Int. J. Nonlinear Anal. Appl. 14 (2023) 6, 23–29
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.6849

Some results of domination on the discrete topological graph with its inverse

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(Communicated by Sirous Moradi)

Abstract

Let $G_\tau = (V,\ E)$ be a topological graph which is a finite, simple, undirected, connected graph without isolated vertices. In this paper, several bounds and domination parameters are studied and applied to it: bi-domination, doubly connected bi-domination and pitchfork domination. The dominating set and domination number with its inverse for all these types are calculated. Also, some figures from the topological graph are introduced.

Keywords: Topological graph, discrete topology, dominating set, domination number
2020 MSC: 05C69

1 Introduction

Let $G = (V,\ E)$ be a graph where the set of vertices of G is $V(G)$ and the set of edges of G is $E(G)$. The vertex u is adjacent to a vertex v if there is an edge between them. The order of a graph G is the number of all elements in $V(G)$, denoted by $|V(G)|$. The size of a graph G is the number of all elements in $E(G)$. The subgraph H of G is induced subgraph denoted by $G[H]$ and constructed by all vertices of $H \subseteq V(G)$ and all edges between vertices of H. A graph G is connected graph if every two vertices are joined by a path, see [32]. The subset D is dominating set if for each vertex of $V-D$ is adjacent to one or more vertices of D. The domination number denoted by $\gamma(G)$ is the cardinality of the minimum dominating set [18]. The inverse dominating set in a graph G is a minimum dominating set exist in the set $V-D$, denoted by D^{-1}. The inverse domination number denoted by $\gamma^{-1}(G)$ is the cardinality of the minimum inverse dominating set [29]. The subset D is called bi-dominating set if every vertex in D is adjacent to exactly two vertices in $V-D$. The bi-domination number denoted by $\gamma_{bi}(G)$ [16]. The subset D is a doubly connected bi-dominating set if D is bi-dominating set and both $G[D]$ and $G[V-D]$ are connected. The doubly connected bi-domination number denoted by $\gamma_{bi}^{cc}(G)$ [2]. The subset D is a pitchfork dominating set if every vertex in D dominates at least $j=1$ and at most $k=2$ vertices of $V-D$. The pitchfork domination number denoted by $\gamma_{pf}(G)$ [1]. For more information about domination see [1]-[15], [17, 30, 31]. The discrete topology is denoted by $(X,\ \tau)$ such that X is a non-empty set and τ is a family of all subsets of X, where $\tau = P(X)$ [33]. There are many papers to linking the graph to topology, see [19]-[28]. In this paper, some types of domination are studied on the discrete topological graph and calculate the inverse domination for it.

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Received: March 2022 *Accepted:* May 2022

2 Main Results

In this section, the definition that form a topological graphs is written with different properties and theorems of this graphs are studied.

Definition 2.1. [26] Let X be a non-empty set and τ be a discrete topology on X. The discrete topological graph denoted by $G_\tau = (V, E)$ is a graph of the vertex set $V = \{A; A \in \tau \text{ and } A \neq \emptyset, X\}$, and the edge set $E = \{A\ B; A \nsubseteq B \text{ and } B \nsubseteq A\}$.

Proposition 2.2. [26] Let X be a non-empty set of order n and let τ be a discrete topology on X. If $n = 2$, then $G_\tau \cong K_2$.

Proposition 2.3. [26] Let X be a non-empty set of order n and let τ be a discrete topology on X. If $n = 3$, then $G_\tau \cong \overline{C_6}$.

Proposition 2.4. [26] Let $|X| = n$ and G_τ be a discrete topological graph. Then, the graph G_τ has $n - 1$ complete induced subgraphs K_t such that $t \geq n$.

Theorem 2.5. Let G_τ be a discrete topological graph of a non-empty set X. Then, G_τ is a connected graph.

Proof . Assume that u_1 and u_2 are any two vertices in a graph G_τ, let S be a set of all vertices of singleton element. Then, there are three cases as follows:

Case 1: If $u_1,\ u_2 \in S$, since $G[S] = K_n$ from proof of Proposition 2.4. Then, u_1 adjacent to u_2 for all elements of S. So, there is an edge $u_1\ u_2 \in E(G_\tau)$ in a graph G_τ.

Case 2: If $u_1 \in S$ and $u_2 \notin S$, if $u_1 \nsubseteq u_2 \ \wedge\ u_2 \nsubseteq u_1$ then $u_1\ u_2 \in E(G_\tau)$. If u_1 not adjacent to u_2. Then, there is at least one vertex in S say v adjacent to u_2 such that $v \nsubseteq u_2$ and $u_2 \nsubseteq v$. Since v adjacent with u_1 from proof of Proposition 2.4, so that v adjacent to u_1 and u_2. Thus, $u_1 - v - u_2$ is a path in a graph G_τ.

Case 3: If $u_1,\ u_2 \notin S$ and u_1 not adjacent to u_2. If there is a vertex $t \in S$ such that $u_1 \nsubseteq t \ \wedge\ t \nsubseteq u_1$, also $u_2 \nsubseteq t \ \wedge\ t \nsubseteq u_2$. Then, $u_1\ t \in E(G_\tau)$ and $u_2\ t \in E(G_\tau)$ and $u_1 - t - u_2$ is a path in G_τ. Otherwise, there is $t_1,\ t_2 \in S$ where $u_1\ t_1 \in E(G_\tau)$ and $t_2\ u_2 \in E(G_\tau)$, then $u_1 - t_1 - t_2 - u_2$ is a path in G_τ. Hence, G_τ is a connected graph. □

Proposition 2.6. [26] Let $|X| = n$, then the order of discrete topological graph G_τ is $2^n - 2$.

Corollary 2.7. [28] Let $|X| = n$, then the order of the topological graph G_τ is $\sum_{i=1}^{n-1} \binom{n}{i}$.

3 Domination on the Topological Graph

In this section, many results of domination are found on the discrete topological graph.

Observation 3.1. Let G_τ be a discrete topological graph of order $2^n - 2$ has a bi-dominating set. If $\gamma_{bi}(G_\tau) > \frac{2^n - 2}{2}$, thus it has no inverse bi-dominating set.

Observation 3.2. For any topological graph G_τ of order $2^n - 2$ has a pitchfork domination. If $\gamma_{pf}(G_\tau) > \frac{2^n - 2}{2}$, then G_τ has no inverse pitchfork domination.

Proposition 3.3. [28] Let $|X| = 3$ and G_τ be a discrete topological graph. Then, G_τ has a bi-dominating set and $\gamma_{bi}(G_\tau) = 2$.

Theorem 3.4. [28] Let $|X| = n\ (n \geq 4)$ and G_τ be a discrete topological graph. Then, G_τ has bi-dominating set and $\gamma_{bi}(G_\tau) = \sum_{i=1}^{n-1} \binom{n}{i} - 4$.

Proposition 3.5. [28] Let $|X| = n\ (n \geq 4)$ and G_τ be a discrete topological graph. Then, G_τ has no inverse bi-dominating set.

Proposition 3.6. Let $|X| = 3$, then G_τ has a doubly connected bi-dominating set and $\gamma_{bi}^{cc}(G_\tau) = 2$.

Proof . If $|X| = 2$, then $G_\tau \cong K_2$ by Proposition 2.2, and it is clear K_2 has no bi-dominating set, also it has no doubly connected bi-dominating set. If $|X| = 3$ by the same technique of proof of Proposition 3.3. Let $D = \{u, u^c\}$ such that this two vertices of D dominate only two vertices of $V - D$ and it is bi-dominating set. Now, if we take $D = \{\{1\}, \{2, 3\}\}$ since $\{1\} \nsubseteq \{2, 3\} \bigwedge \{2, 3\} \nsubseteq \{1\}$. Then, there is an edge between them so $G[D]$ form a path and it is connected. Let $V - D = \{\{2\}, \{3\}, \{1, 2\}, \{1, 3\}\}$ since $\{2\}$ adjacent with $\{3\}$ and $\{1, 2\}$ adjacent with $\{1, 3\}$ from proof of Proposition 2.3. Also, since $\{3\} \nsubseteq \{1, 2\} \bigwedge \{1, 2\} \nsubseteq \{3\}$ so there is an edge between them. Again, since $\{2\} \nsubseteq \{1, 3\} \bigwedge \{1, 3\} \nsubseteq \{2\}$ also there is an edge between them. Now, since $\{2\}$ adjacent to $\{3\}$, $\{3\}$ adjacent to $\{1, 2\}$, $\{1, 2\}$ adjacent to $\{1,3\}$ and $\{1,3\}$ adjacent to $\{2\}$. Hence, $G[V - D]$ form a cycle so that it is connected. Since both $G[D]$ and $G[V - D]$ are connected. Hence, D is a doubly connected bi-dominating set and $\gamma_{bi}^{cc}(G_\tau) = 2$. See Figure 1 (a). □

Proposition 3.7. Let $|X| = 3$, then G_τ has inverse doubly connected bi-dominating set and $\gamma_{bi}^{-cc}(G_\tau) = 2$.

Proof . By the same technique of proof of Proposition 3.6. Let $D^{-1} = \{\{2\}, \{1, 3\}\}$ such that $G\left[D^{-1}\right]$ form a path so it is connected. Also, let $V - D^{-1} = \{\{1\}, \{3\}, \{1, 2\}, \{2, 3\}\}$ where $G\left[V - D^{-1}\right]$ form a cycle and it is connected. Since both $G\left[D^{-1}\right]$ and $G\left[V - D^{-1}\right]$ are connected. Thus, D^{-1} is an inverse doubly connected bi-dominating set and $\gamma_{bi}^{-cc}(G_\tau) = 2$. See Figure 1 (b). □

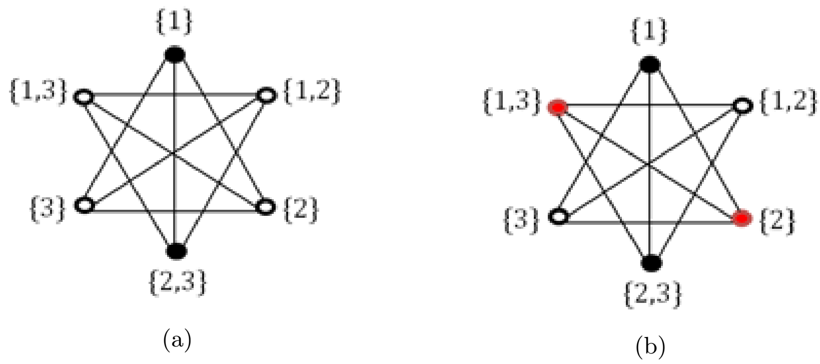

Figure 1: D and D^{-1} of doubly connected bi-domination for $\overline{C_6}$.

Theorem 3.8. Let $|X| = n$ $(n \geq 4)$, then G_τ has a doubly connected bi-dominating set and $\gamma_{bi}^{cc}(G_\tau) = \sum_{i=1}^{n-1}\binom{n}{i} - 4$.

Proof . By the same technique of proof of Theorem 3.4. Let $V - D = \{w, v, w^c, v^c\}$ where each vertex of D dominates only two vertices of $V - D$, and it is a bi-dominating set. Now, in $G[D]$ and in similar proof of Theorem 2.5 we get it is connected. The remaining vertices in $V - D = \{w, v, w^c, v^c\}$. Such that let w, v be two vertices of singleton element then w^c, v^c are two vertices have $n - 1$ elements. Since $w\ v \in E(G_\tau)$ and $w^c\ v^c \in E(G_\tau)$ from proof of Proposition 2.4. Also, since $w \nsubseteq w^c \bigwedge w^c \nsubseteq w$, so w adjacent with w^c. Again, since $v \nsubseteq v^c \bigwedge v^c \nsubseteq v$, thus v adjacent with v^c. Now, since w adjacent to v, v adjacent to v^c, v^c adjacent to w^c and w^c adjacent to w. Then, $G[V - D]$ form a cycle and it is connected. Therefore, D is a doubly connected bi-dominating set and $\gamma_{bi}^{cc}(G_\tau) = \sum_{i=1}^{n-1}\binom{n}{i} - 4$. See Figure 2. □

Corollary 3.9. Let $|X| = n$ $(n \geq 4)$ and G_τ be a discrete topological graph defined on a set X. Then, G_τ has a doubly connected bi-dominating set and $\gamma_{bi}^{cc}(G_\tau) = 2^n - 6$.

Proof . From proof of Theorem 3.8. Since D is a doubly connected bi-dominating set and has all vertices of G_τ unless four vertices of $V - D$. In addition the order of G_τ which is $2^n - 2$ by Proposition 2.6. Hence, $\gamma_{bi}^{cc}(G_\tau) = 2^n - 6$. □

Proposition 3.10. Let $|X| = n$ $(n \geq 4)$, then G_τ has no inverse doubly connected bi-dominating set.

Proof . Since G_τ has no inverse bi-dominating set for $n \geq 4$ by Proposition 3.5, G_τ has no inverse doubly connected bi-dominating set. □

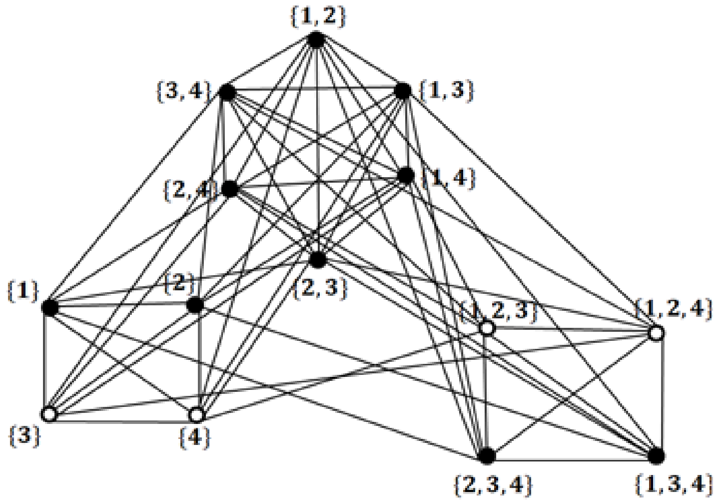

Figure 2: The minimum doubly connected bi-domination when $|X| = 4$.

Proposition 3.11. Let $|X| = n$, then G_τ has a pitchfork dominating set and

$$\gamma_{pf}(G_\tau) = \begin{cases} 1, & if\ n = 2 \\ 2, & if\ n = 3. \end{cases}$$

Proof . If $n = 2$, then $G_\tau \cong K_2$ by Proposition 2.2 and it is clear the pitchfork domination number of K_2 is one, where $\gamma_{pf}(G_\tau) = 1$. See Figure 3 (a). If $n = 3$ by the same technique of proof of Proposition 3.3, let $D = \{u, u^c\}$. Since each vertex in D dominates only two vertices in $V - D$. Thus, D is a minimum pitchfork dominating set and $\gamma_{pf}(G_\tau) = 2$. See Figure 1 (a). □

Proposition 3.12. Let $|X| = n$, then G_τ has inverse pitchfork dominating set and

$$\gamma_{pf}^{-1}(G_\tau) = \begin{cases} 1, & if\ n = 2 \\ 2, & if\ n = 3. \end{cases}$$

Proof . If $n = 2$, then $G_\tau \cong K_2$ by Proposition 2.2 and it is clear the inverse pitchfork domination number of K_2 is one, where $\gamma_{pf}^{-1}(G_\tau) = 1$. See Figure 3 (b). If $n = 3$ in similar proof of Proposition 3.3, let $D^{-1} = \{v, v^c\}$ such that the vertices of D^{-1} dominate only two vertices in $V - D^{-1}$. Thus, D^{-1} is a minimum inverse pitchfork dominating set and $\gamma_{pf}^{-1}(G_\tau) = 2$. See Figure 1 (b). □

Figure 3: D and D^{-1} of pitchfork domination for K_2.

Theorem 3.13. Let $|X| = n$ $(n \geq 4)$, then G_τ has pitchfork dominating set and $\gamma_{pf}(G_\tau) = \sum_{i=1}^{n-1}\binom{n}{i} - 4$.

Proof . By the same technique of proof of Theorem 3.4, let $V - D = \{u,\ w,\ u^c,\ w^c\}$. Since each vertex in D dominates only two vertices of $V - D$, D is a minimum pitchfork dominating set and $\gamma_{pf}(G_\tau) = \sum_{i=1}^{n-1}\binom{n}{i} - 4$. See Figure 2 and Figure 4. □

Corollary 3.14. Let $|X| = n$ $(n \geq 4)$ and G_τ be a discrete topological graph. Then, G_τ has a pitchfork dominating set where $\gamma_{pf}(G_\tau) = 2^n - 6$.

Proof . From proof of Theorem 3.13, since D is a pitchfork dominating set and has all vertices of G_τ unless four vertices of $V-D$ such that the order of G_τ which is 2^n-2 by Proposition 2.6, we have $\gamma_{pf}(G_\tau)=2^n-6$. □

Proposition 3.15. Let $|X|=n$ $(n\geq 4)$, then G_τ has no inverse pitchfork dominating set.

Proof . Since the order of G_τ is 2^n-2 by Proposition 2.6 and $\gamma_{pf}(G_\tau)>\frac{2^n-2}{2}$, by Observation 3.2 the graph G_τ has no inverse pitchfork dominating set. □

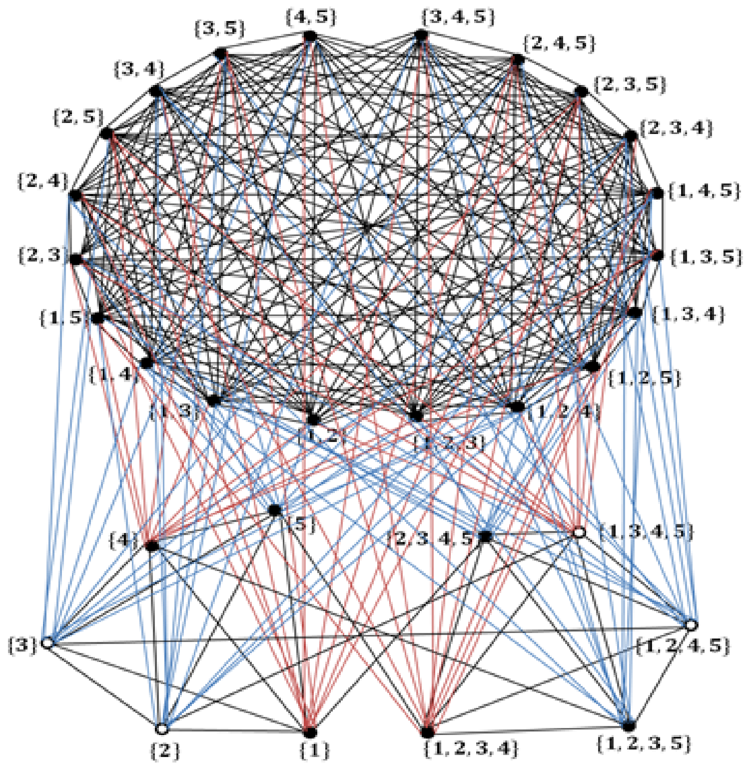

Figure 4: The pitchfork domination for $|X|=5$.

4 Conclusions

Many results of domination with it's inverse are applied on the topological graphs and introduced some figures for it.

5 Open problems

Applying other types of domination parameters on the topological graph such as: total pitchfork domination, arrow domination, Hn-domination, co-even domination.

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