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Some inequalities for the growth of rational functions with prescribed poles

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Abstract

Let $\mathcal{R}_n$ be the set of all rational functions of the type $r(z) = f(z)/w(z)$, where $f(z)$ is a polynomial of degree at most n and $w(z) = \prod_{j=1}^{n}(z - \beta_j)$, $|\beta_j| > 1$ for $1 \leq j \leq n$. In this paper, we prove some results concerning the growth of rational functions with prescribed poles by involving some of the coefficients of polynomial $f(z)$. Our results not only improve the results of N. A. Rather et al. [8], but also give the extension of some recent results concerning the growth of polynomials by Kumar and Milovanovic [3] to the rational functions with prescribed poles and we obtain the analogous results for such rational functions with restricted zeros.

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1 Introduction

Let $\mathcal{P}_n$ be the class of all complex polynomials of degree at most n. If $f \in \mathcal{P}_n$, then

$$\max_{|z|=\nu} |f(z)| \leq \nu^n \max_{|z|=1} |f(z)|, \quad \nu \geq 1. \tag{1.1}$$

Inequality (1.1) is a simple consequence of Maximum Modulus Principle (see [5], [6], [9]). The reverse analogue of inequality (1.1) whenever $\nu \leq 1$ was given by Varga [11], and he proved that if $f \in \mathcal{P}_n$, then

$$\max_{|z|=\eta} |f(z)| \geq \eta^n \max_{|z|=1} |f(z)|, \tag{1.2}$$

whenever $0 \leq \eta \leq 1$. The equality in (1.1) and (1.2) holds whenever $f(z) = \lambda z^n$, $\lambda \neq 0$.

For the class of polynomials having no zeros inside the unit circle, T. J. Rivlin [10] proved the following result:

Theorem 1.1. If $f \in \mathcal{P}_n$ does not vanish in $|z| < 1$, then for $0 \leq \eta \leq 1$ and $|z| = 1$,

$$|f(\eta z)| \geq \left(\frac{\eta + 1}{2}\right)^n |f(z)|. \tag{1.3}$$

The result is best possible and equality holds for $f(z) = (z + \zeta)^n, |\zeta| = 1$.

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As a generalization of inequality (1.3), A. Aziz [1] established the following result:

Theorem 1.2. If $f \in \mathcal{P}_n$ has no zeros in $|z| < k$, then for $|z| = 1$,

$$|f(\eta z)| \geq \left(\frac{k+\eta}{k+1}\right)^n |f(z)|, \quad k \geq 1 \quad and \quad 0 \leq \eta \leq 1, \tag{1.4}$$

and

$$|f(\eta z)| \geq \left(\frac{k+\eta}{k+1}\right)^n |f(z)|, \quad k \leq 1 \quad and \quad 0 < \eta \leq k^2. \tag{1.5}$$

The result is sharp and equality holds for $f(z) = (z+k)^n$.

Recently Kumar and Milovanovic [3] sharpened the inequalities (1.3), (1.4) and (1.5) by involving some of the coefficients of underlying polynomial and obtained the following result:

Theorem 1.3. If $f(z) = \sum\limits_{j=1}^{n} \alpha_j z^j$ is a polynomial of degree n having no zeros in $|z| < k$, then for $|z| = 1$,

$$|f(\eta z)| \geq \left\{ \left(\frac{k+\eta}{k+1}\right)^n + \frac{1}{k^{n-1}} \left[\frac{|\alpha_0| - |\alpha_n| k^n}{|\alpha_0| + |\alpha_n|}\right] \left(\frac{1-\eta}{k+1}\right)^n \right\} |f(z)|, \quad k \geq 1 \quad and \quad 0 \leq \eta \leq 1, \tag{1.6}$$

and

$$|f(\eta z)| \geq \left\{ \left(\frac{k+\eta}{k+1}\right)^n + \left[\frac{|\alpha_0| - |\alpha_n| k^n}{|\alpha_0| + |\alpha_n|}\right] \left(\frac{\rho}{k+1}\right)^n \right\} |f(z)|, \quad k \leq 1, \quad 0 < \eta \leq k^2 \quad and \tag{1.7}$$

$\rho = \min\{1-\eta, k+\eta\}$. The result is sharp and equality holds for $f(z) = (z+k)^n$ and also for $f(z) = z + \gamma$ for any γ with $|\gamma| \geq k$.

For $\beta_j \in \mathbb{C}$, j = 1, 2, … , n, we define

$$w(z) := \prod_{j=1}^{n} (z - \beta_j), \qquad B(z) := \prod_{j=1}^{n} \left(\frac{1 - \overline{\beta_j} z}{z - \beta_j}\right)$$

and

$$\mathcal{R}_n := \mathcal{R}_n(\beta_1, \beta_2, \ldots, \beta_n) = \left\{ \frac{f(z)}{w(z)}; f \in \mathcal{P}_n \right\}.$$

Then $\mathcal{R}_n$ is the set of all rational functions with poles $\beta_j, j = 1, 2, \ldots, n$ at most and with finite limit at infinity. It is clear that $B(z) \in \mathcal{R}_n$ and $|B(z)| = 1$ for $|z| = 1$. Throughout this paper, we shall assume that all the poles β_j, j = 1, 2, … , n lie in $|z| > 1$.

The problem concerning estimation of the inequalities for the rational functions has been evolved subsequently over the last many years. Li, Mohapatra and Rodriguez [4] were the first mathematicians who obtained Bernstein-type inequalities for rational functions. For the latest publications concerning to the growth estimates for the rational functions, one can refer the papers [2], [7] and [12]. Very recently N. A. Rather et al. [8] extended the inequalities (1.3), (1.4) and (1.5) to the rational functions and they proved the following result:

Theorem 1.4. Let $r \in \mathcal{R}_n$ have no zeros in $|z| < k$, then for $|z| = 1$,

$$|r(\eta z)| \geq \left(\frac{\eta+k}{1+k}\right)^n \prod_{j=1}^{n} \left(\frac{|\beta_j| - 1}{|\beta_j| + \eta}\right) |r(z)|, \quad k \geq 1 \quad and \quad 0 \leq \eta \leq 1, \tag{1.8}$$

and

$$|r(\eta z)| \geq \left(\frac{\eta+k}{1+k}\right)^n \prod_{j=1}^{n} \left(\frac{|\beta_j| - 1}{|\beta_j| + \eta}\right) |r(z)|, \quad k \leq 1 \quad and \quad 0 < \eta \leq k^2. \tag{1.9}$$

2 Main results

In this section, we establish some results concerning to the rational functions of the type $r(z) = f(z)/w(z)$, where $f(z) = \sum_{j=1}^{n} \alpha_j z^j$ and $w(z) = \prod_{j=1}^{n}(z - \beta_j)$, $|\beta_j| > 1$ for $1 \le j \le n$ by involving some coefficients of $f(z)$. The obtained results bring forth extensions of inequalities (1.6) and (1.7) to the rational functions with prescribed poles and also sharpen the inequalities (1.8) and (1.9). We begin by presenting the following result:

Theorem 2.1. Let $r \in \mathcal{R}_n$ have no zeros in $|z| < k$, $k \ge 1$, then for $0 \le \eta \le 1$ and $|z| = 1$,

$$|r(\eta z)| \ge \left\{ \left(\frac{k+\eta}{k+1}\right)^n + \frac{1}{k^{n-1}} \left[\frac{|\alpha_0| - |\alpha_n|k^n}{|\alpha_0| + |\alpha_n|}\right] \left(\frac{1-\eta}{k+1}\right)^n \right\} \prod_{j=1}^{n} \left(\frac{|\beta_j| - 1}{|\beta_j| + \eta}\right) |r(z)|. \tag{2.1}$$

Remark 2.2. Since $r(z) = f(z)/w(z)$, where $f(z) = \sum_{j=1}^{n} \alpha_j z^j$ has all its zeros in $|z| \ge k$, $k \ge 1$, we always have the situation

$$\frac{|\alpha_0| - |\alpha_n|k^n}{|\alpha_0| + |\alpha_n|} \ge 0.$$

Therefore, for all rational functions satisfying the hypothesis of Theorem 2.1 excepting those satisfying $|\alpha_0| = |\alpha_n|k^n$, our above inequality (2.1) sharpens the inequality (1.8).

Remark 2.3. Take $w(z) = (z-\beta)^n, |\beta| > 1$ in Theorem 2.1. Then inequality (2.1) reduces to the following inequality

$$|f(\eta z)| \ge \left\{ \left(\frac{k+\eta}{k+1}\right)^n + \frac{1}{k^{n-1}} \left[\frac{|\alpha_0| - |\alpha_n|k^n}{|\alpha_0| + |\alpha_n|}\right] \left(\frac{1-\eta}{k+1}\right)^n \right\} \left(\frac{|\beta| - 1}{|\beta| + \eta}\right)^n \left|\frac{\eta z - \beta}{z - \beta}\right|^n |f(z)|. \tag{2.2}$$

Letting $|\beta| \to \infty$ in inequality (2.2), we immediately get inequality (1.6).

Theorem 2.4. Let $r \in \mathcal{R}_n$ have no zeros in $|z| < k$, $k \le 1$, then for $|z| = 1$,

$$|r(\eta z)| \ge \left\{ \left(\frac{k+\eta}{k+1}\right)^n + \left[\frac{|\alpha_0| - |\alpha_n|k^n}{|\alpha_0| + |\alpha_n|}\right] \left(\frac{\rho}{k+1}\right)^n \right\} \prod_{j=1}^{n} \left(\frac{|\beta_j| - 1}{|\beta_j| + \eta}\right) |r(z)|, \tag{2.3}$$

whenever $0 < \eta \le k^2$ and $\rho = \min\{1 - \eta, k + \eta\}$.

Remark 2.5. As before, it can be easily seen that inequality (2.3) sharpens the inequality (1.9).

Remark 2.6. Take $w(z) = (z-\beta)^n, |\beta| > 1$ in Theorem 2.4. Then inequality (2.3) reduces to the following inequality

$$|f(\eta z)| \ge \left\{ \left(\frac{k+\eta}{k+1}\right)^n + \left[\frac{|\alpha_0| - |\alpha_n|k^n}{|\alpha_0| + |\alpha_n|}\right] \left(\frac{\rho}{k+1}\right)^n \right\} \left(\frac{|\beta| - 1}{|\beta| + \eta}\right)^n \left|\frac{\eta z - \beta}{z - \beta}\right|^n |f(z)|. \tag{2.4}$$

Letting $|\beta| \to \infty$ in inequality (2.4), we immediately obtain inequality (1.7).

3 Preliminaries

In order to establish our results stated above, we need the following two lemmas due to Kumar and Milovanovic [3] .

Lemma 3.1. For any $0 \le \eta \le 1$ and $\eta_j \ge k \ge 1$, $1 \le j \le n$, we have

$$\prod_{j=1}^{n} \frac{\eta + \eta_j}{1 + \eta_j} \ge \left(\frac{k+\eta}{k+1}\right)^n + \frac{1}{k^{n-1}} \left[\frac{\eta_1\eta_2\ldots\eta_n - k^n}{\eta_1\eta_2\ldots\eta_n + 1}\right] \left(\frac{1-\eta}{k+1}\right)^n.$$

Lemma 3.2. For any $0 \le \eta \le 1$ and $\eta_j \ge k$, $1 \le j \le n$, $k > 0$ we have

$$\prod_{j=1}^{n} \frac{\eta + \eta_j}{1 + \eta_j} \ge \left(\frac{k+\eta}{k+1}\right)^n + \left[\frac{\eta_1\eta_2\ldots\eta_n - k^n}{\eta_1\eta_2\ldots\eta_n + k^n}\right] \left(\frac{\rho}{k+1}\right)^n,$$

where $\rho = \min\{1 - \eta, k + \eta\}$.

4 Proofs of the theorems

Proof . [Proof of Theorem 2.1] By assumption $r \in \mathcal{R}_n$ with no zero in $|z| < k$, $k \geq 1$, we have $r(z) = \frac{f(z)}{w(z)}$, where $f(z) = c\prod_{j=1}^{n}(z - z_j)$, $|z_j| \geq k$, $k \geq 1$ and $w(z) = \prod_{j=1}^{n}(z - \beta_j)$, $|\beta_j| > 1$. Since all the zeros of $f(z)$ lie in $|z| \geq k$, $k \geq 1$, we write $f(z) = c\prod_{j=1}^{n}(z - \eta_j e^{i\theta_j})$, where $\eta_j \geq k \geq 1$, $j = 1, 2, \dots, n$. Hence, for $0 \leq \eta \leq 1$ and $0 \leq \theta < 2\pi$, we have

$$\begin{aligned}\left|\frac{r(\eta e^{i\theta})}{r(e^{i\theta})}\right| &= \left|\frac{f(\eta e^{i\theta})}{w(\eta e^{i\theta})}\right| \Big/ \left|\frac{f(e^{i\theta})}{w(e^{i\theta})}\right| \\ &= \left|\frac{f(\eta e^{i\theta})}{f(e^{i\theta})}\right| \left|\frac{w(e^{i\theta})}{w(\eta e^{i\theta})}\right| \\ &= \prod_{j=1}^{n}\left|\frac{\eta e^{i\theta} - \eta_j e^{i\theta_j}}{e^{i\theta} - \eta_j e^{i\theta_j}}\right| \prod_{j=1}^{n}\left|\frac{e^{i\theta} - \beta_j}{\eta e^{i\theta} - \beta_j}\right|. \end{aligned} \tag{4.1}$$

Now,

$$\begin{aligned}\prod_{j=1}^{n}\left|\frac{\eta e^{i\theta} - \eta_j e^{i\theta_j}}{e^{i\theta} - \eta_j e^{i\theta_j}}\right| &= \prod_{j=1}^{n}\left|\frac{\eta e^{i(\theta-\theta_j)} - \eta_j}{e^{i(\theta-\theta_j)} - \eta_j}\right| \\ &= \prod_{j=1}^{n}\left(\frac{\eta^2 + \eta_j^2 - 2\eta\eta_j\cos(\theta - \theta_j)}{1 + \eta_j^2 - 2\eta_j\cos(\theta - \theta_j)}\right)^{1/2} \\ &\geq \prod_{j=1}^{n}\frac{\eta + \eta_j}{1 + \eta_j}.\end{aligned}$$

Thus we have,

$$\prod_{j=1}^{n}\left|\frac{\eta e^{i\theta} - \eta_j e^{i\theta_j}}{e^{i\theta} - \eta_j e^{i\theta_j}}\right| \geq \prod_{j=1}^{n}\frac{\eta + \eta_j}{1 + \eta_j}. \tag{4.2}$$

Now employing Lemma 3.1 to the right hand side of the inequality (4.2) and using the fact that

$$\eta_1\eta_2\dots\eta_n = \frac{|\alpha_0|}{|\alpha_n|},$$

we get

$$\prod_{j=1}^{n}\left|\frac{\eta e^{i\theta} - \eta_j e^{i\theta_j}}{e^{i\theta} - \eta_j e^{i\theta_j}}\right| \geq \left(\frac{k+\eta}{k+1}\right)^n + \frac{1}{k^{n-1}}\left[\frac{|\alpha_0| - |\alpha_n|k^n}{|\alpha_0| + |\alpha_n|}\right]\left(\frac{1-\eta}{k+1}\right)^n. \tag{4.3}$$

Also for $|\beta_j| > 1$, $j = 1, 2, \dots, n$, we have

$$\prod_{j=1}^{n}\left|\frac{e^{i\theta} - \beta_j}{\eta e^{i\theta} - \beta_j}\right| \geq \prod_{j=1}^{n}\frac{|\beta_j| - 1}{|\beta_j| + \eta}. \tag{4.4}$$

Using inequalities (4.3) and (4.4) in equation (4.1), we obtain for $0 \leq \theta < 2\pi$,

$$\left|\frac{r(\eta e^{i\theta})}{r(e^{i\theta})}\right| \geq \left\{\left(\frac{k+\eta}{k+1}\right)^n + \frac{1}{k^{n-1}}\left[\frac{|\alpha_0| - |\alpha_n|k^n}{|\alpha_0| + |\alpha_n|}\right]\left(\frac{1-\eta}{k+1}\right)^n\right\}\prod_{j=1}^{n}\left(\frac{|\beta_j| - 1}{|\beta_j| + \eta}\right).$$

That is, for $|z| = 1$ and $0 \leq \eta \leq 1$, we have

$$|r(\eta z)| \geq \left\{\left(\frac{k+\eta}{k+1}\right)^n + \frac{1}{k^{n-1}}\left[\frac{|\alpha_0| - |\alpha_n|k^n}{|\alpha_0| + |\alpha_n|}\right]\left(\frac{1-\eta}{k+1}\right)^n\right\}\prod_{j=1}^{n}\left(\frac{|\beta_j| - 1}{|\beta_j| + \eta}\right)|r(z)|.$$

This completes the proof of Theorem 2.1. $\square$

Proof .[Proof of Theorem 2.4] By hypothesis $r \in \mathcal{R}_n$ with no zero in $|z| < k$, $k \leq 1$, we have $r(z) = \frac{f(z)}{w(z)}$, where $f(z) = c\prod_{j=1}^{n}(z - z_j)$, $|z_j| \geq k$, $k \leq 1$ and $w(z) = \prod_{j=1}^{n}(z - \beta_j)$, $|\beta_j| > 1$. Since all the zeros of $f(z)$ lie in $|z| \geq k$, $k \geq 1$, we write $f(z) = c\prod_{j=1}^{n}(z - \eta_j e^{i\theta_j})$, where $\eta_j \geq k$, $k \leq 1$, $j = 1, 2, \ldots, n$. Hence for $0 < \eta \leq k^2$ and $0 \leq \theta < 2\pi$, we have

$$\left|\frac{r(\eta e^{i\theta})}{r(e^{i\theta})}\right| = \prod_{j=1}^{n}\left|\frac{\eta e^{i\theta} - \eta_j e^{i\theta_j}}{e^{i\theta} - \eta_j e^{i\theta_j}}\right| \prod_{j=1}^{n}\left|\frac{e^{i\theta} - \beta_j}{\eta e^{i\theta} - \beta_j}\right|. \tag{4.5}$$

Now,

$$\begin{aligned}
\prod_{j=1}^{n}\left|\frac{\eta e^{i\theta} - \eta_j e^{i\theta_j}}{e^{i\theta} - \eta_j e^{i\theta_j}}\right| &= \prod_{j=1}^{n}\left|\frac{\eta e^{i(\theta-\theta_j)} - \eta_j}{e^{i(\theta-\theta_j)} - \eta_j}\right| \\
&= \prod_{j=1}^{n}\left(\frac{\eta^2 + \eta_j^2 - 2\eta\eta_j\cos(\theta - \theta_j)}{1 + \eta_j^2 - 2\eta_j\cos(\theta - \theta_j)}\right)^{1/2} \\
&\geq \prod_{j=1}^{n}\frac{\eta + \eta_j}{1 + \eta_j}.
\end{aligned}$$

Therefore, we have

$$\prod_{j=1}^{n}\left|\frac{\eta e^{i\theta} - \eta_j e^{i\theta_j}}{e^{i\theta} - \eta_j e^{i\theta_j}}\right| \geq \prod_{j=1}^{n}\frac{\eta + \eta_j}{1 + \eta_j}. \tag{4.6}$$

Now applying Lemma 3.2 to the right hand side of the inequality (4.6) and using the fact that

$$\eta_1\eta_2\ldots\eta_n = \frac{|\alpha_0|}{|\alpha_n|},$$

we get

$$\prod_{j=1}^{n}\left|\frac{\eta e^{i\theta} - \eta_j e^{i\theta_j}}{e^{i\theta} - \eta_j e^{i\theta_j}}\right| \geq \left(\frac{k+\eta}{k+1}\right)^n \left[\frac{\eta_1\eta_2\ldots\eta_n - k^n}{\eta_1\eta_2\ldots\eta_n + k^n}\right]\left(\frac{\rho}{k+1}\right)^n, \tag{4.7}$$

where $\rho = \min\{1 - \eta, k + \eta\}$. Again as before, for $|\beta_j| > 1$, $j = 1, 2, \ldots, n$, we have

$$\prod_{j=1}^{n}\left|\frac{e^{i\theta} - \beta_j}{\eta e^{i\theta} - \beta_j}\right| \geq \prod_{j=1}^{n}\frac{|\beta_j| - 1}{|\beta_j| + \eta}. \tag{4.8}$$

Using inequalities (4.7) and (4.8) in equation (4.5), we have for $|z| = 1$ and $0 < \eta \leq k^2$,

$$|r(\eta z)| \geq \left\{\left(\frac{k+\eta}{k+1}\right)^n + \left[\frac{|\alpha_0| - |\alpha_n|k^n}{|\alpha_0| + |\alpha_n|}\right]\left(\frac{\rho}{k+1}\right)^n\right\}\prod_{j=1}^{n}\left(\frac{|\beta_j| - 1}{|\beta_j| + \eta}\right)|r(z)|,$$

where $\rho = \min\{1 - \eta, k + \eta\}$. This completes the proof. $\square$

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