[p_n^*]\|_\infty \longrightarrow 0$, k_1 is continuous and especially $\{p_n^*\}$ is uniformly bounded, then the sequence $\{p_n^*\}$ is equicontinuous. Considering that $\{p_n^*\}$ is uniformly bounded and equicontinuous, according to Arzelà–Ascoli theorem proof is completed. □

Proving the convergence of Volterra-Fredholm integral equation is the same as Theorem 5.4. Therefore **Main Conclusion** in the previous section for nonlinear integral equations is satisfied.

6 Numerical results

In this section, five numerical examples will be solved using the proposed method. These examples confirm the accuracy and high performance of the method, and all of them are carried out using programs written in MAPLE. Also, when the results are compared with those of [6], [10], and [16] it is indicated that the presented method is more accurate.
Absolute error between the exact solution and its approximate solution is defined by

$$\begin{gathered} e_n(x) = |y_n^*(x) - y^*(x)|, \\ \textit{and} \\ \|e_n\|_\infty = \max_{x\in[0,1]} |e_n(x)|, \end{gathered} \tag{6.1}$$

where y_n^* is an approximate solution of the proposed method and y^* is the exact solution. On the other hand, it is assumed that $\mathrm{p}_\mathrm{n}^{**}$ is the best polynomial approximation of degree at most n, for y^* by using Maple software. y_n^* in Examples 6.1, 6.2 and 6.3 is a polynomial (same p_n^*), the comparison between the two graphs y_n^* and p_n^{**}, reveals that they are identical and it does not display minor differences between them. To show this and the rate of convergence of this method, the graphs $y^* - y_n^*$ and $y^* - p_n^{**}$ are plotted in Figures 1, 2 and 3. From these figures one can see that y_n^* and $\mathrm{p}_\mathrm{n}^{**}$ are almost the same. Especially, it is clear that by increasing n, the difference between y_n^* and $\mathrm{p}_\mathrm{n}^{**}$ decreases. This confirms our theoretical results.

Example 6.1. Consider the following Volterra integral equation

$$y(x) = \sin(x) - e^{\sin(x)} + 1 + \int_0^x \cos(t) e^{y(t)} dt, \tag{6.2}$$

with the exact solution $y(x) = \sin(x)$. The comparison of $y^* - y_n^*$ and $y^* - p_n^{**}$ for $n = 6$ and $n = 8$, is presented in Fig.1. Overlapping of these two ghraphs in Fig.1 reveals that y_n^* is a best polynomial approximation for y^*.

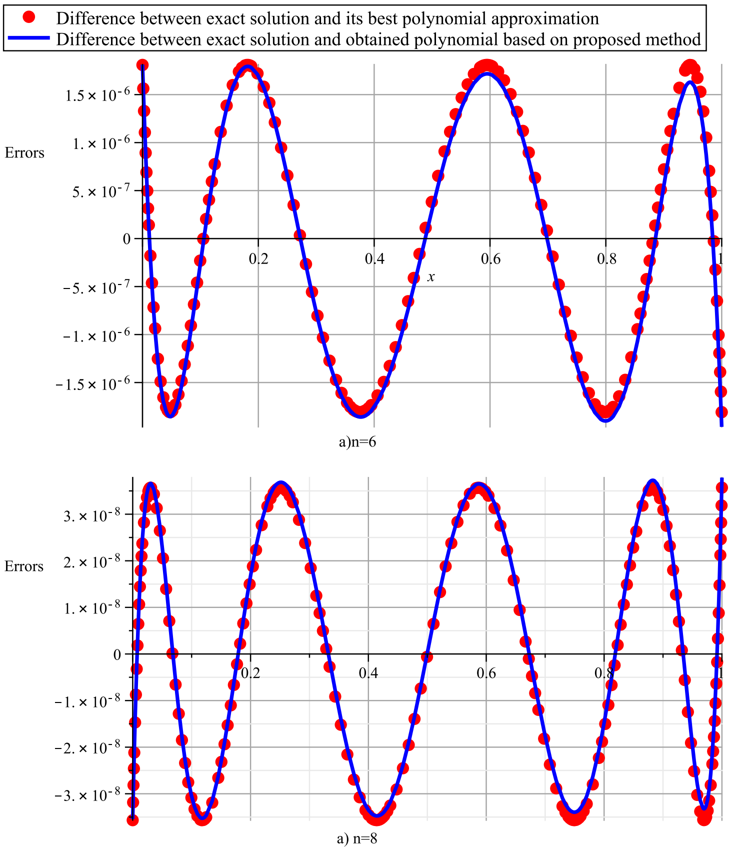

Figure 1: The comparison between differences the exact solution with its best polynomial approximation and obtained polynomial based on the proposed method for Example 6.1.

Example 6.2. Consider the following Volterra integral equation [6]:

$$y(x) = \frac{3}{2} - \frac{1}{2} e^{-2x} - \int_0^x [y^2(t) + y(t)] dt, \tag{6.3}$$

with the exact solution $y(x) = e^{-x}$. The errors obtained by the proposed method and those of the method in [6] are shown in Table 1. This table demonstrates that presented method has smaller errors than those of the method [6]. So, the graphs of $y^* - y_n^*$ and $y^* - p_n^{**}$ for $n = 12$ and $n = 15$ in Fig.2 show that y_n^* is a best polynomial approximation for y^*.

Table 1: Absolute errors based on the proposed method and the method in [6] for Example 6.2.

n	$\|e_n\|_\infty$ Proposed method	$\|e_n\|_\infty$ Method in [6]
4	$1E-05$	$2.39E-05$
8	$1.5E-11$	$1.99E-10$

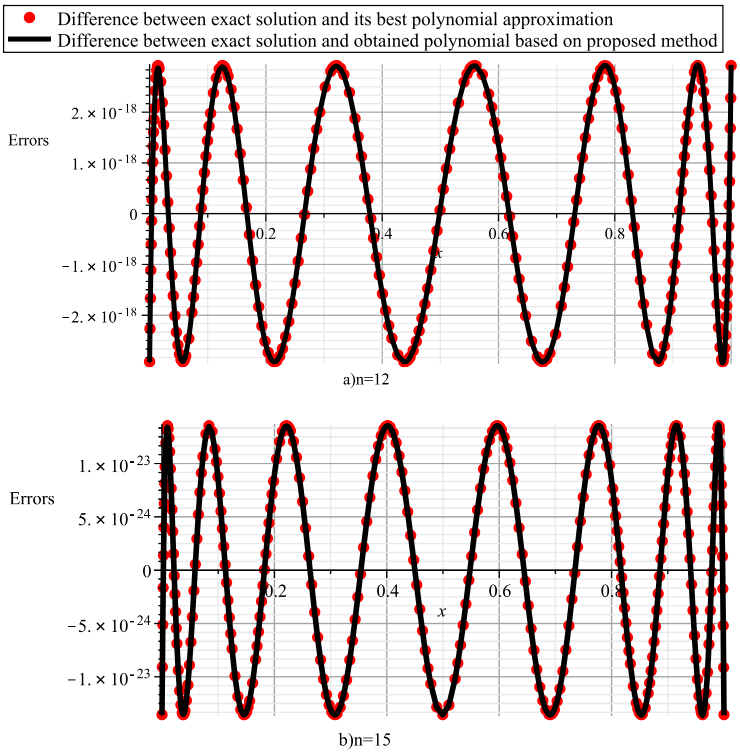

Figure 2: The comparison between differences the exact solution with its best polynomial ap- proximation and obtained polynomial based on the proposed method for Example 6.2.

Example 6.3. Consider the following Volterra-Fredholm integral equation[16]

$$y(x) = 2\cos(x) - 2 + 3\int_0^x \sin(x-t)y^2(t)dt + \frac{6}{7-6\cos(1)}\int_0^1 (1-t)\cos^2(x)(t+y(t))dt, \tag{6.4}$$

with the exact solution $y(x) = \cos(x)$. Table 3 indicates that the results of our method have more rapid rate of convergence than the method [16]. The comparison of $\mathrm{y}^* - \mathrm{p}_\mathrm{n}^{**}, \mathrm{y}^* - \mathrm{y}_\mathrm{n}^*$ for $n = 10, 12$ are exhibited in Fig. 2. Maximum absolute errors 10^{-14} and 10^{-17} show that y_n^* is a good approximation for y^*. Also, by increasing n, y_n^* is very close to best approximation.

Table 2: Comparison of absolute errors for Example 6.3 in case $n = 8$.

x_i	$e_n(x_i)$ in [16]	$e_n(x_i)$ in presented method
0	0	7.12E-12
0.2	2.81E-14	0
0.4	2.88E-11	1.38E-11
0.6	1.66E-09	6.51E-12
0.8	2.94E-08	5.23E-12
1	2.73E-08	3.28E-12

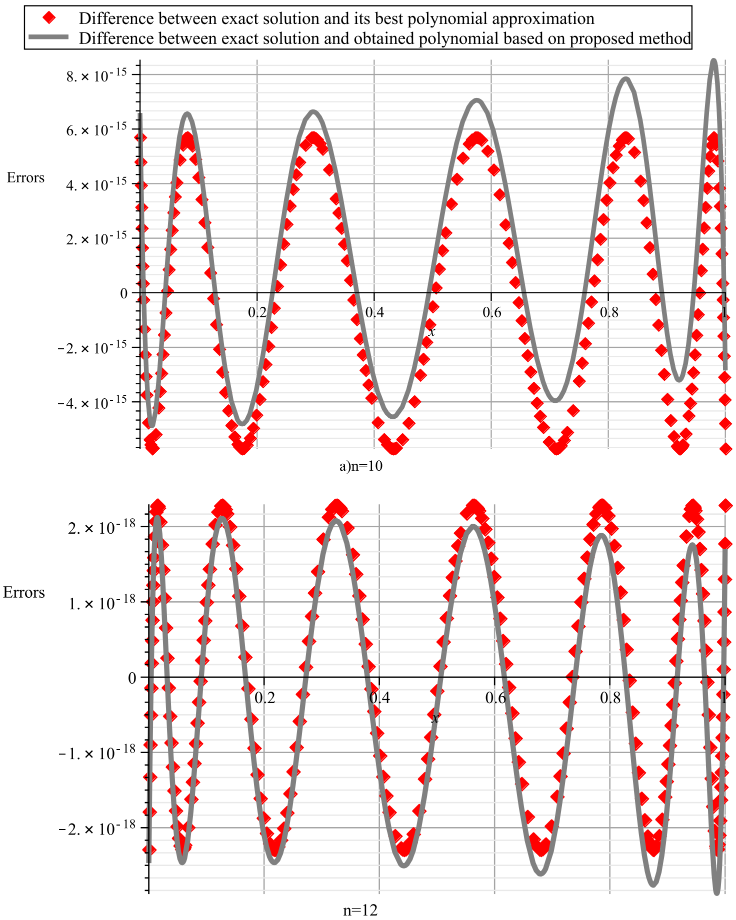

Figure 3: Comparison between differences the exact solution with its best polynomial ap- proximation and obtained polynomial based on the proposed method for Example 6.3.

Example 6.4. Consider the following Volterra-Fredholm integral equation[10]:

$$y(x) = f(x) + \int_0^x \sin(y(t))dt + \int_0^1 2x^2 t \ln(y(t))dt, \tag{6.5}$$

with the exact solution $y(x) = 1 + x$. In this example, accoding to (5.1) nonpolynomial approximate solution is as follows:

$$y_n^*(x) = f(x) + \int_0^x p_n^*(t)dt + \int_0^1 2x^2 t q_n^*(t)dt, \tag{6.6}$$

Table 3: The relative error for 6.1, 6.2, 6.3 and 6.4 in case n=11 , n=13.

n	e_{nr} Example (6.1)	e_{nr} Example (6.2)	e_{nr} Example (6.3)	e_{nr} Example (6.4)
11	2.7E-13	1.5 E-16	4.5E-16	1.75E-12
13	6.4E-17	3.4E-20	8.3E-20	4.2E-15

where polynomials p_n^* and q_n^* are obtained from the system 5.5. The comparison of absolute errors in Fig. 4 with the figure of example 7 of [10] confirms the high efficiency of this method.

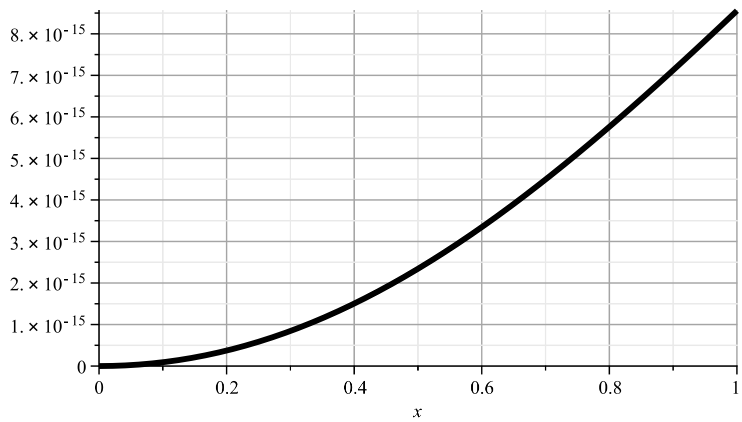

Figure 4: Graph of $e_n(x)$ for example (6.4) with $n = 13$.

Finally, to ensure the convergence and efficiency of the proposed method, a relative error of the approximate solution is obtained for each of the presented examples for n=11 , 13 in the table 3.

7 Conclusions

This study discussed how the Remez algorithm can be applied for obtaining the approximate solution of integral equations. The most important contribution in this work is that the implementation of this method for integral equations yielded the best polynomial approximations for the solution of the integral equation. So, a great advantage of the recommended method from a computational point of view is simplicity and quick reduction of an integral equation to a system of algebraic equations. The proof of essential theorems and the presentation of numerical examples confirm the application of this method for obtaining the approximate solution of integral equations with a high convergence rate. So, it could be possible to extend this method to integrodifferential integral equations, multi-dimensional integral equations, ordinary and partial differential equations, which will be investigated in the future studies.

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