Int. J. Nonlinear Anal. Appl. 14 (2023) 2, 233–246
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.26862.3431

Solvability of infinite system of general order differential equations via generalized Meir-Keeler condensing operator and semi-analytic method

Anupam Das[a], Mohsen Rabbani[b,*], Bipan Hazarika[c], S. A. Mohiuddine[d]

[a] Department of Mathematics, Cotton University, Panbazar, Guwahati-781001, Assam, India

[b] Department of Mathematics, Sari Branch, Islamic Azad University, Sari, Iran

[c] Department of Mathematics, Gauhati University, Guwahati 781014, Assam, India

[d] Operator Theory and Applications Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia

(Communicated by Abdolrahman Razani)

Abstract

In this article, a new fixed point theory and generalized condensing operator have been established to prove the existence of solutions for an infinite system of differential equations of n^{th} order. Also, some interesting examples are employed to support the findings. To validate our discussion the solutions of the examples are approximated by an iterative algorithm with high accuracy. The algorithm is convergent and constructed based on the modified homotopy perturbation method.

Keywords: Measure of noncompactness(MNC), Meir-Keeler condensing(MKC), ordinary differential equations (ODE), Green's function, Banach sequence spaces, Modified homotopy perturbation
2020 MSC: 47H08, 34A34, 46B45, 46B99

In Banach spaces, the study of infinite system of ODE is one of the fundamental and widely studied. The theory of infinite system of ODE describes many factual problems that are existing in mechanics, branching processes etc. The MNC shows an imperative role in the theory of infinite system of ODE (see[3, 17]). Some application of MNC in space ℓ_p $(1 < p < \infty)$ to solve infinite system of ODE can be seen in [15]. Banaś and Lecko [5] and Mursaleen et al. [16] initiated the study of the existence of solutions of infinite system of ODE in the spaces c_0, c, ℓ_1 and ℓ_p, respectively. Mursaleen and Rizvi [17] considered the same differential equations and solved it in spaces c_0 and ℓ_1 MKC. In our discussion, we aim to study an infinite system of ODE problem with order n in spaces c, l_p $(1 < p < \infty)$ and $n(\psi)$ by using MKC operator and Green's function (see [7]).

*Corresponding author
Email addresses: math.anupam@gmail.com (Anupam Das), mrabbani@iausari.ac.ir (Mohsen Rabbani), bh_rgu@yahoo.co.in (Bipan Hazarika), mohiuddine@gmail.com (S. A. Mohiuddine)

Received: April 2022 *Accepted:* July 2022

1 Preliminaries

Let us consider a real Banach space C with the norm $\| \, . \, \|$. Assume that $B(x_0, r) = \{x \in C : \| x - x_0 \| \leq r_0\}$. Let $\mathtt{Q} \subseteq C$ is a nonempty, then $\bar{\mathtt{Q}}$ and Conv $\mathtt{Q}$ are denoted closure and convex closure of $\mathtt{Q}$, respectively. Further, $\mathcal{M}_C$ is a family of all nonempty and bounded subsets of C and by the notation $\mathcal{N}_C$ we denotes its subfamily consisting of all relatively compact sets.

Definition 1.1. Mapping $\mathit{\Delta} : \mathcal{M}_C \to [0, \infty)$ is called a MNC [5] if:

(i) ker $\mathit{\Delta} = \{\mathtt{Q} \in \mathcal{M}_C : \mathit{\Delta}(\mathtt{Q}) = 0\}$ be nonempty, ker $\mathit{\Delta} \subset \mathcal{N}_C$,
(ii) $\mathtt{Q} \subset \mathtt{R} \implies \mathit{\Delta}(\mathtt{Q}) \leq \mathit{\Delta}(\mathtt{R})$,
(iii) $\mathit{\Delta}(\text{Conv } \mathtt{Q}) = \mathit{\Delta}(\mathtt{Q})$,
(iv) $\mathit{\Delta}(\lambda \mathtt{Q} + (1-\lambda)\mathtt{R}) \leq \lambda \mathit{\Delta}(\mathtt{Q}) + (1-\lambda)\mathit{\Delta}(\mathtt{R})$ for $\lambda \in [0,1]$,
(v) $\mathit{\Delta}(\bar{\mathtt{Q}}) = \mathit{\Delta}(\mathtt{Q})$,
(vi) if $\mathtt{Q}_n \in \mathcal{M}_C$, $\mathtt{Q}_n = \bar{\mathtt{Q}}_n$, $\mathtt{Q}_{n+1} \subset \mathtt{Q}_n$ for $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \mathit{\Delta}(\mathtt{Q}_n) = 0$ then $\bigcap_{n=1}^{\infty} \mathtt{Q}_n \neq \emptyset$.

Definition 1.2. [1] Let $\mathtt{M}$ be a nonempty subset of a Banach space C and let $\mathit{\Delta}$ be an arbitrary MNC on C. Thus, $\mathtt{T} : \mathtt{M} \to \mathtt{M}$ is a Meir-Keeler condensing operator if,

$$\forall \epsilon > 0, \exists \delta > 0; \epsilon \leq \mathit{\Delta}(\mathtt{Q}) < \epsilon + \delta \implies \mathit{\Delta}(\mathtt{T}(\mathtt{Q})) < \epsilon$$

for any bounded subset $\mathtt{Q}$ of $\mathtt{M}$.

Theorem 1.3. [1] Let $\mathtt{M}$ be a nonempty, bounded, closed and convex subset of a Banach space C and let $\mathit{\Delta}$ be an arbitrary MNC on C. If $\mathtt{T} : \mathtt{M} \to \mathtt{M}$ is a continuous and MKC operator, then $\mathtt{T}$ has at least one fixed point and the set of all fixed points of $\mathtt{T}$ in $\mathtt{M}$ is compact.

Now, we must define a generalized version of operator MKC and also establish a new fixed point theorem by employing this new condensing operator.

Definition 1.4. Let $\mathtt{M} \subseteq C$ be a nonempty of a Banach space C and $\mathit{\Delta}$ be an arbitrary MNC on C. Also, let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing mapping. The operator $\mathtt{T} : \mathtt{M} \to \mathtt{M}$ is called a generalized operator MKC if,

$$\forall \epsilon > 0, \exists \delta > 0; \epsilon \leq \beta(\mathit{\Delta}(X))\,[\mathit{\Delta}(X) + \phi(\mathit{\Delta}(X))] < \epsilon + \delta \implies \alpha(\mathit{\Delta}(\mathtt{T}X))\,[\mathit{\Delta}(\mathtt{T}X) + \phi(\mathit{\Delta}(\mathtt{T}X))] < \epsilon, \tag{1.1}$$

for any bounded $X \subseteq \mathtt{M}$ where $\alpha : \mathbb{R}_+ \to [1, \infty)$, $\beta : \mathbb{R}_+ \to (0, 1]$ are mappings.

2 Main Results

Theorem 2.1. Let $\mathtt{M}$ be a nonempty, bounded, compact, closed subset of a Banach space C and let $\mathit{\Delta}$ be an arbitrary MNC C. If $\mathtt{T} : \mathtt{M} \to \mathtt{M}$ is continuous and generalized operator MKC then $\mathtt{T}$ admits fixed point in $\mathtt{M}$.

Proof . Let $\{\mathtt{M}_n\}_{n=1}^{\infty}$ be a sequence satisfying $\mathtt{M}_1 = \mathtt{M}$, $\mathtt{M}_{n+1} = Conv(\mathtt{T}\mathtt{M}_n)$, $n \geq 1$. If $\mathit{\Delta}(\mathtt{M}_N) = 0$ for some integer $N \geq 1$, then $\mathtt{M}_N$ is compact. Schauder's theorem implies $\mathtt{T}$ has a fixed point.

If $\mathit{\Delta}(\mathtt{M}_n) > 0$ for any $n \geq 1$. Take $\epsilon_n = \beta(\mathit{\Delta}(\mathtt{M}_n))\,[\mathit{\Delta}(\mathtt{M}_n) + \phi(\mathit{\Delta}(\mathtt{M}_n))] > 0$ and consider $\delta_n = \delta(\epsilon_n)$ such that (2.1) holds.

Therefore by (1.1) we obtain

$$\alpha(\mathit{\Delta}(\mathtt{T}\mathtt{M}_n))\,[\mathit{\Delta}(\mathtt{T}\mathtt{M}_n) + \phi(\mathit{\Delta}(\mathtt{T}\mathtt{M}_n))] < \beta(\mathit{\Delta}(\mathtt{M}_n))\,[\mathit{\Delta}(\mathtt{M}_n) + \phi(\mathit{\Delta}(\mathtt{M}_n))]$$

for each $n \in \mathbb{N}$. By using (1.1) we get,

$$\begin{aligned}
\epsilon_{n+1} &= \beta(\mathit{\Delta}(\mathtt{M}_{n+1}))\,[\mathit{\Delta}(\mathtt{M}_{n+1}) + \phi(\mathit{\Delta}(\mathtt{M}_{n+1}))] \\
&\leq \mathit{\Delta}(\mathtt{M}_{n+1}) + \phi(\mathit{\Delta}(\mathtt{M}_{n+1})) \\
&= \mathit{\Delta}(Conv(\mathtt{T}\mathtt{M}_n)) + \phi(\mathit{\Delta}(Conv(\mathtt{T}\mathtt{M}_n))) \\
&\leq \alpha(\mathit{\Delta}(\mathtt{T}\mathtt{M}_n))\,[\mathit{\Delta}(\mathtt{T}\mathtt{M}_n) + \phi(\mathit{\Delta}(\mathtt{T}\mathtt{M}_n))] \\
&< \beta(\mathit{\Delta}(\mathtt{M}_n))\,[\mathit{\Delta}(\mathtt{M}_n) + \phi(\mathit{\Delta}(\mathtt{M}_n))] \\
&= \epsilon_n,
\end{aligned}$$

which implies that $\{\epsilon_n\}$ is positive strictly decreasing sequence. Thus there exists $r \geq 0$ with $\lim_{n\to\infty} \epsilon_n = r$. If $r > 0$ then $\delta(r) > 0$ exists satisfying (1.1) so $N_0 > 0$ exists such that

$$r \leq \epsilon_n = \beta(\mathit{\Delta}(\mathtt{M}_n))\left[\mathit{\Delta}(\mathtt{M}_n) + \phi(\mathit{\Delta}(\mathtt{M}_n))\right] < r + \delta(r)$$

for $n \geq N_0$. By (1.1) we get,

$$\alpha(\mathit{\Delta}(\mathtt{TM}_n))\left[\mathit{\Delta}(\mathtt{TM}_n) + \phi(\mathit{\Delta}(\mathtt{TM}_n))\right] < r$$

for each $n \geq N_0$. Hence $\epsilon_{n+1} < r$ for any $n \geq N_0$ which is contradiction so $r = 0$. From this we get

$$\lim_{n\to\infty} \mathit{\Delta}(\mathtt{M}_n) = 0.$$

Since $\mathtt{M}_n \supseteq \mathtt{M}_{n+1}$, from Definition 1.1, concludes that $\mathtt{M}_\infty = \bigcap_{n=1}^{\infty} \mathtt{M}_n \subseteq \mathtt{M}$ is nonempty, closed and convex. Furthermore, $\mathtt{M}_\infty$ and is invariant under $\mathtt{T}_1$. By Shauder theorem, $\mathtt{T}$ has at least a fixed point in $\mathtt{M}$. □

Corollary 2.2. For $\phi \equiv 0$, $\alpha \equiv 1$ and $\beta \equiv 1$ the generalized operator MKC transforms into Meir-Keeler theorem so this theorem is a extended of Theorem 1.3.

In $\left(\ell_p, \|\ .\ \|_{\ell_p}\right)$ for $1 < p < \infty$, Hausdorff MNC χ is in the form(see [4]):

$$\chi\left(\mathtt{V}\right) = \lim_{n\to\infty}\left[\sup_{u\in\mathtt{V}}\left(\sum_{k=n}^{\infty} \mid u_k \mid^p\right)^{1/p}\right]$$

where $u(\zeta) = (u_i(\zeta))_{i=1}^{\infty} \in \ell_p$, $\zeta \in [0, \mathtt{T}]$ and $\mathtt{V} \in \mathcal{M}_{\ell_p}$. Therefore, the most suitable MNC $\mathit{\Delta}$ in $(c, \|\ .\ \|_c)$, is in the form:

$$\mathit{\Delta}\left(\mathtt{V}\right) = \lim_{p\to\infty}\left[\sup_{u\in\mathtt{V}}\left\{\sup\left\{\mid u_n - u_m \mid : n, m \geq p\right\}\right\}\right] = \lim_{n\to\infty}\left[\sup_{u\in\mathtt{V}}\left\{\sup_{k\geq n}\left\{\mid u_k - \lim_{m\to\infty} u_m \mid\right\}\right\}\right],$$

where $u(\zeta) = (u_i(\zeta))_{i=1}^{\infty} \in c$, $\zeta \in [0, \zeta]$ and $\mathtt{V} \in \mathcal{M}_c$. Note that, the measure $\mathit{\Delta}$ is regular. We use that standard symbol ω to denotes the set of all complex sequences $x = (x_k)$. For any $x \in \omega$, one writes $\Delta x = \Delta x_k = x_k - x_{k-1}$. Assume that space $\mathcal{C}$ is,

$$\mathcal{C} = \left\{\varsigma : \varsigma' s \text{ are finite sets of distinct positive integers}\right\},$$

Furthermore,

$$\mathcal{C}_{S'} = \left\{\varsigma \in \mathcal{C} : \sum_{n=1}^{\infty} c_n(\varsigma) \leq S'\right\}; c_n(\varsigma) = 1,\ n \in \varsigma \text{ and } c_n(\varsigma) = 0, \quad \text{in otherwise.}$$

Also,define

$$\Phi = \left\{\psi = (\psi_k) \in \omega : 0 < \psi_1 \leq \psi_n \leq \psi_{n+1}, (n+1)\psi_n \geq n\psi_{n+1}\right\}.$$

For $\psi \in \Phi$, the author named Sargent [20] introduced the sequence space $n(\psi)$ which was also considered in ([11, 12]), defined by

$$n(\psi) = \left\{x = (x_k) \in \omega : \| x \|_{n(\psi)} = \sup_{u\in S(x)}\left(\sum_{k=1}^{\infty} \mid u_k \mid \Delta\psi_k\right) < \infty\right\},$$

where $S(x)$ is the space of all sequences that are rearrangements of x. In the Banach space $\left(n(\psi), \|\ .\ \|_{n(\psi)}\right)$, Hausdorff MNC χ is given as (see [12]):

$$\chi\left(\mathtt{V}\right) = \lim_{k\to\infty} \sup_{u\in\mathtt{V}}\left(\sup_{v\in S(u)}\left(\sum_{n=k}^{\infty} \mid v_n \mid \Delta\psi_n\right)\right),$$

where $u(\zeta) = (u_i(\zeta))_{i=1}^{\infty} \in n(\psi)$ for each $\zeta \in [0, \mathtt{T}]$ and $\mathtt{V} \in \mathcal{M}_{n(\psi)}$. In our paper, we study the following infinite system,

$$y_i^{(n)}(\zeta) = f_i\left(\zeta, y(\zeta)\right),\ n \geq 1 \tag{2.1}$$

where $y(\zeta) = (y_i(\zeta))_{i=1}^{\infty}$, $y_i(0) = y_i^{'}(0) = y_i^{''}(0) = ... = y_i^{(n-1)}(0) = 0$, $i \in \mathbb{N}$ and $\zeta \in [0, \mathtt{T}] = I$.

A function $y \in C^n(I,\mathbb{R})$ is a solution of (2.1) if and only if $y \in C(I,\mathbb{R})$ is a solution of the following system:

$$y_i(\zeta) = \int_0^{\zeta} \mathcal{G}(\zeta,\eta) f_i(\eta, y(\eta)) d\eta \qquad (i = 1, 2, 3, ...), \tag{2.2}$$

where $f_i(\zeta, y) \in C(I,\mathbb{R})$ and $\zeta \in I$, and Green's function in (2.2) is the following form (see [7])

$$\mathcal{G}(\zeta,\eta) = \frac{1}{(n-1)!} (\zeta - \eta)^{n-1}, \ 0 \le \eta \le \zeta \le \mathtt{T}. \tag{2.3}$$

Thus

$$y_i^{'}(\zeta) = \int_0^{\zeta} \frac{\partial}{\partial \zeta} \mathcal{G}(\zeta,\eta) f_i(\eta, y(\eta)) d\eta, ..., \ y_i^{(n-1)}(\zeta) = \int_0^{\zeta} \frac{\partial^{n-1}}{\partial \zeta^{n-1}} \mathcal{G}(\zeta,\eta) f_i(\eta, y(\eta)) d\eta.$$

In the next three sections, we will establish existence results for an infinite system (2.1) in ℓ_p $(1 < p < \infty)$, $n(\psi)$, c.

3 Solvability of (2.1) in $\ell_p\, (1 < p < \infty)$

Mursaleen at el. [14] established existence of solution of following systems,

$$\begin{cases} \frac{d^2 y_i}{d\zeta^2} + p(\zeta)\frac{dy_i}{d\zeta} + q(\zeta) y_i = f_i(\zeta, y_1(\zeta), y_2(\zeta), \ldots), \ 0 < \zeta < \mathtt{T}, \\ i = 1, 2, 3, \ldots. \end{cases} \tag{3.1}$$

with $p, q \in C\left([0,\mathtt{T}],\mathbb{R}\right)$ and $y_i(0) = y_i(\mathtt{T}) = 0$ in ℓ_p by Meir-Keeler fixed point theorem. To formulate our result, let us assume the following assumptions in a similar manner as done in [14]:

(i) The mapping $f_i : I \times \ell_p \to \mathbb{R},\ i = 1, 2, ...$ and consider the following mapping

$$(\zeta, y) \to (fy)(\zeta) = (f_1(\zeta, y), f_2(\zeta, y), f_3(\zeta, y), ...)$$

maps $I \times \ell_p$ into ℓ_p and is such that $((fy)(\zeta))_{\zeta \in I}$ is equicontinuous in ℓ_p.

(ii) For any $i \in \mathbb{N}$, there exist functions $g_i, h_i : I \to \mathbb{R}_+$ satisfying:

$$|\ f_i(\zeta, y_1, y_2, y_3, ...)\ |^p \le g_i(\zeta) + h_i(\zeta)\ |\ y_i(\zeta)\ |^p$$

for $\zeta \in I$ and $y = (y_i)$ in ℓ_p. Suppose also that the function series $\sum_{k \ge 1} g_k(\zeta)$ converges uniformly on I and the sequence $(h_i(\zeta))$ is equibounded on I.

Consider

$$\mathcal{G} = \sup_{\zeta \in I} \left\{ \sum_{k \ge 1} g_k(\zeta) \right\}$$

and

$$H = \sup_{i \in \mathbb{N}, \zeta \in I} \{h_i(\zeta)\}$$

with

$$\frac{T^n H^{1/p}}{(n-1)!} < 1.$$

Theorem 3.1. If conditions (i)-(ii) hold, the system (2.1) admits a solution $y(\zeta) = (y_i(\zeta))$ in ℓ_p for all $\zeta \in I$.

Proof . With (2.2) and (ii), and for arbitrary fixed $\zeta \in I$, one writes

$$\begin{aligned}\| y(\zeta) \|_{\ell_p}^p &= \sum_{i=1}^{\infty} \left| \int_0^{\zeta} \mathcal{G}(\zeta,\eta) f_i(\eta, y(\eta)) d\eta \right|^p \\ &\leq \sum_{i=1}^{\infty} \left[\left\{ \int_0^{\zeta} |\mathcal{G}(\zeta,\eta) f_i(\eta, y(\eta))|^p \, d\eta \right\}^{1/p} \left(\int_0^{\zeta} d\eta \right)^{1/q} \right]^p \quad , \frac{1}{p} + \frac{1}{q} = 1 \\ &\leq \mathtt{T}^{p/q} \sum_{i=1}^{\infty} \int_0^{\zeta} |\mathcal{G}(\zeta,\eta)|^p \{ g_i(\eta) + h_i(\eta) \, |y_i(\eta)|^p \} \, d\eta\end{aligned}$$

Since $y(\zeta) \in \ell_p$ therefore we have

$$\sum_{i=1}^{\infty} |y_i(\zeta)|^p \leq M < \infty \text{ (say) and } |\mathcal{G}(\zeta,\eta)| \leq \frac{\mathtt{T}^{n-1}}{(n-1)!}.$$

Hence

$$\begin{aligned}\| y(\zeta) \|_{\ell_p}^p &\leq \frac{\mathtt{T}^{p/q}.\mathtt{T}^{p(n-1)}}{\{(n-1)!\}^p} \int_0^{\zeta} \left[\sum_{i=1}^{\infty} g_i(\eta) + \sum_{i=1}^{\infty} h_i(\eta) \, |y_i(\eta)|^p \right] d\eta \\ &\leq \frac{\mathtt{T}^{p/q}.\mathtt{T}^{p(n-1)}}{\{(n-1)!\}^p} \int_0^{\mathtt{T}} \left(G + H \sum_{i=1}^{\infty} |y_i(\eta)|^p \right) d\eta \\ &\leq \frac{\mathtt{T}^{p/q}.\mathtt{T}^{p(n-1)}}{\{(n-1)!\}^p} (G + HM)\mathtt{T} \\ &= \frac{(G + HM) \, \mathtt{T}^{np}}{\{(n-1)!\}^p} = r^p (say).\end{aligned}$$

Thus, $\| y(\zeta) \|_{\ell_p} \leq r$. Let $y^0(\zeta) = \left(y_i^0(\zeta)\right)$ where $y_i^0(\zeta) = 0$, $\forall \, \zeta \in I$. Moreover, assume that $\mathtt{V} = \mathtt{V}\left(y^0, r\right) = \left\{ y \in \ell_p : \| y - y^0 \| \leq r \right\}$, then $\mathtt{V}$ is a non-empty, closed, bounded and convex subset of ℓ_p. Let the operator $\mathcal{F} = (\mathcal{F}_i)$ on $C\left(I, \mathtt{V}\right)$ defined by

$$(\mathcal{F}y)(\zeta) = \{(\mathcal{F}_i y)(\zeta)\} = \left\{ \int_0^{\zeta} \mathcal{G}(\zeta,\eta) f_i(\eta, y(\eta)) d\eta \right\} \qquad (\forall \zeta \in I),$$

where $y(\zeta) = (y_i(\zeta)) \in \mathtt{V}$ and $y_i(\zeta) \in C(I, \mathbb{R})$. As $(\mathcal{F}y)(\zeta) = ((\mathcal{F}_i y)(\zeta)) \in \ell_p$ for every $\zeta \in I$. Since $(f_i(\zeta, y(\zeta))) \in \ell_p$ for each $\zeta \in I$, we get

$$\sum_{i=1}^{\infty} |(\mathcal{F}_i y)(\zeta)|^p = \sum_{i=1}^{\infty} \left| \int_0^{\zeta} \mathbb{G}(\zeta,\eta) f_i(\eta, y(\eta)) d\eta \right|^p \leq r^p < \infty.$$

Clearly, $(\mathcal{F}_i y)(\zeta)$ satisfies boundary conditions, that is,

$$(\mathcal{F}_i y)(0) = (\mathcal{F}_i y)^{'}(0) = ... = (\mathcal{F}_i y)^{(n-1)}(0) = 0.$$

Since $\| (\mathcal{F}y)(\zeta) - y^0(\zeta) \|_{\ell_p} \leq r$, $\mathcal{F}$ is self function on $\mathtt{V}$. Consequently, $\mathcal{F}$ is continuous on $C\left(I, \mathtt{V}\right)$ by the hypothesis (i).

For $\epsilon > 0$, we obtain $\delta > 0$ such that the following relation holds:

$$\epsilon \leq \chi(\mathtt{V}) < \epsilon + \delta \Rightarrow \chi(\mathcal{F}\mathtt{V}) < \epsilon.$$

We can write for arbitrary fixed $\zeta \in I$,

$$\begin{aligned}
\chi(\mathcal{F}\mathtt{V}) &= \lim_{m\to\infty}\left[\sup_{y(\zeta)\in\mathtt{V}}\left\{\sum_{k\geq m}\left|\int_0^{\zeta}\mathcal{G}(\zeta,\eta)f_k(\eta,y(\eta))d\eta\right|^p\right\}^{\frac{1}{p}}\right]\\
&\leq \frac{\mathtt{T}^{1/q+n-1}}{(n-1)!}\lim_{m\to\infty}\left[\sup_{y(\zeta)\in\mathtt{V}}\left\{\sum_{k\geq m}\int_0^{\zeta}\left(g_k(\eta)+h_{k(\eta)}\left|y_k(\eta)\right|^p\right)d\eta\right\}^{\frac{1}{p}}\right]\\
&\leq \frac{\mathtt{T}^{1/q+n-1}}{(n-1)!}\lim_{m\to\infty}\left[\sup_{y(\zeta)\in\mathtt{V}}\left\{\int_0^{\mathtt{T}}\left(\sum_{k\geq m}g_k(\eta)+H\sum_{k\geq m}\left|y_k(\eta)\right|^p\right)d\eta\right\}^{\frac{1}{p}}\right]\\
&\leq \frac{\mathtt{T}^{1/q+n-1}H^{1/p}}{(n-1)!}\lim_{m\to\infty}\left[\sup_{y(\zeta)\in\mathtt{V}}\left\{\int_0^{\mathtt{T}}\left(\sum_{k\geq m}\left|y_k(\eta)\right|^p\right)d\eta\right\}^{\frac{1}{p}}\right].
\end{aligned}$$

Since for arbitrary fixed $\zeta \in [0,\mathtt{T}]$ and $y(\zeta) = (y_i(\zeta))_{i=1}^{\infty} \in \mathtt{V} \subset \ell_p$, we have as $n \to \infty$ that

$$\chi(\mathtt{V}) \geq \left(\sum_{k=n}^{\infty} \mid y_k(\zeta)\mid^p\right)^{1/p}.$$

Thus, it hods for all $\zeta \in [0,\mathtt{T}]$. Hence as $m \to \infty$ and $\eta \in [0,\mathtt{T}],\ y(\eta) = (y_i(\eta))_{i=1}^{\infty} \in \mathtt{V}$, we have

$$\chi(\mathtt{V})^p \geq \sum_{k\geq m}^{\infty} \mid y_k(\eta)\mid^p,$$

i.e.

$$\int_0^{\mathtt{T}}\left(\sum_{k\geq m}\left|y_k(\eta)\right|^p\right)d\eta \leq \int_0^{\mathtt{T}}\chi(\mathtt{V})^p\,d\eta = \mathtt{T}\chi(\mathtt{V})^p.$$

which yields

$$\chi(\mathcal{F}\mathtt{V}) \leq \frac{\mathtt{T}^{\frac{1}{q}+n-1}H^{\frac{1}{p}}\mathtt{T}^{\frac{1}{p}}}{(n-1)!}\chi(\mathtt{V}) = \frac{\mathtt{T}^nH^{1/p}}{(n-1)!}\chi(\mathtt{V}).$$

Let $\epsilon > 0$ and $\delta = \frac{\epsilon(n-1)!-\epsilon H^{1/p}\mathtt{T}^n}{H^{1/p}\mathtt{T}^n} > 0;\ \ \epsilon \leq \chi(\mathtt{V}) < \epsilon+\delta$. Then,

$$\chi(\mathcal{F}\mathtt{V}) \leq \frac{\mathtt{T}^nH^{1/p}}{(n-1)!}.(\epsilon+\delta) = \frac{\mathtt{T}^nH^{1/p}}{(n-1)!}.\frac{\epsilon(n-1)!}{\mathtt{T}^nH^{1/p}} = \epsilon.$$

Therefore, $\mathcal{F}$ satisfies conditions of corollary 2.2 on the set $\mathtt{V} \subset \ell_p$ for arbitrary fixed $\zeta \in [0,\mathtt{T}]$. Thus $\mathcal{F}$ has a fixed point in $\mathtt{V}$. Hence, the proof is completed. □

4 Solvability of (2.1) in the space $n(\psi)$

Alotaibi at el. [2] established solvability of following systems,

$$\frac{d^2y_i}{d\zeta^2} = -f_i(\zeta, y_1(\zeta), y_2(\zeta), \ldots),\ \zeta \in [0,\mathtt{T}],\ i \in \mathbb{N}, \tag{4.1}$$

and $y_i(0) = y_i(\mathtt{T}) = 0$ are the boundary conditions in $n(\phi)$ by Meir-Keeler fixed point theorem. To formulate our result, let us assume the following assumptions in a similar manner as done in [2]:

(i) The maps $f_i : I \times \mathbb{R}^\infty \to \mathbb{R}$ $(i \in \mathbb{N})$ and f defined on $I \times n(\psi)$ as

$$(\zeta, y) \to (fy)(\zeta) = (f_1(\zeta, y), f_2(\zeta, y), f_3(\zeta, y), ...)$$

transform $I \times n(\psi)$ into $n(\psi)$ and $((fy)(\zeta))_{\zeta \in I}$ is equicontinuous at each point of $n(\psi)$.

(ii) For any $i \in \mathbb{N}$, the following inequality holds true:

$$| f_i(\zeta, y_1, y_2, y_3, ...) | \leq \hat{g}_i(\zeta) + \hat{h}_i(\zeta) \mid y_i(\zeta) \mid$$

for $\zeta \in I$ and $y = (y_i)$ in $n(\psi)$, and functions $\hat{g}_i, \hat{h}_i : I \to \mathbb{R}_+$ which $\hat{g}_i$ $(i = 1, 2, ...)$ is continuous and the mapping series $\sum\limits_{k \geq 1} \hat{g}_k(\zeta) \Delta\psi_k$ converges uniformly, while the sequence $\left(\hat{h}_i(\zeta)\right)$ is equibounded on I.

Let us assume

$$\hat{\mathcal{G}} = \sup_{\zeta \in I} \left\{ \sum_{k \geq 1} \hat{g}_k(\zeta) \Delta\psi_k \right\}$$

and

$$\hat{H} = \sup_{i \in \mathbb{N}, \zeta \in I} \left\{ \hat{h}_i(\zeta) \right\}$$

such that

$$\frac{\hat{H} \mathtt{T}^n}{(n-1)!} < 1.$$

Theorem 4.1. Under the conditions (i)-(ii), the system (2.1) admits a solution $y(\zeta) = (y_i(\zeta)) \in n(\psi)$ i.e., $y(\zeta) = (y_i(\zeta))$ in $n(\psi)$ for each $\zeta \in I$.

Proof . Let $S(y(\zeta))$ be the space of all sequences that are rearrangements of $y(\zeta)$. If $v(\zeta) \in S(y(\zeta))$ then there exists finite real $M > 0$ for all $y(\zeta) = (y_i(\zeta))$ $(\zeta \in I)$ in $n(\psi)$ such that $\sum\limits_{i=1}^{\infty} \mid v_i(\zeta) \mid \Delta\psi_i \leq M < \infty$. With the help of (2.2) and (ii), and for arbitrary fixed $\zeta \in I$, one obtains

$$\begin{aligned}
\| y(\zeta) \|_{n(\psi)} &= \sup_{v \in S(y(\zeta))} \left[\sum_{i=1}^{\infty} \left| \int_0^\zeta \mathcal{G}(\zeta, \eta) f_i(\eta, v(\eta)) d\eta \right| \Delta\psi_i \right] \\
&\leq \sup_{v \in S(y(\zeta))} \left[\sum_{i=1}^{\infty} \left\{ \int_0^\zeta |\mathcal{G}(\zeta, \eta) f_i(\eta, v(\eta))| \, d\eta \right\} \Delta\psi_i \right] \\
&\leq \sup_{v \in S(y(\zeta))} \left[\sum_{i=1}^{\infty} \left\{ \int_0^\zeta |\mathcal{G}(\zeta, \eta)| \left\{ \hat{g}_i(\eta) + \hat{h}_i(\eta) \mid v_i(\eta) \mid \right\} d\eta \right\} \Delta\psi_i \right] \\
&\leq \frac{\mathtt{T}^{n-1}}{(n-1)!} \sup_{v \in S(y(\zeta))} \left[\int_0^{\mathtt{T}} \left\{ \sum_{i=1}^{\infty} g_i(\eta) \Delta\psi_i \right\} d\eta + \hat{H} \int_0^{\mathtt{T}} \left\{ \sum_{i=1}^{\infty} \mid v_i(\eta) \mid \Delta\psi_i \right\} d\eta \right] \\
&\leq \frac{\hat{\mathtt{G}} \mathtt{T}^n}{(n-1)!} + \frac{\hat{H} M \mathtt{T}^n}{(n-1)!} = r_1 \text{ (say)},
\end{aligned}$$

i.e. to say $\| y(\zeta) \|_{n(\psi)} \leq r_1$. Let $y^0(\zeta) = \left(y_i^0(\zeta)\right)$ where $y_i^0(\zeta) = 0$, $\forall \zeta \in I$. Therefore, by $\mathtt{V}_1 = \mathtt{V}_1\left(y^0, r_1\right) = \left\{ y \in n(\psi) : \| y - y^0 \| \leq r_1 \right\}$ it is clear that $\mathtt{V}_1$ is an non-empty, closed, bounded and convex subset of $n(\psi)$. Assume $\mathcal{F} = (\mathcal{F}_i)$ on $C(I, \mathtt{V}_1)$ given by

$$(\mathcal{F}y)(\zeta) = \{(\mathcal{F}_i y)(\zeta)\} = \left\{ \int_0^\zeta \bar{\mathcal{G}}(\zeta, \eta) f_i(\eta, y(\eta)) d\eta \right\} \quad (\forall \zeta \in I),$$

where $y(\zeta) = (y_i(\zeta)) \in \mathtt{V}_1$ and $y_i(\zeta) \in C(I, \mathbb{R})$. Also $(\mathcal{F}y)(\zeta) = ((\mathcal{F}_i y)(\zeta)) \in n(\psi)$ for each $\zeta \in I$. Since $(f_i(\zeta, y(\zeta))) \in n(\psi)$ for each $\zeta \in I$, therefore

$$\sup_{v \in S(y(\zeta))} \left[\sum_{i=1}^{\infty} |(\mathcal{F}_i v)(\zeta)| \, \Delta\psi_i \right] \leq r_1 < \infty.$$

Also, $(\mathcal{F}_i y)(\zeta)$ satisfies boundary conditions given by

$$(\mathcal{F}_i y)(0) = (\mathcal{F}_i y)^{'}(0) = ... = (\mathcal{F}_i y)^{(n-1)}(0) = 0.$$

Since $\| \, (\mathcal{F}y)(\zeta) - y^0(\zeta) \, \|_{n(\psi)} \leq r$, therefore $\mathcal{F}$ is self function on $\mathtt{V}_1$. It is clear by the assumption (i) that the operator $\mathcal{F}$ is continuous on $C(I, \mathtt{V}_1)$. For $\epsilon > 0$, we find $\delta > 0$ such that the following implication holds:

$$\epsilon \leq \chi(\mathtt{V}_1) < \epsilon + \delta \Rightarrow \chi(\mathcal{F}\mathtt{V}_1) < \epsilon.$$

One writes

$$\begin{aligned}
&\chi(\mathcal{F}\mathtt{V}_1) \\
&= \lim_{k\to\infty} \left[\sup_{y(\zeta)\in\mathtt{V}_1} \left\{ \sup_{v\in S(y(\zeta))} \left(\sum_{i\geq k} \left| \int_0^\zeta \mathcal{G}(\zeta,\eta) f_i(\eta, v(\eta)) d\eta \right| \Delta\psi_i \right) \right\} \right] \\
&\leq \lim_{k\to\infty} \left[\sup_{y(\zeta)\in\mathtt{V}_1} \left\{ \sup_{v\in S(y(\zeta))} \left(\sum_{i\geq k} \int_0^\zeta \left| \mathcal{G}(\zeta,\eta) f_i(\eta, v(\eta)) \right| d\eta \Delta\psi_i \right) \right\} \right] \\
&\leq \lim_{k\to\infty} \left[\sup_{y(\eta)\in\mathtt{V}_1} \left\{ \sup_{v\in S(y(\eta))} \left(\sum_{i\geq k} \int_0^\eta \mathcal{G}(\zeta,\eta) g_i(\eta) \Delta\psi_i d\eta + \sum_{i\geq k} \int_0^\zeta \mathcal{G}(\zeta,\eta) h_i(\eta) \mid v_i(\eta) \mid \Delta\psi_i d\eta \right) \right\} \right] \\
&\leq \frac{\mathtt{T}^{n-1}}{(n-1)!} \lim_{k\to\infty} \left[\sup_{y(\zeta)\in\mathtt{V}_1} \left\{ \sup_{v\in S(y(\zeta))} \left(\int_0^{\mathtt{T}} \left(\sum_{i\geq k} g_i(\eta)\Delta\psi_i \right) d\eta + \hat{H} \int_0^{\mathtt{T}} \left(\sum_{i\geq k} \mid v_i(\eta) \mid \Delta\psi_i \right) d\eta \right) \right\} \right].
\end{aligned}$$

Analogous to Theorem 3.1 it can be shown that as $k \to \infty$, we have

$$\chi(\mathtt{V}_1) \geq \sum_{i\geq k} |v_i(\zeta)| \Delta\psi_i,$$

i.e.

$$\int_0^{\mathtt{T}} \left(\sum_{i\geq k} \mid v_i(\zeta) \mid \Delta\psi_i \right) d\zeta \leq T\chi(\mathtt{V}_1).$$

Again, since $\sum_{i\geq 1} \hat{g}_i(\zeta)\Delta\psi_i$ converges uniformly,therefore as $k \to \infty$, we get

$$\sum_{i\geq k} \hat{g}_i(\zeta)\Delta\psi_i \to 0.$$

Hence

$$\chi(\mathcal{F}\mathtt{V}_1) \leq \frac{\hat{H}\mathtt{T}^n}{(n-1)!} \chi(\mathtt{V}_1).$$

Let $\epsilon > 0$ and $\delta = \frac{\epsilon}{\hat{H}\mathtt{T}^n} \left((n-1)! - \hat{H}\mathtt{T}^n \right) > 0$ such that $\epsilon \leq \chi(\mathtt{V}_1) < \epsilon + \delta$. Then,

$$\chi(\mathcal{F}\mathtt{V}_1) \leq \frac{\hat{H}\mathtt{T}^n}{(n-1)!}(\epsilon + \delta) = \frac{\hat{H}\mathtt{T}^n}{(n-1)!} . \frac{\epsilon(n-1)!}{\hat{H}\mathtt{T}^n} = \epsilon.$$

Therefore, $\mathcal{F}$ satisfies conditions of corollary 2.2 on $\mathtt{V}_1 \subset n(\psi)$, $\zeta \in I$. Thus $\mathcal{F}$ has a fixed point in $\mathtt{V}_1$ for all $\zeta \in I$ and proof is complete. $\square$

5 Solvability of (2.1) in the space c

Mursaleen at el. [13] established solvability of the following system,

$$\frac{d^2y_i(\zeta)}{\zeta^2} = a_i(\zeta)y_i(\zeta) + g_i(\zeta, y_1(\zeta), y_2(\zeta), \ldots),\ 0 < \zeta < \mathtt{T},\ i = 1, 2, 3, \ldots \tag{5.1}$$

with the boundary conditions $y_i(0) = y_i(\mathtt{T}) = 0$ in c by Darbo's theorem. To formulate our result, let us assume the following assumptions in a similar manner as done in [13]:

(i) The maps $f_i : I \times \mathbb{R}^\infty \to \mathbb{R},\ i \in \mathbb{N}$ and f defined on $I \times c$ by

$$(\zeta, y) \to (fy)(\zeta) = (f_1(\zeta, y), f_2(\zeta, y), f_3(\zeta, y), ...)$$

maps $I \times c$ into c and $((fy)(\zeta))_{\zeta\in I}$is equicontinuous for all $y \in c$.

(ii) For any $i \in \mathbb{N}$, the following formula holds true:

$$f_i(\zeta, y(\zeta)) = p_i(\zeta, y(\zeta)) + q_i(\zeta)y_i(\zeta) \qquad (\zeta \in I \text{ and } y = (y_i) \in c),$$

where both the real maps $p_i(\zeta, y(\zeta))$ and $q_i(\zeta)$ are continuous on $I \times c$ and I, respectively. Moreover, there exist a sequence $\{P_i\}$ converges to zero with $\mid p_i(\zeta, y(\zeta)) \mid\leq P_i$ for any $\zeta \in I, y(\zeta) \in c$ and the function sequence $(q_i(\zeta))$ is uniformly convergent on I.

Consider,

$$P = \sup_{i\in\mathbb{N}} \{P_i\}$$

and

$$Q = \sup_{\zeta\in I, i\in\mathbb{N}} \{q_i(\zeta)\}.$$

such that

$$\frac{Q\mathtt{T}^n}{(n-1)!} < 1.$$

Theorem 5.1. If conditions(i)-(ii) hold, system (2.1) admits a solution $y(\zeta) = (y_i(\zeta))$ in c for every $\zeta \in I$.

Proof . Let $M > 0$ with $\sup_{i\in\mathbb{N}} \mid y_i(\zeta) \mid\leq M < \infty$ for all $y(\zeta) = (y_i(\zeta)) \in c$ and $\zeta \in I$. With the help of (2.2) and (ii), and for arbitrary fixed $\zeta \in I$, one obtains

$$\begin{aligned}
\| y(\zeta) \|_c &= \sup_{k\geq1} \left| \int_0^\zeta \mathcal{G}(\zeta, \eta) f_k(\zeta, y(\eta)) d\eta \right| \\
&\leq \sup_{k\geq1} \int_0^\zeta |\mathcal{G}(\zeta, \eta)| \, |p_k(\eta, y(\eta)) + q_k(\eta)y_k(\eta)| \, d\eta \\
&\leq \sup_{k\geq1} \int_0^\zeta |\mathcal{G}(\zeta, \eta)| \{|p_k(\eta, y(\eta))| + |q_k(\eta)| \mid y_k(\eta) \mid\} \, d\eta \\
&\leq \frac{\mathtt{T}^{n-1}}{(n-1)!} \sup_{k\geq1} \left\{ \int_0^{\mathtt{T}} (P_k + QM) ds \right\} \\
&\leq \frac{(P + QM)\mathtt{T}^n}{(n-1)!} = r_2 \ (say)
\end{aligned}$$

Let $y^0(\zeta) = \left(y_i^0(\zeta)\right)$ where $y_i^0(\zeta) = 0,\ \forall\, \zeta \in I$. According to $\mathtt{V}_2 = \mathtt{V}_2\left(y^0, r_2\right) = \left\{y \in c :\| y - y^0 \|\leq r_2\right\}$, then $\mathtt{V}_2$ is an non-empty, bounded, closed and convex subset of c. Consider $\mathcal{F} = (\mathcal{F}_i)$ on $C(I, \mathtt{V}_2)$ defined by:

$$(\mathcal{F}y)(\zeta) = \{(\mathcal{F}_i y)(\zeta)\} = \left\{ \int_0^\zeta \mathcal{G}(\zeta, \eta) f_i(\eta, y(\eta)) d\eta \right\} \qquad (\zeta \in I),$$

where $y(\zeta) = (y_i(\zeta)) \in \mathtt{V}_2$ and $y_i(\zeta) \in C(I, \mathbb{R})$. Since $(f_i(\zeta, y(\zeta))) \in c$ for each $\zeta \in I$, thus

$$\lim_{i\to\infty} (\mathcal{F}_i y)(\zeta) = \lim_{i\to\infty} \int_0^\zeta \mathcal{G}(\zeta,\eta) f_i(\eta, y(\eta))d\eta = \int_0^\zeta \mathcal{G}(\zeta,\eta) \lim_{i\to\infty} f_i(\eta, y(\eta))d\eta$$

is unique and finite. It follows that $(\mathcal{F}y)(\zeta) \in c$. Further on, $(\mathcal{F}_i y)(\zeta)$ satisfies the following boundary conditions:

$$(\mathcal{F}_i y)(0) = (\mathcal{F}_i y)^{'}(0) = \ldots = (\mathcal{F}_i y)^{(n-1)}(0) = 0.$$

Since $\| (\mathcal{F}y)(\zeta) - y^0(\zeta) \|_c \le r_2$, the operator $\mathcal{F}$ is self function on $\mathtt{V}_2$. Clearly, by (i) of this section, $\mathcal{F}$ is continuous on $C(I, \mathtt{V}_2)$. For $\epsilon > 0$, we therefore write

$$\begin{aligned}\mathit{\Delta}(\mathcal{F}\mathtt{V}_2) &= \lim_{p\to\infty}\left[\sup_{y(\zeta)\in\mathtt{V}_2}\left\{\sup_{k\ge p}\left|\int_0^\zeta \mathcal{G}(\zeta,\eta)f_k(\eta,y(\eta))d\eta - \lim_{m\to\infty}\int_0^\eta \mathcal{G}(\zeta,\eta)f_m(\eta,y(\eta))d\eta\right|\right\}\right]\\
&\le \frac{\mathtt{T}^{n-1}}{(n-1)!}\lim_{p\to\infty}\left[\sup_{y(\zeta)\in\mathtt{V}_2}\left\{\sup_{k\ge p}\left|\int_0^\zeta f_k(\eta,y(\eta))d\eta - \lim_{m\to\infty}\int_0^\zeta f_m(\eta,y(\eta))d\eta\right|\right\}\right]\\
&\le \frac{\mathtt{T}^{n-1}}{(n-1)!}\lim_{p\to\infty}\left[\sup_{y(\zeta)\in\mathtt{V}_2}\left\{\sup_{k\ge p}\int_0^\zeta \left|p_k(\eta,y(\eta)) + q_k(\eta)y_k(\eta) - \lim_{m\to\infty}(p_m(\eta,y(\eta)) + q_m(\eta)y_m(\eta))\right| d\eta\right\}\right]\\
&\le \frac{Q\mathtt{T}^{n-1}}{(n-1)!}\lim_{p\to\infty}\left[\sup_{y(\zeta)\in\mathtt{V}_2}\left\{\sup_{k\ge p}\int_0^\mathtt{T} \left|y_k(\eta) - \lim_{m\to\infty} y_m(\eta)\right| d\eta\right\}\right].\end{aligned}$$

Analogous to Theorem 3.1 it can be shown that as $p \to \infty$, we have

$$\mathit{\Delta}(\mathtt{V}_2) \ge \left|y_k(\eta) - \lim_{m\to\infty} y_m(\eta)\right|,$$

i.e.

$$\int_0^\mathtt{T} \left|y_k(\eta) - \lim_{m\to\infty} y_m(\eta)\right| d\eta \le \mathtt{T}\mathit{\Delta}(\mathtt{V}_2).$$

Hence

$$\mathit{\Delta}(\mathcal{F}\mathtt{V}_2) \le \frac{\mathtt{T}^n Q \mathit{\Delta}(\mathtt{V}_2)}{(n-1)!}.$$

Let $\epsilon > 0$ and $\delta = \frac{\epsilon}{\mathtt{T}^n Q}((n-1)! - \mathtt{T}^n Q) > 0$, such that $\epsilon \le \mathit{\Delta}(\mathtt{V}_2) < \epsilon + \delta$. Then

$$\mathit{\Delta}(\mathcal{F}\mathtt{V}_2) \le \frac{\mathtt{T}^n Q(\epsilon+\delta)}{(n-1)!} = \frac{\mathtt{T}^n Q}{(n-1)!}.\frac{\epsilon(n-1)!}{\mathtt{T}^n Q} = \epsilon.$$

It follows that $\mathcal{F}$ satisfies all conditions of corollary 2.2 on $\mathtt{V}_2 \subset c$ for arbitrary fixed $\zeta \in I$. Thus $\mathcal{F}$ has a fixed point in $\mathtt{V}_2$ for all $\zeta \in I$. This is a required solution of (2.1). □

6 Practical Examples

To justify of our results proved in previous sections, we present some examples.

Example 6.1. Consider the fifth-order differential equations system:

$$\frac{d^5 y_i(\zeta)}{d\zeta^5} = f_i(\zeta, y(\zeta)) \tag{6.1}$$

with $y_i(0) = y_i^{'}(0) = 0$ *and* $y_i^{''}(0) = y_i^{'''}(0) = y_i^{iv}(0) = 0$, where

$$f_i(\zeta, y(\zeta)) = \frac{\zeta}{(1+i)^4} + \frac{e^\zeta \cos(\zeta) y_i(\zeta)}{i^2} \qquad (\forall i \in \mathbb{N}, \zeta \in I = [0,1]).$$

Then, $\sum\limits_{k=1}^{\infty}|f_k(\zeta,y(\zeta))|^p \leq 2^p\sum\limits_{k=1}^{\infty}\frac{1}{i^{4p}}+2^pe^p\sum\limits_{k=1}^{\infty}|y_k(\zeta)|^p<\infty$ if $y(\zeta)=(y_i(\zeta))\in\ell_p$, where $1<p<\infty$ i.e., $(f_i(\zeta,y(\zeta)))\in\ell_p$.

Let $\epsilon>0$ be given and $z(\zeta)\in\ell_p$. Considering $z(\zeta)\in\ell_p$ with the strict inequality $\| y(\zeta)-z(\zeta)\|_{\ell_p}<\delta=\left(\frac{\epsilon}{e^p}\right)^{1/p}$, then

$$\begin{aligned}|f_i(\zeta,y(\zeta))-f_i(\zeta,z(\zeta))|^p &= \left|\frac{e^\zeta\cos(\zeta)y_i(\zeta)}{i^2}-\frac{e^\zeta\cos(\zeta)z_i(\zeta)}{i^2}\right|^p\\ &\leq e^p\| y(\zeta)-z(\zeta)\|_{\ell_p}^p .\end{aligned}$$

Consequently,

$$|f_i(\zeta,y(\zeta))-f_i(\zeta,z(\zeta))|<\epsilon$$

which gives the equicontinuity of $((fy)(\zeta))_{\zeta\in I}$ on Banach space ℓ_p. Moreover, one writes

$$|f_i(\zeta,y(\zeta))|^p\leq\frac{2^p}{i^{4p}}+\frac{2^pe^{p\zeta}}{i^{2p}}|y_i(\zeta)|^p=g_i(\zeta)+h_i(\zeta)|y_i(\zeta)|^p \qquad (\forall i\in\mathbb{N},\zeta\in I),$$

where $g_i(\zeta)=\frac{2^p}{i^{4p}}$ and $h_i(\zeta)=\frac{2^pe^{p\zeta}}{i^{2p}}$ functions are real on I while the function series $\sum\limits_{k\geq1}g_k(\zeta)=\sum\limits_{k\geq1}\frac{2^p}{i^{4p}}$ converges uniformly on I and the function sequence $\{h_i(\zeta)\}$ are converges uniformly and equibounded, respectively, on I. We also obtain $H=2^pe^p$ and $\frac{\mathtt{T}^nH^{1/p}}{(n-1)!}=\frac{e}{12}<1$. By taking Theorem 3.1 into account, system (6.1) has unique solution in ℓ_p.

Example 6.2. Consider the fourth-order differential equations system in the form:

$$\frac{d^4y_i(\zeta)}{d\zeta^4}=f_i(\zeta,y(\zeta)) \tag{6.2}$$

with $y_i^{(n)}(0)=0,n=0,1,2,3$ where

$$f_i(\zeta,y(\zeta))=\frac{\zeta+1}{i^2}+\sum_{m=1}^{i}\frac{y_i(\zeta))}{m^2}, \quad (\forall i\in\mathbb{N},\zeta\in I=[0,1]).$$

If $y(\zeta)\in c$ then

$$\lim_{i\to\infty}f_i(\zeta,y(\zeta))=\lim_{i\to\infty}\left[\frac{\zeta+1}{i^2}+\sum_{m=1}^{i}\frac{y_i(\zeta)}{m^2}\right]$$

is unique and finite. Consequently, $(f_i(\zeta,y(\zeta)))\in c$. Let $\epsilon>0$ be given, and also let $z(\zeta)\in c$ be such that $\|y(\zeta)-z(\zeta)\|_c\leq\delta=\frac{6\epsilon}{\pi^2}$. Therefore, one obtains

$$\begin{aligned}|\,f_i(\zeta,y(\zeta))-f_i(\zeta,z(\zeta))\,| &= \left|\sum_{m=1}^{i}\frac{y_i(\zeta)-z_i(\zeta)}{m^2}\right|\leq\sum_{m=1}^{i}\frac{1}{m^2}|y_i(\zeta)-z_i(\zeta)|\leq\delta\sum_{m=1}^{i}\frac{1}{m^2}\\ &<\delta\frac{\pi^2}{6}<\epsilon\end{aligned}$$

for any fixed i. Hence, $(f_i(\zeta,y(\zeta)))_{\zeta\in I}$ is equicontinuous on the space of convergent sequence c. Further on, one writes

$$\begin{aligned}f_i(\zeta,y(\zeta)) &= \frac{\zeta+1}{i^2}+\sum_{m=1}^{i}\frac{y_i(\zeta)}{m^2}\\ &= \frac{\zeta+1}{i^2}+y_i(\zeta)\sum_{m=1}^{i}\frac{1}{m^2}\\ &= p_i(\zeta,y(\zeta))+q_i(\zeta)y_i(\zeta),\end{aligned}$$

where $P_i=\frac{2}{i^2}$, $p_i(\zeta,y(\zeta))=\frac{\zeta+1}{i^2}$ and $q_i(\zeta)=\sum\limits_{m=1}^{i}\frac{1}{m^2}$ defined, P_i is convergent to zero, $q_i(\zeta)$ and $(p_i(\zeta,y(\zeta)))$ are continuous and $\{q_i(\zeta)\}$ is uniformly convergent on I. Moreover, we have $Q=\frac{\pi^2}{6}$, $P=2$ and $\mathtt{T}=1$. Consequently, $\frac{Q\mathtt{T}^n}{(n-1)!}=\frac{\pi^2}{36}<1$. Hence, by Theorem 5.1, we conclude that (6.2) has unique solution in c.

7 Constructing an iterative algorithm to approximate solution of Eq.(6.2)

To obtain an approximation of solution of (6.2), we construct an iterative algorithm via a modified homotopy perturbation method. Some improved and modified homotopy perturbation methods to solve non-linear integral and differential equations were applied by Rabbani et al. in [18, 19] respectively. Also coupled modified homotopy perturbation and Adomian decomposition method to solve infinite system of nonlinear integral equations can be seen in [6, 9]. Consider a general form of nonlinear system of equations

$$\begin{cases} A(y_i(\zeta)) - f_i(\zeta) = 0, \\ y_i^{(k)}(0) = 0, k = 0, 1, \cdots, n-1, \\ \zeta \in \Omega = [0,1], \quad i \in \mathbb{N}, \end{cases} \tag{7.1}$$

where A is a general nonlinear operator and f_i's are known analytic functions. We convert operator A to nonlinear operators as N_1 and N_2 which in special case N_1 or N_2 can be linear operator. Also f_i's can be converted to functions $f_{i,1}$ and $f_{i,2}$, thus a modified homotopy perturbation for the above infinite system of equations is defined to this form,

$$\begin{cases} H(\nu_i, p) = N_1(\nu_i(\zeta)) - f_{i,1}(\zeta) + p\Big(N_2(\nu_i(\zeta)) - f_{i,2}(\zeta)\Big) = 0, p \in [0,1] \\ i = 1, 2, 3, \dots. \end{cases} \tag{7.2}$$

where ν_i's are approximations of y_i's for $i \in \mathbb{N}$ and p is an embedding parameter. By variations of $p = 0$ to $p = 1$ it concludes that $N_1(\nu_i(\zeta)) = f_{1,i}(\zeta)$ to $A(\nu_i(\zeta)) - f_i(\zeta) = 0$. In fact in (7.2) for $p = 1$ we approach the solution of (7.1). Therefore the solution of (7.1) is approximated by the following series

$$\begin{cases} y_i(\zeta) \approx \nu_i(\zeta) = \sum\limits_{k=0}^{\infty} p^k \nu_{i,k}(\zeta), i \in \mathbb{N} \\ y_i(\zeta) = \lim\limits_{p \to 1} \nu_i(\zeta). \end{cases} \tag{7.3}$$

The system of fourth-order differential equations(6.2) may be written in this form,

$$\begin{cases} y_i(\zeta) - \lambda y_i^{(4)}(\zeta) + \lambda \frac{\zeta+1}{i^2} = 0, \quad (\forall i \in \mathbb{N}, \zeta \in I = [0,1]) \\ y_i^{(k)}(0) = 0, k = 0, 1, 2, 3, \quad and \quad \lambda = (\sum\limits_{m=1}^{i} \frac{1}{m^2})^{-1}. \end{cases} \tag{7.4}$$

Let us to define operatorsN_1 and N_2 and functions f's for E.q.(7.4)

$$\begin{aligned} &N_1(y_i(\zeta)) = y_i(\zeta), \qquad N_2(y_i(\zeta)) = -\lambda y_i^{(4)}(\zeta) \\ &f_{i,1}(\zeta) + f_{i,2}(\zeta) = f_i(\zeta) = -\lambda \frac{\zeta+1}{i^2}, \\ &y_i^{(k)}(0) = 0, k = 0, 1, 2, 3. \end{aligned} \tag{7.5}$$

Substituting (7.5) and (7.3) in (7.2) yields

$$\Big(\sum_{k=0}^{\infty} p^k \nu_{i,k}(\zeta) - f_{1,i}(\zeta)\Big) + p\Big(-\lambda \sum_{k=0}^{\infty} p^k \nu_{i,k}^{(4)}(\zeta) + \lambda \frac{\zeta+1}{i^2} + f_{1,i}(\zeta)\Big) = 0, \tag{7.6}$$

Rearranging (7.6) respect to p powers, leads to an iterative algorithm.

Algorithm:

$$\begin{aligned} &\nu_{i,0}(\zeta) = f_{1,i}(\zeta), \ \text{ subject to: } \nu_{i,0}^{(k)}(0) = 0, k = 0, 1, 2, 3. \\ &\nu_{i,1}(\zeta) = \lambda \nu_{i,0}^{(4)}(\zeta) - \lambda \frac{\zeta+1}{i^2} - \nu_{i,0}(\zeta) \\ &\nu_{i,j}(\zeta) = \lambda \nu_{i,j-1}^{(4)}(\zeta), \quad i \in \mathbb{N}, \ j = 2, 3, \cdots \end{aligned} \tag{7.7}$$

Convergence of the above algorithm could be proved similar to [10]. Now, we compute terms of sequence $\{y_1(\zeta), y_2(\zeta), ..\}$ to introduce the solution corresponding to(6.2) by the above algorithm. To choose a suitable start point of the algorithm (7.7), we solve the first equation of (7.4) for $i = 1$ analytically and the solution can be given easily in the form,

$$y_1(\zeta) = \frac{1}{2}e^{\zeta} + \frac{1}{2}(\cos(\zeta) + \sin(\zeta)) - \zeta - 1. \tag{7.8}$$

This solution is an effective start point in the above algorithm, because it satisfies in the initial conditions of the above problem (7.4) and absolute error is zero. Therefore the solution of i-th ($i = 1, 5, 25, 50, 100$) equations of (7.4) can be given as,

$$\begin{aligned}
y_1(\zeta) &\approx \sum_{k=0}^{25} \nu_{1,k}(\zeta) = -1 + 0.5e^{\zeta} - \zeta + 0.5(\cos(\zeta) + \sin(\zeta)),\\
y_5(\zeta) &\approx \sum_{k=0}^{25} \nu_{5,k}(\zeta) = -0.0273297 + 0.0000365871e^{\zeta} - 0.0273297\zeta + 0.0000365871(\cos(\zeta) + \sin(\zeta)),\\
y_{25}(\zeta) &\approx \sum_{k=0}^{25} \nu_{25,k}(\zeta) = -0.000996436 + 3.60746 \times 10^{-6}e^{\zeta} - 0.000996436\zeta\\
&\qquad + 3.60746 \times 10^{-6}(\cos(\zeta) + \sin(\zeta)),\\
y_{50}(\zeta) &\approx \sum_{k=0}^{25} \nu_{50,k}(\zeta) = -0.000246134 + 2.67146 \times 10^{-6}e^{\zeta} - 0.000246134\zeta + 2.67146 \times 10^{-6}(\cos(\zeta) + \sin(\zeta)),\\
y_{100}(\zeta) &\approx \sum_{k=0}^{25} \nu_{100,k}(\zeta) = -0.0000611627 + 2.29685 \times 10^{-6}e^{\zeta} - 0.0000611627\zeta\\
&\qquad + 2.29685 \times 10^{-6}(\cos(\zeta) + \sin(\zeta)).
\end{aligned} \tag{7.9}$$

According to (7.9), substituting i-th solution in i-th equation ($i = 1, 5, 25, 50, 100$) of E.q.(7.4) and comparing both sides of the equations, the absolute errors in some points are shown in the table.1.

Table 1: Absolute errors

ζ	$y_1(\zeta)$ for first E.q	$y_5(\zeta)$ for 5-th E.q	$y_{25}(\zeta)$ for the 25-th E.q	$y_{50}(\zeta)$ for the 50-th E.q	$y_{100}(\zeta)$ for the 100-th E.q
0.0	0	2.3×10^{-5}	2.7×10^{-6}	2.0×10^{-6}	1.7×10^{-6}
0.1	0	2.5×10^{-5}	2.9×10^{-6}	2.2×10^{-6}	1.9×10^{-6}
0.2	0	2.7×10^{-5}	3.2×10^{-6}	2.4×10^{-6}	2.1×10^{-6}
0.3	0	3.0×10^{-5}	3.5×10^{-6}	2.6×10^{-6}	2.3×10^{-6}
0.4	0	3.2×10^{-5}	3.8×10^{-6}	2.8×10^{-6}	2.4×10^{-6}
0.5	0	3.4×10^{-5}	4.0×10^{-6}	3.0×10^{-6}	2.6×10^{-6}
0.6	0	3.7×10^{-5}	4.3×10^{-6}	3.3×10^{-6}	2.8×10^{-6}
0.7	0	3.9×10^{-5}	4.6×10^{-6}	3.5×10^{-6}	3.0×10^{-6}
0.8	0	4.2×10^{-5}	4.9×10^{-6}	3.7×10^{-6}	3.2×10^{-6}
0.9	0	4.4×10^{-5}	5.2×10^{-6}	3.9×10^{-6}	3.4×10^{-6}
1.0	0	4.7×10^{-5}	5.5×10^{-6}	4.2×10^{-6}	3.6×10^{-6}

8 Conclusions

In this research work, existence of solution for infinite system of n^{th} order ODE is proved. Some examples are presented to clarify the reliability of our results. Moreover, we constructed an iterative algorithm in order to find an approximate solution of the examples with high accuracy.

References

[1] A. Aghajani, M. Mursaleen and A.S. Haghighi, *Fixed point theorems for Meir-Keeler condensing operators via MNC*, Acta. Math. Sci. **35** (2015), no. 3, 552–566.

[2] A. Alotaibi, M. Mursaleen, Badriah A.S. Alamri, *Solvability of second order linear differential equations in the sequence space $n(\phi)$*,, Adv. Differ. Equ. **2018** (2018), no. 1, 1–8.

[3] A. Alotaibi, M. Mursaleen and S.A. Mohiuddine, *Application of measure of noncompactness to infinite system of linear equations in sequence spaces*, Bull. Iran. Math. Soc. **41** (2015), 519–527.

[4] J. Banaś and K. Goebel, *Measure of Noncompactness in Banach Spaces*, Lecture Notes in Pure and Applied Mathematics, 60, Marcel Dekker, New York, 1980.

[5] J. Banaś and M. Lecko, *Solvability of infinite systems of differential equations in Banach sequence spaces*, J. Comput. Appl. Math. **137** (2001), 363–375.

[6] A. Das, M. Rabbani, B. Hazarika and R. Arab, *Solvability of infinite systems of nonlinear singular integral equations in the $C(I \times I, c)$ space and using semi-analytic method to find a closed-form of solution*, Int. J. Nonlinear Anal. Appl. **10** (2019), no. 1, 63–76.

[7] D.G. Duffy, *Green's function with applications*, Chapman and Hall/CRC, London, (2001).

[8] L.S. Goldenštein and A.S. Markus, *On a measure of noncompactness of bounded sets and linear operators*, Studies in Algebra and Mathematical Analysis, Kishinev, 1965.

[9] Bipan Hazarika, H.M. Srivastava, Reza Arab and M. Rabbani, *Existence of solution for an infinite system of nonlinear integral equations via measure of noncompactness and homotopy perturbation method to solve it*, J. Comput. Appl. Math. **343** (2018), 341—352.

[10] B. Hazarika, H.M. Srivastava, R. Arab and M. Rabbani, *Application of simulation function and measure of noncompactness for solvability of nonlinear functional integral equations and introduction to an iteration algorithm to find solution*, Appl. Math. Comput. **360** (2019), 131—146.

[11] E. Malkowsky and M. Mursaleen, *Matrix transformations between FK-spaces and the sequence spaces $m(\phi)$ and $n(\phi)$*, J. Math. Anal. Appl. **196** (1995), no. 2, 659–665.

[12] E. Malkowsky and M. Mursaleen, *Compact matrix operators between the spaces $m(\phi)$ and $n(\phi)$*, Bull. Korean Math. Soc. **48** (2011), no. 5, 1093–1103.

[13] M. Mursaleen, S.M. H. Rizvi and B. Samet, *Solvability of a class of boundary value problems in the space of convergent sequences*, Appl. Anal. **97** (2018), no. 11, 1829–1845.

[14] M. Mursaleen, E. Pourhadi and R. Saadati, *Solvability of infinite systems of second-order differential equations with boundary conditions in ℓ_p*, Q. Math. **43** (2020), no. 9, 1311–1330.

[15] S.A. Mohiuddine, H.M. Srivastava and A. Alotaibi, *Application of measures of noncompactness to the infinite system of second-order differential equations in ℓ_p spaces*, Adv. Difference Equ. **2016** (2016), Article 317.

[16] M. Mursaleen and S.A. Mohiuddine, *Applications of measures of noncompactness to the infinite system of differential equations in ℓ_p spaces*, Nonlinear Anal. **75** (2012), 2111–2115.

[17] M. Mursaleen and S.M.H. Rizvi, *Solvability of infinite systems of second order differential equations in c_0 and ℓ_1 by Meir-Keeler condensing operators*, Proc. Amer. Math. Soc. **144** (2016), no. 10, 4279–4289.

[18] M. Rabbani, *Modified homotopy method to solve non-linear integral equations*, Int. J. Nonlinear Anal. Appl. **6** (2015), no. 2, 133–136.

[19] M. Rabbani and R. Arab, *Extension of some theorems to find solution of nonlinear integral equation and homotopy perturbation method to solve it*, Math. Sci. **11** (2017), no. 2, 87—94.

[20] W.L.C. Sargent, *On compact matrix transformations between sectionally bounded BK-spaces*, J. London Math. Soc. **41** (1966), 79–87.