Int. J. Nonlinear Anal. Appl. 14 (2023) 4, 387–390
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.21051.2225

Slowly oscillating functions on semitopological semigroup

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(Communicated by Madjid Eshaghi Gordji)

Abstract

The slowly oscillating functions have been used by I. V. Protasov to study of the algebraic structure of the Stone-Česh compactification. M. Filali and P. Salmi developed the concept of slowly oscillating functions to arbitrary locally compact topological groups. In this paper, we study the structure of Lmc-compactification of a semitopological semigroup by the slowly oscillating functions. In fact, we develop the concept of slowly oscillating functions to semitopological semigroups.

Keywords: Lmc-Compactification, The Stone-Čech Compactification, e-filter, Slowly oscillating functions
2020 MSC: Primary 22A20; Secondary 54D80

1 Introduction

The concept of slowly oscillating functions was introduced by Higson for metric spaces (see for example [8, p. 29]). Protasov used them to study the algebraic structure of the Stone−Čech compactification of a countable discrete group G (see [6]), and developed by Filali and Salmi to any locally compact group, see [4].

To study left ideals of βG for countable discrete group G, this notion was extended further in [3] to slowly oscillating functions in the direction of filters in the following way: Let $f : G \to \mathbb{C}$ be a function and let ϕ be a filter on G such that ϕ contains the complement of every finite subset of G. Then f is slowly oscillating in the direction of ϕ if for every $\epsilon > 0$ and for every finite subset F of G containing the identity of G, there exists $A \in \phi$ such that $diam f(Ft) < \epsilon$ for every $t \in A$. Here $diam X = sup\{|x-y| : x, y \in X\}$ for any $X \subseteq \mathbb{C}$. This includes both definitions given in [6] and [4]. In fact, the functions used in [6] are slowly oscillating in the direction of the filter of co-finite subsets of G, i.e., the sets A with $|G \setminus A| < \omega$, and those used in [4] are slowly oscillating in the direction of the filter of the sets A with $|G \setminus A| < |G|$.

A ball structure is a triplet $\mathbb{B} = (X, P, B)$, where X, P are nonempty sets and $B(x, \alpha)$ is a subset of X which is called a ball of radius α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$ and $\alpha \in P$. The set X is the support of $\mathbb{B}$. Given any $x \in X$, $A \subseteq X$, $\alpha \in P$, put

$$B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\} \text{ and } B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha).$$

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Received: August 2020 *Accepted:* June 2021

A ball structure is symmetric if for every $\alpha \in P$, there exists $\beta \in P$ such that $B(x, \alpha) \subseteq B^*(x, \beta)$ for every $x \in X$, and vice versa. A ball structure is multiplicative if for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that $B(B(x, \alpha), \beta) \subseteq B(x, \gamma)$ for every $x \in X$. A ball structure is connected if for any $x, y \in X$, there exists $\alpha \in P$ such that $y \in B(x, \alpha)$. We say that a filter φ on X is thick if for any $A \in \varphi$ and $\alpha \in P$ there exists $H \in \varphi$ such that, $B(H, \alpha) \subseteq A$.

Let S be a semigroup and $P_f(S)$ denote the collection of all non-empty finite subsets of S. Then the ball structure $\mathbb{B}_r(S) = (S, P_f(S), B_r)$, where $B_r(t, F) = Ft \cup \{t\}$ for each $F \in P_f(S)$ and $t \in S$, is always multiplicative.

Let S be a non-compact Hausdorff semitopological semigroup and let S_d denote semigroup S with discrete topology. The collection of all bounded complex valued continuous functions on S with uniform norm is denoted by $\mathcal{CB}(S)$, and $l^\infty(S_d)$ denotes the algebra of all complex valued bounded functions on S_d. Let $\mathcal{F}$ be a C^*-subalgebra of $\mathcal{CB}(S)$ containing the constant functions, then the set of all multiplicative means of $\mathcal{F}$ (the spectrum of $\mathcal{F}$) is denoted by $S^{\mathcal{F}}$. $S^{\mathcal{F}}$, equipped with the Gelfand topology, is a compact Hausdorff topological space. For each $s \in S$, the maps r_s and λ_s are defined by $r_s(t) = ts$ and $\lambda_s(t) = st$ for all $t \in S$. Let $R_s f = f \circ r_s \in \mathcal{F}$ and $L_s f = f \circ \lambda_s \in \mathcal{F}$ for all $s \in S$ and $f \in \mathcal{F}$, and the function $s \mapsto (T_\mu f(s)) = \mu(L_s f)$ be in $\mathcal{F}$ for all $f \in \mathcal{F}$ and $\mu \in S^{\mathcal{F}}$, then $S^{\mathcal{F}}$ under the multiplication $\mu\nu = \mu \circ T_\nu$ $(\mu, \nu \in S^{\mathcal{F}})$, furnished with the Gelfand topology, makes $(\varepsilon, S^{\mathcal{F}})$ a semigroup compactification (called the $\mathcal{F}$-compactification) of S, where $\varepsilon : S \to S^{\mathcal{F}}$ is the evaluation mapping. (For more details see [2]).

A function $f \in \mathcal{CB}(S)$ is left multiplicative continuous if and only if $\mathbf{T}_\mu f \in \mathcal{CB}(S)$ for all $\mu \in \beta S = S^{\mathcal{CB}(S)}$. The collection of all left multiplicative continuous functions on S is denoted by $Lmc(S)$. (ε, S^{Lmc}) is the universal semigroup compactification of S (Definition 4.5.1 and Theorem 4.5.2 in [2]).

Now we quote some prerequisite material from [9] for the description of e-filters and relative concepts with Lmc-compactification. For $f \in Lmc(S)$ and $\epsilon > 0$, we define $E_\epsilon(f) = \{x \in S : |f(x)| \leq \epsilon\}$. For $I \subseteq Lmc(S)$, we write $E(I) = \{E_\epsilon(f) : f \in I, \epsilon > 0\}$. Finally, for any family $\mathcal{A}$ of zero sets, we define

$$E^{\leftarrow}(\mathcal{A}) = \{f \in Lmc(S) : E_\epsilon(f) \in \mathcal{A} \text{ for each } \epsilon > 0\}.$$

Now let $\mathcal{A}$ be a $z-$filter, $\mathcal{A}$ is called an e-filter if and only if $E(E^{\leftarrow}(\mathcal{A})) = \mathcal{A}$. Hence, $\mathcal{A}$ is an e-filter if and only if, whenever $Z \in \mathcal{A}$, there exist $f \in Lmc(S)$ and $\epsilon > 0$, such that $Z = E_\epsilon(f)$ and $E_\delta(f) \in \mathcal{A}$ for every $\delta > 0$. For any $z-$filter $\mathcal{A}$, $E(E^{\leftarrow}(\mathcal{A}))$ is the largest e-filter contained in $\mathcal{A}$. A maximal e-filter is called an e-ultrafilter. Zorn's Lemma implies that every e-filter is contained in an e-ultrafilter, (see [9]).

The collection of all e-ultrafilters is denoted by $\mathcal{E}(S)$. The collection of all $A^\dagger = \{p \in \mathcal{E}(S) : A \in p\}$ for each $A \in Z(Lmc(S))$ is a basis for open topology of $\mathcal{E}(S)$. For each $a \in S$, $e(a) = \{E_\epsilon(f) : f(a) = 0, \epsilon > 0\}$ is an e-ultrafilter. Finally, S^{Lmc} and $\mathcal{E}(S)$ are topologically isomorphism. So for tow e-ultrafilters p and q in S^{Lmc}, $p + q = lim_\alpha lim_\beta e(x_\alpha)e(y_\beta)$, where $lim_\alpha e(x_\alpha) = p$ and $lim_\beta e(y_\beta) = q$. For more details see [9].

Let $\mathcal{A}$ be an e-filter, we define $\overline{\mathcal{A}} = \{p \in \mathcal{E}(S) : \mathcal{A} \subseteq p\}$. Then $\overline{\mathcal{A}}$ is a closed subset of $\mathcal{E}(S)$. Also, $A \in \mathcal{A}$ if and only if $\overline{\mathcal{A}} \subseteq A^\dagger$, see [9].

When S_d is a discrete semigroup, every e-ultrafilter is ultrafilter and above concepts coincide with the similar concepts of ultrafilters. The collection of all ultrafilters on S_d is denoted by βS_d and is called the Stone-Čech compactification of S_d. The collection $\{A^\dagger : A \subseteq S_d\}$ is a basis for topology on βS_d. For $p, q \in \beta S_d$, $p + q$ is an ultrafilter and $A \in p + q$ if and only if $\{x \in S_d : \lambda_x^{-1}(A) \in q\} \in p$. Also, if $\mathcal{A}$ and $\mathcal{B}$ are two filters, then

$$\mathcal{A} + \mathcal{B} = \{A \subseteq S_d : \{x \in S_d : x^{-1}A \in \mathcal{B}\} \in \mathcal{A}\},$$

is a filter. A family $\mathcal{A}$ is a filter base for a filter $\mathcal{U}$ on S_d if $\mathcal{A} \subseteq \mathcal{U}$ and for each $B \in \mathcal{U}$ there is some $A \in \mathcal{A}$ such that $A \subseteq B$. Let $\mathcal{A}$ be a family with finite intersection property on S_d, then $\overline{\mathcal{A}}$ the collection of all ultrafilters containing $\mathcal{A}$ is nonempty subset of βS_d, and $\bigcap \overline{\mathcal{A}}$ is the smallest filter containing $\mathcal{A}$.

The Stone-Čech compactification of S_d is $l^\infty(S_d)$-compactification of S_d. In fact, it is the universal compactification of S_d. For more details see [5] and [9].

2 Applications

Let S be a semitopological semigroup and let $P_f(S)$ denote the collection of all non-empty finite subset of S. For each $t \in S$ and each $F \in P_f(S)$ we define $B(t, F) = Ft \cup \{t\}$. Then $\mathbb{B}(S) = (S, P_f(S), B)$ is ball structure. $\mathbb{B}(S)$ is always multiplicative. An e-filter $\mathcal{A}$ on S is called thick if for each $A \in \mathcal{A}$ and for each $s \in S$, there exists $B \in \mathcal{A}$ such that $sB \subseteq A$. Clearly, if $\mathcal{A}$ is a thick e-filter on S, for $A \in \mathcal{A}$, and $F \in P_f(S)$ then there exists $B \in \mathcal{A}$ such that $FB \subseteq A$. Two e-ultrafilters $r, q \in S^{Lmc}$ are parallel $(r \parallel q)$ if there exists $F \in P_f(S)$ such that, for every $R \in r$, we have $FR \in q$.

Lemma 2.1. Let φ be a e-filter, then $\overline{\varphi}$ is a closed left ideal of S^{Lmc} if and only if φ is thick.

Proof . Let $\overline{\varphi}$ be a left ideal of S^{Lmc}. So $S\overline{\varphi} \subseteq \overline{\varphi}$. Pick $t \in S$ and $A \in \varphi$. Then for every $p \in \overline{\varphi}$, $tp \in \overline{\varphi}$. This implies that $A \in tp$, and so $B_p = t^{-1}A \in p$. Now $\{B_p^{\dagger} : p \in \overline{\varphi}\}$ is an open covering of $\overline{\varphi}$ of $\overline{\varphi}$. So there exist $p_1, \cdots, p_k$ in $\overline{\varphi}$ such that $\overline{\varphi} \subseteq \bigcup_{i=1}^{k} B_{p_i}^{\dagger}$. Now let $B = \bigcup_{i=1}^{k} B_{p_i}$. Then $\overline{\varphi} \subseteq \overline{B}$. This implies that for every $p \in \overline{\varphi}$, $B \in p$. Therefore $B \in \varphi$. Since $tB \subseteq A$ we conclude that φ is thick.

Now let φ be thick, pick $p \in \overline{\varphi}$ and $t \in S$. So for every $A \in \varphi$ and every $t \in S$, there exists $B \in \varphi$ such that $tB \subseteq A$. Now $B \subseteq t^{-1}A$ and $B \in p$. So $t^{-1}A \in p$ and hence $A \in tp$. Therefore $tp \in \overline{\varphi}$ for every $t \in S$ and every $p \in \overline{\varphi}$. This implies that $\overline{\varphi}$ is left ideal. □

Definition 2.2. (i) Let $\mathcal{A}$ be an e-filter on S. A function $f : S \to \mathbb{C}$ is slowly oscillating in the direction of $\mathcal{A}$ if for every $\varepsilon > 0$ and for every $F \in P_f(S)$, there exists $A \in \mathcal{A}$ such that $diamf(Ft \cup \{t\}) < \varepsilon$ for every $t \in A$.
(ii) Let $f : S \to \mathbb{C}$ be a function. For every $\varepsilon > 0$ and for every $F \in P_f(S)$, define

$$S(f, \varepsilon, F) = \{t \in S : diamf(Ft \cup \{t\}) < \varepsilon\}.$$

We say that f is very oscillating if there exists $\varepsilon > 0$ and $F \in P_f(S)$ such that $S(f, \varepsilon, F) = \emptyset$. If f is very oscillating on a non-empty proper subset of S, then f is called oscillating.

For the next definition, note that if $F_1, F_2 \in P_f(S)$ and $\varepsilon_1, \varepsilon_2 > 0$, then $S(f, \varepsilon, F) \subseteq S(f, \varepsilon_1, F_1) \cap S(f, \varepsilon_2, F_2)$ for any function $f : S \to \mathbb{C}$, where $F = F_1 \cup F_2$ and $\varepsilon = min\{\varepsilon_1, \varepsilon_2\}$.

Definition 2.3. Let $f : S \to \mathbb{C}$ be a function which is not very oscillating. Then the filter generated by

$$\{S(f, \varepsilon, F) : \varepsilon > 0, F \in P_f(S)\}$$

is denoted by $so(f)$.

Remark 2.4. (a) Let $f : S \to \mathbb{C}$ be a function which is not very oscillating. Then $tS(f, \varepsilon, Ft \cup \{t\}) \subseteq S(f, \varepsilon, F)$ for any $F \in P_f(S)$, $\varepsilon > 0$, and $t \in S$. Hence, $so(f)$ is thick.
(b) The filter $so(f)$ is smallest filter φ on S such that f is slowly oscillating in the direction of φ, and it is a unique filter on S such that f is slowly oscillating in the direction of φ.
(c) For every non-empty proper A in $so(f)$, there are $\varepsilon > 0$ and $F \in P_f(S)$ such that $diamf(Ft \cup \{t\}) \geq \varepsilon$ for every $t \in S \backslash A$. In fact f is oscillating on $S \backslash A$ for every $A \in \varphi \setminus \{S\}$.
(d) If φ is the smallest filter generated by an e-filter $\mathcal{A}$, we write $\varphi = \langle \mathcal{A} \rangle$.

Theorem 2.5. A function $f : X \to \mathbb{C}$ is slowly oscillating in direction of an e-filter $\mathcal{A}$ if and only if f is slowly oscillating in direction of every e-ultrafilter q containing $\mathcal{A}$.

Proof . If f is slowly oscillating in direction of $\mathcal{A}$, then f is slowly oscillating in direction of every e-filter $\mathcal{A}'$ containing $\mathcal{A}$.

Assume that f is slowly oscillating in direction of every e-ultrafilter $q \in \overline{\mathcal{A}}$. Fix $\epsilon > 0$ and $F \in P_f(S)$. For each $q \in \overline{\mathcal{A}}$, pick $A_q \in q$ such that $diamf(Fx \cup \{x\}) < \epsilon$ for each $x \in A_q$. Then we consider the open covering $\{A_q^{\dagger} : q \in \overline{\mathcal{A}}\}$ of the compact space $\overline{\mathcal{A}}$ and choose some finite subcovering $\{A_{q_1}, \cdots, A_{q_n}\}$. Then $A = A_{q_1} \cup \cdots \cup A_{q_n} \in \mathcal{A}$. Since $diamf(Fx \cup \{x\}) < \epsilon$ for each $x \in A_{q_1} \cup \cdots \cup A_{q_n} \in \mathcal{A}$, we see that f is slowly oscillating in direction of $\mathcal{A}$. □

Theorem 2.6. Let $\mathcal{A}$ be a thick e-filter and $q \in \overline{\mathcal{A}}$. If $r \| q$ for some e-ultrafilter r, then $r \in \overline{\mathcal{A}}$.

Proof . Let $r \notin \overline{\mathcal{A}}$. So there exists $A \in \mathcal{A}$ such that $A \cap R = \emptyset$ for some $R \in r$. Since $r \| q$ so there exists $F \in P_f(S)$ such that $FR \cup R = \bigcup_{t \in R} Ft \cup \{t\} \in q$, this implies that $FR \in q$. Also $\mathcal{A}$ is thick, so there exists $B \in \mathcal{A}$ such that $FB \subseteq A$. Since $FB \in q$, we have $FB \cap FR = F(B \cap R) = \emptyset$. It is a contradiction. □

Theorem 2.7. Let S be a semitopological semigroup. Suppose that S contains a point s such that λ_s has no fixed point in S. Let $\mathcal{A}$ be a thick e-filter with a countable basis on S. Then there exists a bounded function $f : S \to \mathbb{C}$ such that $so(f) = \langle \mathcal{A} \rangle$.

Proof . Since $\mathcal{A}$ is thick with a countable basis, so $\varphi = \langle \mathcal{A} \rangle$ is thick with a countable basis. Let $\langle A_n \rangle_{n=0}^{\infty}$ be a basis of φ such that $A_0 = S$ and $A_{n+1} \subseteq A_n$ for every $n \geq 0$. Since λ_s has no fixed point in S, by [1, Lemma 3.33], there exist a partition $S = T_1 \cup T_2 \cup T_3$ such that $sT_i \cap T_i = \emptyset$ for every $i \in \{1, 2, 3\}$. Now define $f(x) = \frac{\sqrt{3}}{2n}$ for $x \in T_1 \cap (A_n \setminus A_{n+1})$, $f(x) = \frac{1}{2n}$ for $x \in T_2 \cap (A_n \setminus A_{n+1})$ and $f(x) = \frac{-1}{2n}$ for $x \in T_3 \cap (A_n \setminus A_{n+1})$ for some $n > 0$.

First, f is slowly oscillating. Let $\epsilon > 0$ and let $F \in P_f(S)$. Pick $n \geq 0$ such that $\frac{1}{n} < \epsilon$. Since φ is thick, so there exists $A_m \in \varphi$ such that $FA_m \subseteq A_n$. For each $x \in A_m$, $Fx \cup \{x\} \subseteq A_n$. So $f(Fx \cup \{x\}) \subseteq f(A_n)$. Hence $diamf(Fx \cup \{x\}) \leq diamf(A_n)$ and

$$diam(f(Fx \cup \{x\}) \leq \frac{1}{n} < \epsilon.$$

Now we show that f is oscillating on $S \setminus A_{n+1}$ for each $n \in \mathbb{N}$. Let $\epsilon = \frac{1}{2n}$ and $F = \{s\}$, then for each $x \in S \setminus A_{n+1}$, there exists $m \in \{1, \cdots, n\}$ such that $x \in A_m \setminus A_{m+1}$, and so

$$\begin{aligned} diamf(B(x, \{s\})) &= diamf(\{sx, x\}) \\ &= |f(sx) - f(x)| \\ &\geq \frac{1}{2m} \\ &\geq \frac{1}{2n}. \end{aligned}$$

Now by Remark 2.4, $so(f) = \varphi$. □

Theorem 2.8. Let S be a semitopological semigroup and suppose that S contains a point s such that λ_s has no fixed point in S. Let $\mathcal{A}$ be an e-filter with a countable basis on S. Then the following statements are equivalent

(i) $\overline{\mathcal{A}}$ is a closed left ideal of S^{Lmc};

(ii) there exists a function $f : S \to \mathbb{C}$ such that $so(f) = \langle \mathcal{A} \rangle$.

Proof . $(i) \Rightarrow (ii)$. We show that $\mathcal{A}$ is thick. Fixed an arbitrary $A \in \mathcal{A}$, $x \in S$. Pick $p \in \overline{\mathcal{A}}$, so $e(x)p \in \overline{\mathcal{A}}$ for each $x \in S$. Therefore $A \in e(x)p$, and by Lemma 3.9 in [9], $\lambda_x^{-1}(A) \in p$. Now let $B = \lambda_x^{-1}(A)$. Hence $xB \subseteq A$. It is obvious that $B \in \mathcal{A}$, so $\mathcal{A}$ is thick, also see Lemma 2.1. Now we can apply Theorem 2.7.

$(ii) \Rightarrow (i)$. Since $so(f)$ is thick, and $so(f) = \langle \mathcal{A} \rangle$, and so $\mathcal{A}$ is thick. Now let $q \in \overline{\mathcal{A}}$, by Theorem 2.6, for each $r \in S^{Lmc}$ that $r \| q$, $r \in \overline{\mathcal{A}}$. So $e(x)q \in \overline{\mathcal{A}}$ for each $x \in S$. Since S^{Lmc} is right topological semigroup and S is dense in S^{Lmc}, we have $pq \in \overline{\mathcal{A}}$ for every $p \in S^{Lmc}$. Hence, $\overline{\mathcal{A}}$ is a closed left ideal of S^{Lmc}. □

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