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Fixed point results for fuzzy mappings in $\mathfrak{F}$-metric spaces

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Abstract

In the present research, we state and prove a common fixed point theorem for fuzzy mappings that satisfy the contractive condition in $\mathfrak{F}$-metric spaces. This theorem generalizes the corresponding results in [5].

Keywords: common fixed point, fuzzy mapping, $\mathfrak{F}$-metric space
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1 Introduction

Fixed point theory plays an important role in various fields of mathematics. It provides very important tools for finding the existence and uniqueness of solution to various mathematical models (integral partial equations and variational inequalities [9, 10, 11]). In 1965, Zadeh [7] initiated the concept of fuzzy set generalizing the concept of crisp set. After that in 1981, Heilpern [2] introduced the concept of fuzzy mapping and proved a fixed point theorem for fuzzy contraction mappings in metric linear space. His theorem is a fuzzy extension of Banach Contraction Principle (BCP). Afterward many authors explored and studied the fixed point for generalized fuzzy contractive mappings in different settings ([12]-[20]). Not long ago, Jleli and Samet [1] introduced the concepts of $\mathfrak{F}$-metric spaces, they generalized the Banach fixed point theorem. Many researchers have improved various outcomes in $\mathfrak{F}$-metric spaces. In 2020, Alansari, et al [5] introduced some fuzzy fixed point theorems in these spaces. Our aim, in this paper, is to establish common fixed point theorem for fuzzy mappings that satisfy the contractive condition in $\mathfrak{F}$-metric spaces. These theorems generalize some results in [5].

2 Preliminaries

In this section, we list the following definitions and lemmae that we will refer to them in our main results.

Definition 2.1. [1] Suppose that $\mathfrak{F}$ is the set of function $f : (0, +\infty) \longrightarrow \mathbb{R}$ satisfying the conditions as below:

$(\mathfrak{F}_1)$ f is nondecreasing, i.e., $0 < t < s \Longrightarrow f(t) \leq f(s)$.

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$(\mathfrak{F}_2)$ for every sequence $\{s_n\} \subset (0, +\infty)$, we get

$$\lim_{n\to+\infty} s_n = 0 \implies \lim_{n\to+\infty} f(s_n) = -\infty.$$

Definition 2.2. [1] Suppose that X is a non-empty set and $d_{\mathfrak{F}} : X \times X \longrightarrow (0, +\infty)$ is a given mapping. Assume that there is $(f, \rho) \in \mathfrak{F} \times [0, \infty)$ so it is

(d_1) $(\eta, \xi) \in X \times X$, $d_{\mathfrak{F}}(\eta, \xi) = 0 \iff \eta = \xi$.

(d_2) $d_{\mathfrak{F}}(\eta, \xi) = d_{\mathfrak{F}}(\xi, \eta), \quad \forall (\eta, \xi) \in X \times X$.

(d_3) for each $(\eta, \xi) \in X \times X$, for each $k \geq 2$ and for each sequence $\{\zeta_i\} \subset X$ with $(\zeta_1, \zeta_k) = (\eta, \xi)$, there is

$$d_{\mathfrak{F}}(\eta, \xi) > 0 \implies f(d_{\mathfrak{F}}(\eta, \xi)) \leq f(\sum_{i=1}^{k-1} d_{\mathfrak{F}}(\zeta_i, \zeta_{i-1})) + \rho.$$

Thus, $d_{\mathfrak{F}}$ is called an $\mathfrak{F}$-metric and the pair $(X, d_{\mathfrak{F}})$ is known as an $\mathfrak{F}$-metric space.

Definition 2.3. [1] Suppose $(X, d_{\mathfrak{F}})$ as an $\mathfrak{F}$-metric space and $\{\eta_n\}_{n\in\mathbb{N}}$ as a sequence in X.

(i) The sequence $\{\eta_n\}$ is $\mathfrak{F}$-convergent to η^* if and only if $\lim_{n\to\infty} d_{\mathfrak{F}}(\eta_n, \eta^*) = 0$.

(ii) The sequence $\{\eta_n\}$ is $\mathfrak{F}$-Cauchy, if and only if

$$\lim_{m,n\to\infty} d_{\mathfrak{F}}(\eta_m, \eta_n) = 0.$$

(iii) $(X, d_{\mathfrak{F}})$ is $\mathfrak{F}$-complete, if every $\mathfrak{F}$-Cauchy sequence in X is $\mathfrak{F}$-convergent to an element in X.

Definition 2.4. [6] The function $\phi : [0, \infty) \longrightarrow [0, \infty)$ is nondecreasing. It is considered as a comparison function, if $\phi^m(r) \to 0$ as $m \to \infty$ for each $r \in [0, \infty)$.

In [2, 7], an element in any fuzzy set has a degree of belonging, a membership function may be used in order to introduce the value of degree of belonging for any element to a set, the value of degree of belonging takes real values on the whole closed interval $[0, 1]$. The **membership function**

$$\mu_A : X \longrightarrow [0, 1].$$

Suppose that (X, d) is a metric linear space. In X a **fuzzy set** is a function $A : X \longrightarrow [0, 1]$. Thus, it is an element of I^X where $I = [0, 1]$. If A is a fuzzy set and $\eta \in X$, then the function value $A(\eta)$ is considered as the grade of membership of η in A. I^X is denoted to The collection of all fuzzy sets in X. The α**-level set of** A, defined by

$$A_\alpha = \{\eta : A(\eta) \geq \alpha\} \quad with \quad \alpha \in (0, 1] \quad \text{and} \quad A_0 = \overline{\{\eta : A(\eta) > 0\}},$$

whenever $\overline{\{\ \}}$ is the closure of set (non fuzzy) $\{\ \}$.

Definition 2.5. [3] Suppose that X is an arbitrary set and Y is a metric space. A mapping F is considered as a **fuzzy mapping** if F is a mapping from the set X into I^X, i.e., $F(\eta) \in I^X$ for each $\eta \in X$. F is fuzzy mapping which is a subset of $X \times Y$ with membership value $F(\eta)(\xi)$.

Definition 2.6. [5] Suppose that $(X, d_{\mathfrak{F}})$ is an $\mathfrak{F}$-metric space. A subset B of X is said to be proximal, if for each $\eta \in X$, there exists $b \in B$ such that

$$d_{\mathfrak{F}}(\eta, b) = d_{\mathfrak{F}}(\eta, B).$$

$\mathcal{P}^r(X)$ is the set of all non-empty bounded proximal sets in X and any proximal set is closed.

Lemma 2.7. [6] Assume that $\phi \in \Omega$. Then there are following properties discussed:

(i) Every iterate $\phi^i \; of \; \phi$, for $i \geq 1$ is a comparison,

(ii) ϕ is continuous at 0,

(iii) $\phi(r) < r \; \forall r > 0$. Where Ω is all comparison function's set.

3 Main Results

First, we rewrite the following lemma without proof.

Lemma 3.1. Suppose $(X, d_{\mathfrak{F}})$ is an $\mathfrak{F}$-metric space. Then any subsequence of $\mathfrak{F}$-convergent sequence in X is $\mathfrak{F}$-convergent.

Following Popa [4], let G be the member of all continuous mappings $g : [0, \infty)^5 \longrightarrow [0, \infty)$ satisfying the properties as below:

(g_1) g is non-decreasing in the 2^{nd}, 3^{rd}, 4^{th} and 5^{th} coordinate variables,

(g_2) there is $k \in (0, 1)$ so for each $\upsilon, \nu \in [0, \infty)$ and $\upsilon \leq g(\nu, \nu, \upsilon, \upsilon + \nu, 0)$ *implies* $\upsilon \leq k\nu$.

(g_3) if $\upsilon \in [0, \infty)$ such that $\upsilon \leq g(\upsilon, 0, 0, \upsilon, \upsilon)$ or $\upsilon \leq g(0, \upsilon, 0, 0, \upsilon)$, then $\upsilon = 0$. Thus, our main theorem can be stated and proved in the following way.

Theorem 3.2. Let $(X, d_{\mathfrak{F}})$ be an $\mathfrak{F}$-complete $\mathfrak{F}$-metric space and $T_1, T_2 : X \longrightarrow I^X$ be fuzzy mappings. Suppose that for each $\eta, \xi \in X$, there is $\alpha_{T_1}(\eta)$, $\beta_{T_2}(\xi) \in (0, 1]$ such that $\{T_1\eta\}_{\alpha_{T_1}(\eta)}, \{T_2\xi\}_{\beta_{T_2}(\xi)} \in \mathcal{P}^r(X)$ and if there is a $g \in G$ satisfies

$$H_{\mathfrak{F}}(T_1\eta, T_2\xi) \leq g(d_{\mathfrak{F}}(\eta, \xi), d_{\mathfrak{F}}(\eta, T_1\eta), d_{\mathfrak{F}}(\xi, T_2\xi), d_{\mathfrak{F}}(\eta, T_2\xi), d_{\mathfrak{F}}(\xi, T_1\eta)), \tag{3.1}$$

then there is $\eta^* \in X$ such that $\eta^* \in \{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)} \cap \{T_2\eta^*\}_{\beta_{T_2}(\eta^*)}$, where $H_{\mathfrak{F}}$ is the hausdorff $\mathfrak{F}$-metric between two sets on $\mathcal{P}^r(X)$.

Proof . Suppose η_0 is an arbitrary point in X. Then there is $\alpha_{T_1}(\eta_0) \in (0, 1]$ such that $\{T_1\eta_0\}_{\alpha_{T_1}(\eta_0)} \in \mathcal{P}^r(X)$. So, there exists $\eta_1 \in \{T_1\eta_0\}_{\alpha_{T_1}(\eta_0)}$ such that

$$d_{\mathfrak{F}}(\eta_0, \eta_1) = d_{\mathfrak{F}}(\eta_0, \{T_1\eta_0\}_{\alpha_{T_1}(\eta_0)}).$$

For $\eta_1 \in X$, then there is $\beta_{T_2}(\eta_1) \in (0, 1]$ such that $\{T_2\eta_1\}_{\beta_{T_2}(\eta_1)} \in \mathcal{P}^r(X)$. So, there is $\eta_2 \in \{T_2\eta_1\}_{\beta_{T_2}(\eta_1)}$ such that

$$d_{\mathfrak{F}}(\eta_1, \eta_2) = d_{\mathfrak{F}}(\eta_1, \{T_2\eta_1\}_{\beta_{T_2}(\eta_1)}).$$

In a similar way, one can obtain a sequence $\{\eta_n\} \subseteq X$ so that

$$\eta_{2n+1} \in \{T_1\eta_{2n}\}_{\alpha_{T_1}(\eta_{2n})} \quad \text{and} \quad \eta_{2n+2} \in \{T_2\eta_{2n+1}\}_{\beta_{T_2}(\eta_{2n+1})}.$$

We find that

$$\begin{aligned}
d_{\mathfrak{F}}(\eta_1, \eta_2) &= d_{\mathfrak{F}}(\eta_1, \{T_2\eta_1\}_{\beta_{T_2}(\eta_1)}) \\
&\leq H_{\mathfrak{F}}(\{T_1\eta_0\}_{\alpha_{T_1}(\eta_0)}), \{T_2\eta_1\}_{\beta_{T_2}(\eta_1)}) \\
&\leq g(d_{\mathfrak{F}}(\eta_0, \eta_1), d_{\mathfrak{F}}(\eta_0, \{T_1\eta_0)\}_{\alpha_{T_1}(\eta_0)}, d_{\mathfrak{F}}(\eta_1, \{T_2\eta_1\}_{\beta_{T_2}(\eta_1)}), \\
&\quad d_{\mathfrak{F}}(\eta_0, \{T_2\eta_1\}_{\beta_{T_2}(\eta_1)}), d_{\mathfrak{F}}(\eta_1, \{T_1\eta_0\}_{\alpha_{T_1}(\eta_0)})) \\
&= g(d_{\mathfrak{F}}(\eta_0, \eta_1), d_{\mathfrak{F}}(\eta_0, \eta_1), d_{\mathfrak{F}}(\eta_1, \eta_2), d_{\mathfrak{F}}(\eta_0, \eta_2), d_{\mathfrak{F}}(\eta_1, \eta_1)) \\
&= g(d_{\mathfrak{F}}(\eta_0, \eta_1), d_{\mathfrak{F}}(\eta_0, \eta_1), d_{\mathfrak{F}}(\eta_1, \eta_2), d_{\mathfrak{F}}(\eta_0, \eta_2), 0)
\end{aligned}$$

From $(\mathfrak{F}_1)$, we have that $d_{\mathfrak{F}}(\eta_0,\eta_2)>0$ and (d_3), then

$$f(d_{\mathfrak{F}}(\eta_0,\eta_2)) \leq f(d_{\mathfrak{F}}(\eta_0,\eta_1)+d_{\mathfrak{F}}(\eta_1,\eta_2))+\rho.$$

For $\rho=0$ and from $(\mathfrak{F}_1)$, we get $d_{\mathfrak{F}}(\eta_0,\eta_2)<d_{\mathfrak{F}}(\eta_0,\eta_1)+d_{\mathfrak{F}}(\eta_1,\eta_2)$

$$\begin{aligned} d_{\mathfrak{F}}(\eta_1,\eta_2) &\leq g(d_{\mathfrak{F}}(\eta_0,\eta_1),d_{\mathfrak{F}}(\eta_0,\eta_1),d_{\mathfrak{F}}(\eta_1,\eta_2),d_{\mathfrak{F}}(\eta_0,\eta_2),0)\\ &\leq g(d_{\mathfrak{F}}(\eta_0,\eta_1),d_{\mathfrak{F}}(\eta_0,\eta_1),d_{\mathfrak{F}}(\eta_1,\eta_2),d_{\mathfrak{F}}(\eta_0,\eta_1)+d_{\mathfrak{F}}(\eta_1,\eta_2),0). \end{aligned}$$

From (g_2), there exists $k\in(0,1]$ such that $d_{\mathfrak{F}}(\eta_1,\eta_2)\leq kd_{\mathfrak{F}}(\eta_0,\eta_1)$. One can deduce that

$$\begin{aligned} d_{\mathfrak{F}}(\eta_2,\eta_3) &\leq kd_{\mathfrak{F}}(\eta_1,\eta_2)\\ &\leq k^2d_{\mathfrak{F}}(\eta_0,\eta_1). \end{aligned}$$

By induction, we obtain $d_{\mathfrak{F}}(\eta_n,\eta_{n+1})\leq k^nd_{\mathfrak{F}}(\eta_0,\eta_1)$. Now, we show that $\{\eta_n\}$ is $\mathfrak{F}$-Cauchy sequence,

$$\begin{aligned} d_{\mathfrak{F}}(\eta_n,\eta_{n+1})+d_{\mathfrak{F}}(\eta_{n+1},\eta_{n+2})+...+d_{\mathfrak{F}}(\eta_{m-1},\eta_m) &\leq k^nd_{\mathfrak{F}}(\eta_0,\eta_1)+k^{n+1}d_{\mathfrak{F}}(\eta_0,\eta_1)+...+\\ &\quad k^{m-1}d_{\mathfrak{F}}(\eta_0,\eta_1)\\ &= k^n[1+k+...]d_{\mathfrak{F}}(\eta_0,\eta_1)\\ &= \frac{k^n}{1-k}d_{\mathfrak{F}}(\eta_0,\eta_1). \end{aligned}$$

for all $m,n\in\mathbb{N}$, $m>n$ so that

$$\sum_{i=n}^{m-1} d_{\mathfrak{F}}(\eta_i,\eta_{i+1})\leq \frac{k^n}{1-k}d_{\mathfrak{F}}(\eta_0,\eta_1). \tag{3.2}$$

Let $(f,\rho)\in\mathfrak{F}\times[0,+\infty)$, (d_3) is satisfied, assume that $\delta>0$ is fixed and from $(\mathfrak{F}_2)$ there is $\epsilon>0$ such that

$$0<t<\epsilon \implies f(t)<f(\delta)-\rho. \tag{3.3}$$

From (3.2) as $n\to\infty$, then $\lim_{n\to\infty}\frac{k^n}{1-k}d_{\mathfrak{F}}(\eta_0,\eta_1)=0$, i.e.,

$$\forall\epsilon>0 \ \ \exists N\in\mathbb{N} \ : \ 0<\frac{k^n}{1-k}d_{\mathfrak{F}}(\eta_0,\eta_1)<\epsilon \ \ \forall n\geq N,$$

from $(\mathfrak{F}_1)$, (3.2) and (3.3), we have that $f(\sum_{i=n}^{m-1} d_{\mathfrak{F}}(\eta_i,\eta_{i+1}))\leq f(\frac{k^n}{1-k}d_{\mathfrak{F}}(\eta_0,\eta_1))$,

$$f(\sum_{i=n}^{m-1} d_{\mathfrak{F}}(\eta_i,\eta_{i+1}))\leq f(\frac{k^n}{1-k}d_{\mathfrak{F}}(\eta_0,\eta_1))\leq f(\delta)-\rho. \tag{3.4}$$

Using (d_3) and (3.4), we get

$$\begin{aligned} f(d_{\mathfrak{F}}(\eta_n,\eta_m)) &\leq f(\sum_{i=n}^{m-1} d_{\mathfrak{F}}(\eta_i,\eta_{i+1}))+\rho\\ &\leq f(\delta). \end{aligned}$$

This means that $d_{\mathfrak{F}}(\eta_n,\eta_m)<\delta$, i.e., $\{\eta_n\}$ is $\mathfrak{F}$-Cauchy sequence. Because X is $\mathfrak{F}$-complete, then we have $\{\eta_n\}$ is $\mathfrak{F}$-convergent to $\eta^*\in X$ such that $\lim_{n\to\infty}\eta_n=\eta^*$. Now, we prove that $\eta^*\in\{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)}$ where

$$\begin{aligned} f(d_{\mathfrak{F}}(\{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)},\eta^*)) &\leq f(d_{\mathfrak{F}}\{(T_1\eta^*\}_{\alpha_{T_1}(\eta^*)},\eta_{2n+2})+d_{\mathfrak{F}}(\eta_{2n+2},\eta^*))+\rho\\ &= f(d_{\mathfrak{F}}\{(T_1\eta^*\}_{\alpha_{T_1}(\eta^*)},\eta_{2n+2})+d_{\mathfrak{F}}(\{T_2\eta_{2n+1}\}_{\beta_{T_2}(\eta_{2n+1})},\eta^*))+\rho, \end{aligned}$$

for $\rho = 0$,

$$\begin{aligned}
d_{\mathfrak{F}}(\{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)}, \eta^*) &\leq d_{\mathfrak{F}}(\{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)}, \{T_2\eta_{2n+1}\}_{\alpha_{T_2}(\eta_{2n+1})}) + d_{\mathfrak{F}}(\eta_{2n+2}, \eta^*) \\
&\leq H_{\mathfrak{F}}(\{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)}, \{T_2\eta_{2n+1}\}_{\beta_{T_2}(\eta_{2n+1})}) + d_{\mathfrak{F}}(\eta_{2n+2}, \eta^*) \\
&\leq g(d_{\mathfrak{F}}(\eta^*, \eta_{2n+1}), d_{\mathfrak{F}}(\eta^*, \{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)}), d_{\mathfrak{F}}(\eta_{2n+1}, \{T_2\eta_{2n+1}\}_{\beta_{T_2}(\eta_{2n+1})}), \\
&\quad d_{\mathfrak{F}}(\eta^*, \{T_2\eta_{2n+1}\}_{\beta_{T_2}(\eta_{2n+1})}), d_{\mathfrak{F}}(\eta_{2n+1}, \{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)})) + d_{\mathfrak{F}}(\eta_{2n+2}, \eta^*) \\
&= g(d_{\mathfrak{F}}(\eta^*, \eta_{2n+1}), d_{\mathfrak{F}}(\eta^*, \{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)}), d_{\mathfrak{F}}(\eta_{2n+1}, \eta_{2n+2}), \\
&\quad d_{\mathfrak{F}}(\eta^*, \eta_{2n+2}), d_{\mathfrak{F}}(\eta^*, \{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)})) + d_{\mathfrak{F}}(\eta_{2n+2}, \eta^*).
\end{aligned}$$

From Lemma 3.1 as $n \to \infty$, then

$$d_{\mathfrak{F}}(\eta^*, \{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)}) \leq g(0, d_{\mathfrak{F}}(\eta^*, \{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)}), 0, 0, d_{\mathfrak{F}}(\eta^*, \{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)})).$$

By (g_3), we have $d_{\mathfrak{F}}(\eta^*, \{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)}) = 0$. Thus $\eta^* \in \{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)}$. Similarly $\eta^* \in \{T_2\eta^*\}_{\beta_{T_2}(\eta^*)}$. Hence T_1 and T_2 have a common fixed point.

Now, to support the generality of the Theorem 3.2 over Theorem 12 [5], we can give an example here.

Example 3.3. Suppose that $X = [0, \infty)$, $(X, d_{\mathfrak{F}})$ is an $\mathfrak{F}$-complete $\mathfrak{F}$-metric space defined by

$$d_{\mathfrak{F}}(\eta, \xi) = \begin{cases} (\eta - \xi)^2 & \text{if } (\eta, \xi) \in [0,5] \times [0,5] \\ |\eta - \xi| & \text{if } (\eta, \xi) \notin [0,5] \times [0,5] \end{cases}$$

Define two fuzzy mappings T_1, T_2 as follows:

$$\{T_1\eta\}_\alpha(t) = \begin{cases} \frac{1}{6} & \text{if } 0 \leq t \leq \frac{\eta^2}{100} \\ \frac{1}{8} & \text{if } \frac{\eta^2}{10} < t \leq \frac{\eta^2}{50} \\ \frac{1}{10} & \text{if } \frac{\eta^2}{50} < t \leq 1 \end{cases}$$

and

$$\{T_2\xi\}_\beta(t) = \begin{cases} \frac{2}{3} & \text{if } 0 \leq t \leq \frac{\xi^2}{100} \\ \frac{1}{2} & \text{if } \frac{\xi^2}{100} < t \leq \xi^2 \\ \frac{2}{5} & \text{if } \xi^2 < t \leq 1 \end{cases}$$

Now, for $\alpha = \frac{1}{6}$, $\{T_1(\eta)\}_{\frac{1}{6}} = [0, \frac{\eta^2}{100}]$ and for $\beta = \frac{2}{3}$, $\{T_2(\xi)\}_{\frac{2}{3}} = [0, \frac{\xi^2}{100}]$,

$$g(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) = \frac{1}{100}\gamma_1 \quad \forall \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 \in [0, \infty).$$

Thus, for $\eta, \xi \in X$ and from definition of $d_{\mathfrak{F}}$, we have

$$\begin{aligned}
H_{\mathfrak{F}}(T_1\eta, T_2\xi) &= (\frac{\eta^2}{100} - \frac{\xi^2}{100})^2 \\
&= |(\frac{\eta + \xi}{100}(\eta - \xi))|^2 \\
&\leq \frac{1}{100}|\eta - \xi|^2 \\
&= \frac{1}{100} d_{\mathfrak{F}}(\eta, \xi) \\
&= g(d_{\mathfrak{F}}(\eta, \xi), d_{\mathfrak{F}}(\eta, T_1\eta), d_{\mathfrak{F}}(\xi, T_2\xi), d_{\mathfrak{F}}(\eta, T_2\xi), d_{\mathfrak{F}}(\xi, T_1\eta)).
\end{aligned}$$

It is clear that all the conditions of Theorem (3.2) hold to find $\eta^* = 0 \in \{T_1 0\}_{\frac{1}{6}} \cap \{T_2 0\}_{\frac{2}{3}}$. In Theorem 12 [5] satisfies conditions of Lemma 2.7 and

$$H_{\mathfrak{F}}(T_1\eta, T_2\xi) \leq \phi(\max\{d_{\mathfrak{F}}(\eta,\xi), d_{\mathfrak{F}}(\eta, T_1\eta), d_{\mathfrak{F}}(\xi, T_2\xi), \frac{d_{\mathfrak{F}}(\eta, T_2\xi) + d_{\mathfrak{F}}(\xi, T_1\eta)}{2}\}).$$

But, when we put

$$\phi(r) = r\max\{\alpha + (1-\alpha), \alpha(1-\alpha), \alpha(1-\alpha)(2\beta), \alpha + (1-\alpha)(2\alpha)\},$$

for $\eta = \frac{41}{100}$, $\xi = \frac{1}{4}$ and $\alpha = \frac{1}{6}, \beta = \frac{2}{3}$, then

$$r = \max\{d_{\mathfrak{F}}(\eta,\xi), d_{\mathfrak{F}}(\eta, T_1\eta), d_{\mathfrak{F}}(\xi, T_2\xi), \frac{d_{\mathfrak{F}}(\eta, T_2\xi) + d_{\mathfrak{F}}(\xi, T_1\eta)}{2}\} = 0.1667, \quad \phi(r) = \frac{23}{18} r > r,$$

this does not imply $\phi(r) < r \ \forall r > 0$. The previous example does not generally satisfy $\phi(r) < r$ of theorem 12 [5].

Remark 3.4.

(I) The Theorem 3.2 is the generalization of Theorem 12 [5] such that the results of Theorem 12 [5] is a specific case of the Theorem 3.2 and we used the contractive condition (3.1) instead of (9) in Theorem 12 [5].

(II) If there is a $g \in G$ such that $\forall\ \eta, \xi \in X$,

$$\delta_{\mathfrak{F}}(T_1\eta, T_2\xi) \leq g(d_{\mathfrak{F}}(\eta,\xi), d_{\mathfrak{F}}(\eta, T_1\eta), d_{\mathfrak{F}}(\xi, T_2\xi), d_{\mathfrak{F}}(\eta, T_2\xi), d_{\mathfrak{F}}(\xi, T_1\eta)),$$

where $\delta_{\mathfrak{F}}(T_1\eta, T_2\xi) = \sup\{d_{\mathfrak{F}}(a,b) : a \in T_1\eta,\ b \in T_2\xi\}$, so we can say that the conclusion of Theorem 3.2 is still valid. This result is considered as special case of Theorem 3.1. Since $H_{\mathfrak{F}}(T_1\eta, T_2\xi) \leq \delta_{\mathfrak{F}}(T_1\eta, T_2\xi)$ ([8], page 414).

Corollary 3.5. Let $(X, d_{\mathfrak{F}})$ be an $\mathfrak{F}$-complete $\mathfrak{F}$-metric space and $T_1, T_2 : X \longrightarrow I^X$ be fuzzy mappings. Suppose that $\eta, \xi \in X$, there is $\alpha_{T_1}(\eta), \alpha_{T_2}(\xi) \in (0,1]$ so $[T_1\eta]_{\alpha_{T_1}(\eta)}, [T_2\eta]_{\beta_{T_2}(\eta)} \in C(2^X)$. Also the following condition satisfies:

$$H_{\mathfrak{F}}(T_1\eta, T_2\xi) \leq \phi(\max\{d_{\mathfrak{F}}(\eta,\xi), d_{\mathfrak{F}}(\eta, T_1\eta), d_{\mathfrak{F}}(\xi, T_2\xi), \frac{d_{\mathfrak{F}}(\eta, T_2\xi) + d_{\mathfrak{F}}(\xi, T_1\eta)}{2}\}),$$

where $\phi \in \Omega$ and $C(2^X)$ is the set of all nonempty compact subsets of X. Then there exists $\eta^* \in X$ such that $\eta^* \in \{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)} \cap \{T_2\eta^*\}_{\beta_{T_2}(\eta^*)}$.

Remark 3.6.

(I) We instead the compact subset of X by the proximal subset of X.

(II) In theorem 3.2, if we put

$$g(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5) = \phi(\gamma_1, \gamma_2, \gamma_3, \frac{\gamma_4 + \gamma_5}{2}),$$

we obtain the conclusion of corollary 3.5.

4 Applications

Let $\hat{G}$ be the member of all continuous mappings $\hat{g} : [0,\infty)^5 \longrightarrow [0,\infty)$ satisfying the properties as below:

$(\hat{g}_1)$ $\hat{g}$ is non-decreasing in the 2^{nd}, 3^{rd}, 4^{th} and 5^{th} coordinate variables,

$(\hat{g_2})$ there is $k \in (0,1)$ so for each $\upsilon, \nu \in [0,\infty)$ and

$$\int_0^{\upsilon} \psi(s)ds \leq \int_0^{\hat{g}\left(\nu,\nu,\upsilon,\upsilon+\nu,0 \right)} \psi(s)ds \quad \textit{implies} \;\; \upsilon \leq k\nu.$$

$(\hat{g_3})$ if $\upsilon \in [0,\infty)$ such that

$$\int_0^{\upsilon} \psi(s)ds \leq \int_0^{\hat{g}\left(\upsilon,0,0,\upsilon,\upsilon \right)} \psi(s)ds,$$

or

$$\int_0^{\upsilon} \psi(s)ds \leq \int_0^{\hat{g}\left(0,\upsilon,0,0,\upsilon \right)} \psi(s)ds,$$

then $\upsilon = 0$.

Theorem 4.1. Let $(X, d_{\mathfrak{F}})$ be an $\mathfrak{F}$-complete $\mathfrak{F}$-metric space and $T_1, T_2 : X \longrightarrow I^X$ be fuzzy mappings. Suppose that for each $\eta, \xi \in X$, there is $\alpha_{T_1}(\eta)$, $\beta_{T_2}(\xi) \in (0,1]$ such that $\{T_1\eta\}_{\alpha_{T_1}(\eta)}, \{T_2\xi\}_{\beta_{T_2}(\xi)} \in \mathcal{P}^r(X)$ and if there is a $g \in G$ satisfies

$$\int_0^{H_{\mathfrak{F}}(T_1\eta,T_2\xi)} \psi(s)ds \leq \int_0^{\hat{g}(d_{\mathfrak{F}}(\eta,\xi),d_{\mathfrak{F}}(\eta,T_1\eta),d_{\mathfrak{F}}(\xi,T_2\xi),d_{\mathfrak{F}}(\eta,T_2\xi),d_{\mathfrak{F}}(\xi,T_1\eta))} \psi(s)ds, \tag{4.1}$$

then there is $\eta^* \in X$ such that $\eta^* \in \{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)} \cap \{T_2\eta^*\}_{\beta_{T_2}(\eta^*)}$.
Where $H_{\mathfrak{F}}$ is the hausdorff $\mathfrak{F}$-metric between two sets on $\mathcal{P}^r(X)$.

Proof . Suppose that η_0 is an arbitrary point in X. Then there is $\alpha_{T_1}(\eta_0) \in (0,1]$ such that $\{T_1\eta_0\}_{\alpha_{T_1}(\eta_0)} \in \mathcal{P}^r(X)$. So, there exists $\eta_1 \in \{T_1\eta_0\}_{\alpha_{T_1}(\eta_0)}$ such that

$$\int_0^{d_{\mathfrak{F}}(\eta_0,\eta_1)} \psi(s)ds = \int_0^{d_{\mathfrak{F}}(\eta_0,\{T_1\eta_0\}_{\alpha_{T_1}(\eta_0)})} \psi(s)ds.$$

For $\eta_1 \in X$, then there is $\beta_{T_2}(\eta_1) \in (0,1]$ such that $\{T_2\eta_1\}_{\beta_{T_2}(\eta_1)} \in \mathcal{P}^r(X)$. So, there is $\eta_2 \in \{T_2\eta_1\}_{\beta_{T_2}(\eta_1)}$ such that

$$\int_0^{d_{\mathfrak{F}}(\eta_1,\eta_2)} \psi(s)ds = \int_0^{d_{\mathfrak{F}}(\eta_1,\{T_2\eta_1\}_{\beta_{T_2}(\eta_1)})} \psi(s)ds.$$

In a similar way, one can obtain a sequence $\{\eta_n\} \subseteq X$ so that

$$\eta_{2n+1} \in \{T_1\eta_{2n}\}_{\alpha_{T_1}(\eta_{2n})} \quad \text{and} \quad \eta_{2n+2} \in \{T_2\eta_{2n+1}\}_{\beta_{T_2}(\eta_{2n+1})}.$$

We find that

$$\begin{aligned}
\int_0^{d_{\mathfrak{F}}(\eta_1,\eta_2)} \psi(s)ds &= \int_0^{d_{\mathfrak{F}}(\eta_1,\{T_2\eta_1\}_{\beta_{T_2}(\eta_1)})} \psi(s)ds \\
&\leq \int_0^{H_{\mathfrak{F}}(\{T_1\eta_0\}_{\alpha_{T_1}(\eta_0)}),\{T_2\eta_1\}_{\beta_{T_2}(\eta_1)})} \psi(s)ds \\
&\leq \int_0^{\hat{g}\left(\begin{array}{c} d_{\mathfrak{F}}(\eta_0,\eta_1), d_{\mathfrak{F}}(\eta_0,\{T_1\eta_0)\}_{\alpha_{T_1}(\eta_0)}, d_{\mathfrak{F}}(\eta_1,\{T_2\eta_1\}_{\beta_{T_2}(\eta_1)}), \\ d_{\mathfrak{F}}(\eta_0,\{T_2\eta_1\}_{\beta_{T_2}(\eta_1)}), d_{\mathfrak{F}}(\eta_1,\{T_1\eta_0\}_{\alpha_{T_1}(\eta_0)}) \end{array} \right)} \psi(s)ds \\
&= \int_0^{\hat{g}(d_{\mathfrak{F}}(\eta_0,\eta_1),d_{\mathfrak{F}}(\eta_0,\eta_1),d_{\mathfrak{F}}(\eta_1,\eta_2),d_{\mathfrak{F}}(\eta_0,\eta_2),d_{\mathfrak{F}}(\eta_1,\eta_1))} \psi(s)ds \\
&= \int_0^{\hat{g}(d_{\mathfrak{F}}(\eta_0,\eta_1),d_{\mathfrak{F}}(\eta_0,\eta_1),d_{\mathfrak{F}}(\eta_1,\eta_2),d_{\mathfrak{F}}(\eta_0,\eta_2),0)} \psi(s)ds.
\end{aligned}$$

From $(\mathfrak{F}_1)$, we have that $d_{\mathfrak{F}}(\eta_0, \eta_2) > 0$ and (d_3), then

$$f(d_{\mathfrak{F}}(\eta_0, \eta_2)) \leq f(d_{\mathfrak{F}}(\eta_0, \eta_1) + d_{\mathfrak{F}}(\eta_1, \eta_2)) + \rho.$$

For $\rho = 0$ and from $(\mathfrak{F}_1)$, we get $d_{\mathfrak{F}}(\eta_0, \eta_2) < d_{\mathfrak{F}}(\eta_0, \eta_1) + d_{\mathfrak{F}}(\eta_1, \eta_2)$

$$\begin{aligned}
\int_0^{d_{\mathfrak{F}}(\eta_1,\eta_2)} \psi(s)ds &\leq \int_0^{\hat{g}(d_{\mathfrak{F}}(\eta_0,\eta_1), d_{\mathfrak{F}}(\eta_0,\eta_1), d_{\mathfrak{F}}(\eta_1,\eta_2), d_{\mathfrak{F}}(\eta_0,\eta_2), 0)} \psi(s)ds \\
&\leq \int_0^{\hat{g}(d_{\mathfrak{F}}(\eta_0,\eta_1), d_{\mathfrak{F}}(\eta_0,\eta_1), d_{\mathfrak{F}}(\eta_1,\eta_2), d_{\mathfrak{F}}(\eta_0,\eta_1)+d_{\mathfrak{F}}(\eta_1,\eta_2), 0)} \psi(s)ds.
\end{aligned}$$

From $(\hat{g_2})$, there exists $k \in (0, 1]$ such that $d_{\mathfrak{F}}(\eta_1, \eta_2) \leq k d_{\mathfrak{F}}(\eta_0, \eta_1)$. One can deduce that

$$\begin{aligned}
d_{\mathfrak{F}}(\eta_2, \eta_3) &\leq k d_{\mathfrak{F}}(\eta_1, \eta_2) \\
&\leq k^2 d_{\mathfrak{F}}(\eta_0, \eta_1).
\end{aligned}$$

By induction, we obtain $d_{\mathfrak{F}}(\eta_n, \eta_{n+1}) \leq k^n d_{\mathfrak{F}}(\eta_0, \eta_1)$. Now, we show that $\{\eta_n\}$ is $\mathfrak{F}$-Cauchy sequence,

$$\begin{aligned}
d_{\mathfrak{F}}(\eta_n, \eta_{n+1}) + d_{\mathfrak{F}}(\eta_{n+1}, \eta_{n+2}) + ... + d_{\mathfrak{F}}(\eta_{m-1}, \eta_m) &\leq k^n d_{\mathfrak{F}}(\eta_0, \eta_1) + k^{n+1} d_{\mathfrak{F}}(\eta_0, \eta_1) + ... + \\
&\quad\; k^{m-1} d_{\mathfrak{F}}(\eta_0, \eta_1) \\
&= k^n [1 + k + ...] d_{\mathfrak{F}}(\eta_0, \eta_1) \\
&= \frac{k^n}{1-k} d_{\mathfrak{F}}(\eta_0, \eta_1).
\end{aligned}$$

for all $m, n \in \mathbb{N}$, $m > n$ so that

$$\sum_{i=n}^{m-1} d_{\mathfrak{F}}(\eta_i, \eta_{i+1}) \leq \frac{k^n}{1-k} d_{\mathfrak{F}}(\eta_0, \eta_1). \tag{4.2}$$

Let $(f, \rho) \in \mathfrak{F} \times [0, +\infty)$, (d_3) is satisfied, assume that $\delta > 0$ is fixed and from $(\mathfrak{F}_2)$ there is $\epsilon > 0$ such that

$$0 < t < \epsilon \implies f(t) < f(\delta) - \rho. \tag{4.3}$$

From (4.2) as $n \to \infty$, then $\lim_{n\to\infty} \frac{k^n}{1-k} d_{\mathfrak{F}}(\eta_0, \eta_1) = 0$, i.e.,

$$\forall \epsilon > 0 \;\; \exists N \in \mathbb{N} \; : \; 0 < \frac{k^n}{1-k} d_{\mathfrak{F}}(\eta_0, \eta_1) < \epsilon \;\; \forall n \geq N,$$

from $(\mathfrak{F}_1)$, (4.2) and (4.3), we have that $f(\sum_{i=n}^{m-1} d_{\mathfrak{F}}(\eta_i, \eta_{i+1})) \leq f(\frac{k^n}{1-k} d_{\mathfrak{F}}(\eta_0, \eta_1))$,

$$f(\sum_{i=n}^{m-1} d_{\mathfrak{F}}(\eta_i, \eta_{i+1})) \leq f(\frac{k^n}{1-k} d_{\mathfrak{F}}(\eta_0, \eta_1)) \leq f(\delta) - \rho. \tag{4.4}$$

Using (d_3) and (4.4), we get

$$\begin{aligned}
f(d_{\mathfrak{F}}(\eta_n, \eta_m)) &\leq f(\sum_{i=n}^{m-1} d_{\mathfrak{F}}(\eta_i, \eta_{i+1})) + \rho \\
&\leq f(\delta).
\end{aligned}$$

This means that $d_{\mathfrak{F}}(\eta_n, \eta_m) < \delta$, i.e., $\{\eta_n\}$ is $\mathfrak{F}$-Cauchy sequence. Because X is $\mathfrak{F}$-complete, then we have $\{\eta_n\}$ is $\mathfrak{F}$-convergent to $\eta^* \in X$ such that $\lim\limits_{n\to\infty} \eta_n = \eta^*$. Now, we prove that $\eta^* \in \{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)}$ where

$$\begin{aligned} f(d_{\mathfrak{F}}(\{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)}, \eta^*)) &\leq f(d_{\mathfrak{F}}\{(T_1\eta^*\}_{\alpha_{T_1}(\eta^*)}, \eta_{2n+2}) + d_{\mathfrak{F}}(\eta_{2n+2}, \eta^*)) + \rho \\ &= f(d_{\mathfrak{F}}\{(T_1\eta^*\}_{\alpha_{T_1}(\eta^*)}, \eta_{2n+2}) + d_{\mathfrak{F}}(\{T_2\eta_{2n+1}\}_{\beta_{T_2}(\eta_{2n+1})}, \eta^*)) + \rho, \end{aligned}$$

for $\rho = 0$,

$$\begin{aligned} \int_0^{d_{\mathfrak{F}}(\{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)},\eta^*)} \psi(s)ds &\leq \int_0^{d_{\mathfrak{F}}(\{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)},\{T_2\eta_{2n+1}\}_{\alpha_{T_2}(\eta_{2n+1})})+d_{\mathfrak{F}}(\eta_{2n+2},\eta^*)} \psi(s)ds \\ &\leq \int_0^{H_{\mathfrak{F}}(\{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)},\{T_2\eta_{2n+1}\}_{\beta_{T_2}(\eta_{2n+1})})+d_{\mathfrak{F}}(\eta_{2n+2},\eta^*)} \psi(s)ds \\ &\leq \int_0^{\hat{g}\begin{pmatrix} d_{\mathfrak{F}}(\eta^*, \eta_{2n+1}), d_{\mathfrak{F}}(\eta^*, \{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)}), \\ d_{\mathfrak{F}}(\eta_{2n+1}, \{T_2\eta_{2n+1}\}_{\beta_{T_2}(\eta_{2n+1})}), \\ d_{\mathfrak{F}}(\eta^*, \{T_2\eta_{2n+1}\}_{\beta_{T_2}(\eta_{2n+1})}), \\ d_{\mathfrak{F}}(\eta_{2n+1}, \{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)}) \end{pmatrix}+d_{\mathfrak{F}}(\eta_{2n+2},\eta^*)} \psi(s)ds \\ &= \int_0^{\hat{g}\begin{pmatrix} d_{\mathfrak{F}}(\eta^*, \eta_{2n+1}), \\ d_{\mathfrak{F}}(\eta^*, \{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)}), d_{\mathfrak{F}}(\eta_{2n+1}, \eta_{2n+2}), \\ d_{\mathfrak{F}}(\eta^*, \eta_{2n+2}), d_{\mathfrak{F}}(\eta^*, \{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)}) \end{pmatrix}+d_{\mathfrak{F}}(\eta_{2n+2},\eta^*)} \psi(s)ds \end{aligned}$$

From Lemma 3.1 as $n \to \infty$, then

$$\int_0^{d_{\mathfrak{F}}(\eta^*,\{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)})} \psi(s)ds \leq \int_0^{\hat{g}(0,d_{\mathfrak{F}}(\eta^*,\{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)}),0,0,d_{\mathfrak{F}}(\eta^*,\{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)}))} \psi(s)ds.$$

By $(\hat{g}_3)$, we have $d_{\mathfrak{F}}(\eta^*, \{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)}) = 0$. Thus $\eta^* \in \{T_1\eta^*\}_{\alpha_{T_1}(\eta^*)}$. Similarly $\eta^* \in \{T_2\eta^*\}_{\beta_{T_2}(\eta^*)}$. Hence T_1 and T_2 have a common fixed point.

5 Conclusion

We have discussed the existence of common fixed point theorem for fuzzy mappings that satisfy the contractive condition in $\mathfrak{F}$- metric spaces as a generalization for some fixed point theorems on metric space.

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