Int. J. Nonlinear Anal. Appl. 14 (2023) 1, 927–940
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.27109.3502

Fear effect on a delayed intraguild predation model with the ratio-dependent functional response

S. Magudeeswaran[a], S. Vinoth[c,*], K. Sathiyanathan[b], Kantapon Chaisena[d]

[a]*Department of Mathematics, Sree Saraswathi Thyagaraja College, Coimbatore, 642107, Tamilnadu, India*
[b]*Department of Computer Science, KGiSL Institute of Information Management, Coimbatore 641035, Tamilnadu, India*
[c]*Department of Mathematics, SRMV College of Arts and Science, Coimbatore 641020, Tamilnadu, India*
[d]*Department of Mathematics, Phuket Rajabhat University, Phuket, 83000, Thailand*

(Communicated by Saman Babaie-Kafaki)

Abstract

In this work, we explore the consequences of fear and time delay on the intraguild predation model. Also, the predator consumes its prey in the form of a ratio-dependent type of interaction. We consider the fear in the prey population and the gestation effect on the predator population. We analyze the existence and the local stability of the proposed model without delay near all non-negative equilibrium points. Furthermore, by taking the fear parameter, the condition to satisfy the existence of Hopf-bifurcation near the coexisting equilibrium is derived. Moreover, we also examine the local stability property and Hopf-bifurcation investigation for the corresponding model in the presence of time delay. Some simulation results were also done to support the primary analytical findings.

Keywords: Fear effect, Hopf-bifurcation, Intraguild predation, Local stability, Time delay
2020 MSC: 92D25, 34K18, 34K20, 92D40, 92-XX

---

1 Introduction

The investigation of dynamical interactions among the various species is an important theme in the environment and, most importantly, the study of relationships between predators and preys. In the environment, there are two techniques for capturing the predatory effects of prey-predator interactions: The first one is direct interaction, where the predator kills the prey directly. The role of functional responses is most important in the study of prey-predator interactions. There are various types of functional responses, including Holling types I, II, and III [1, 2, 3], Crowley-Martin [4], Beddington-DeAngelis [5], and ratio-dependent [6]. These functional responses depend on both the prey density and predator density. The analytical behavior of a common ratio-dependent predator-prey model at origin has been investigated by Jost et al. [7].The parametric study of the prey-predator model investigation was done by Berezovskaya et al. [8], where they considered the prey-predator interaction as a ratio-dependent type. Agarwal and Singh [9] investigated the persistence, boundedness, and stability behavior of a ratio-dependent type three species food-chain model. Flores and Gonzalez-Olivares [10] examined a modified Leslie-Gower predator-prey model with a ratio-dependent type of functional response. Jiang et al. [11] explored the structural stability to study the different

---

[*] Corresponding author
Email address: `svinothappu@gmail.com` (S. Vinoth)

Received: May 2022 *Accepted:* August 2022

types of bifurcation for the proposed ratio-dependent predator-prey model. Khajanch and Banerjee [12] discussed the dynamics of a stage-structured prey–predator model with interaction in the form of ratio-dependent type on the effect of prey refuge. They revealed that the presence of a prey refuge has a significant impact on predator–prey interaction. The authors in [13] have discussed the rate of individual predator-prey intake according to the ratio-dependent type of functional response and the formation of a predator-prey model by self and cross-distribution when harvesting prey through a non-linear harvesting strategy. Roy et al. [14] investigated a two-species prey-predator system with a ratio-dependent functional type, where predator species have to engage in intra-specific competition due to limited resources of food. In this paper, we consider the system that incorporates ratio-dependent functional response that was first introduced by Arditi and Ginzburg [6].

The second type of species interaction is indirect interaction. Compared to direct interaction, the behavioral and physiological modifications related to fear of predation are more efficient and longer-lasting evolutionary effects [36]. This interaction may have a significant impact on the behavior of prey because of the possibility of predation. The fear phenomenon was first introduced by Cannon [15]. Preys are always afraid to enter into an open environment because they are scared of predators, and they do not even have a fearless atmosphere for daily behaviors such as breeding. So the fear of predators affects their breeding rate, and it is necessary to consider the effect of panic as a form of breeding decline. The predator-prey model and the influence of fear on prey density have piqued the interest of many scholars. Recently, the authors, Pal et al. [16], investigated the effect of fear on the Beddington-DeAngelis prey-predator model. The authors in [17] investigated the stability and different bifurcation behaviors of the Leslie type predator-prey model with fear and the Allee effect. Hossain et al. [18] discussed the impact of fear on the three-species intraguild predation model. They came to the conclusion that increasing the intensity of fear causes the system to stabilize from chaotic oscillations. They also discovered that various combinations of fear of intraguild prey and fear of intraguild predator could produce bi-stability between planer equilibrium points as well as between interior and planer equilibrium points. Also, they discovered that fear may produce more stable limit cycles. Kumar and Kumari [19] analyzed the dynamics of the three-species food chain model by incorporating two fear effects. They conclude that fear parameters control the chaotic dynamics of the proposed model. This is because the system shows chaotic behavior if the effect of fear is small, and if the effect of fear is large, the system dynamics change from chaotic behavior to stable focus. Recently, Pandy et al. [20] studied the Holling type-II prey-predator model with the influence of fear in prey population. They concluded that the system's chaotic behavior is projected to low values of both fear parameters.

In addition, one of the very most common components found in every biological activity is a time delay. These are accounted for by different types of delays in the biological process, such as incubation delay [21], maturation delay [22], and gestation delay [23]. These types of time delays in the biological process make the system more practical and complex. Pal et al. [24] investigated the local stability and the possibility of Hopf-bifurcation near the interior equilibrium point and determined the requirements for the proposed model's direction and stability. They have discovered that the model shows chaotic behavior by changing the value of gestation delay. Yan and Zhang [25] explored the Hopf-bifurcation analysis in the Lotka-Volterra predator-prey system with time delay. However, they have investigated the linear stability and demonstrated the existence of Hopf-bifurcation. The authors in [26] explored the dynamics of a food chain model with a time delay and an additive Allee effect. Xu and Liao [27] performed a mathematical analysis such as local stability and the appearance of Hopf-bifurcation based on the coexisting equilibrium point for the Holling type-II delayed food-chain model, and concluded that the cost of time delay exhibits chaotic behavior. Some researchers have recently been working on a prey-predator model that takes both time delay and ratio-dependent functional response into account. Song et al. [28] have investigated local stability, Hopf-bifurcation, and the criteria for directional stability around the interior equilibrium point. As a result, they found that delay plays an important role in ecology. The authors in [29] investigated the local and global stability analysis as well as the existence of Hopf-bifurcation conditions for both delayed and non-delayed intraguild predation models with a ratio-dependent functional response. Several authors investigated the dynamical behaviour of the intraguild predation model with various functional responses [30, 31, 32]. To the best of our knowledge, no one has explored the effect of fear on the intraguild model with a ratio-dependent type of functional response in the existing literature. Motivated by this fact, we study the intraguild predation model by considering ratio-dependent type interaction and also by incorporating fear and time delay.

The organization of this paper: Section 1 presents the background, motivations, and objectives of this study. The formulation of the mathematical model for the problem undertaken together with the basic preliminaries are discussed in Section 2, Sections 3 and 4 explore the appearance of positive equilibria and provide a local stability analysis, respectively. We provide the conditions for the occurrence of Hopf-bifurcation for the non-delayed model in Section 5. In Section 6, we discuss the parameters of local stability and the presence of Hopf-bifurcation analysis in the delayed model. Finally, in Sections 7 and 8, we elaborately discussed the numerical simulations and conclusion.

2 Mathematical Model

Putra et al. [32] considered the food chain model, where the intake of prey by its predator is in the form of Holling-II type and it is given by

$$\begin{aligned}\frac{du}{dt} &= r_1 u\left(1-\frac{u}{n_1 w}\right)-\frac{e_1 uv}{p+u},\\ \frac{dv}{dt} &= r_2 v(1-\frac{v}{n_2 w})+\frac{e_2 uv}{p+u},\\ \frac{dw}{dt} &= w\left(\delta-\mu u-\eta v\right),\end{aligned} \tag{2.1}$$

with the non-negative initial conditions $u(0)=u_0>0$, $v(0)=v_0>0$, and $w(0)=w_0>0$, where

- u, v, and w are the sizes of prey, predators, and biotic resources at time t, respectively.
- $r_1, r_2, n_1, n_2, e_1, e_2, p, \delta, \mu, \eta$ are the positive parameters.
- r_1 and r_2 be the per capita growth rate of the prey and predator populations, respectively.
- The terms $n_1 w$ and $n_2 w$ form the environmental carrying capacity for the populations u and v, respectively.
- The parameters n_1 and n_2 are the carrying capacities of each individual, u and v, respectively. We assume $0<n_1<1$ and $0<n_2<1$ with $n_1+n_2=1$, thus $n_1 w+n_2 w$ is the total carrying capacity. If $n_2>n_1$, the prey population u receives a greater share of the biological resources, leading to higher carrying capacity. As a result, u can grow larger than v.
- Here δ be the growth rate of biotic resources, the uptake rate of resource by u is μuw and by v it is ηvw, where μ and η are constants.
- The Holling type-II interaction term is $\frac{uv}{p+u}$, where p is a half-saturation concentration constant. The predation rate is denoted by e_1, whereas e_2 is the predator's growth rate as a result of contact with prey.

Next, we include the fear effect in the intraguild predation model (2.1) to study the dynamical changes. To attain this object, pursue the way of Panday et al. [20]. In this model, we consider the prey population's growth rate reduces due to the fear of its predators. The intraguild prey's development is reduced only by its intraguild predator. It is a well established fact that the continuous fear induced by predators are surmise to affect the basic reproduction of the prey species [33]. Thus, we have modified the growth term by multiplying a decreasing function (w.r.t. cost of fear and availability of predators) as follows:

$$F_1(f,v)=\frac{1}{1+fv}.$$

This refers to the cost of fear in intraguild prey due to the intraguild predators'. Here, f is the level of fear. Considering the biological meaning of f, it is quite reasonable to assume the following conditions [11]:

$$\begin{aligned}&f(0,v)=F_1(f,0)=1,\quad \lim_{f\to\infty}F_1(f,v)=0=\lim_{v\to\infty}F_1(f,v),\\ &\frac{\partial F_1(f,v)}{\partial f}<0,\quad \frac{\partial F_1(f,v)}{\partial v}<0.\end{aligned}$$

In this work, we incorporate a ratio-dependent type interaction term $\frac{uv}{u+v}$ and the fear effect f, then the model (2.1) becomes the following form:

$$\begin{aligned}\frac{du}{dt} &= \frac{r_1 u}{1+fv}\left(1-\frac{u}{n_1 w}\right)-\frac{e_1 uv}{u+v},\\ \frac{dv}{dt} &= r_2 v\left(1-\frac{v}{n_2 w}\right)+\frac{e_2 uv}{u+v},\\ \frac{dw}{dt} &= w\left(\delta-\mu u-\eta v\right),\end{aligned} \tag{2.2}$$

with non-negative initial conditions $u(0)>0$, $v(0)>0$, and $w(0)>0$.

3 Existence of Equilibria

The model (2.2) has the below equilibrium points:

1. The predator free equilibrium point $E_1(\frac{\delta}{\mu}, 0, \frac{\delta}{\mu n_1})$.
2. The prey free equilibrium point $E_2(0, \frac{\delta}{\eta}, \frac{\delta}{\eta n_2})$.
3. Interior equilibrium point $E^*(u^*, v^*, w^*)$, where

$$v^* = \frac{\delta}{\eta} - \frac{\mu u^*}{\eta}, \; w^* = \frac{r_2 v^*(u^* + v^*)}{n_2 r_2(u^* + v^*) + e_2 n_2 u^*},$$

and u^* obtain by solving the below equation

$$A_0 u^{*4} + A_1 u^{*3} + A_2 u^{*2} + A_3 u^* + A_4 = 0, \tag{3.1}$$

with

$$\begin{aligned}
A_0 &= n_1 e_1 \mu^3 \eta f r_2 - n_1 e_1 \mu^4 f r_2, \\
A_1 &= n_2 e_2 \mu \eta^3 r_1 - n_2 e_2 \eta^4 r_1 + n_1 e_1 \mu^3 \eta r_2 - n_1 e_1 \mu^2 \eta^2 r_2 + 4 n_1 e_1 \delta \mu^3 f r_2 - 3 n_1 e_1 \delta \mu^2 \eta f \\
&\quad - n_1 \mu^3 \eta r_1 r_2 + 2 n_1 \mu^2 \eta^2 r_1 r_2 - n_2 \mu^2 \eta^2 r_1 r_2 + 2 n_2 \mu \eta^3 r_1 r_2 - q \eta^4 r_1 r_2, \\
A_2 &= -n_2 e_2 \delta \eta^3 r_1 - 3 n_1 e_1 \delta \mu^2 \eta r_2 + 2 n_1 e_1 \delta \mu \eta^2 r_2 - 6 n_1 e_1 \delta^2 \mu^2 f r_2 + 3 n_1 \delta \mu^2 \eta r_1 r_2 \\
&\quad - 4 n_1 \delta \mu \eta^2 r_1 r_2 + 2 n_2 \delta \mu \eta^2 r_1 r_2 + n_1 \delta \eta^3 r_1 r_2 - 2 n_2 \delta \eta^3 r_1 r_2, \\
A_3 &= 3 n_1 e_1 \delta^2 \mu \eta r_2 - n_1 e_1 \delta^2 \eta^2 r_2 + 4 n_1 e_1 \delta^3 \mu f r_2 - n_1 e_1 \delta^3 \eta f r_2 - 3 n_1 \delta^2 \mu \eta r_1 r_2 \\
&\quad - n_2 \delta^2 \eta^2 r_1 r_2 + 2 n_1 \delta^2 \eta^2 r_1 r_2, \\
A_4 &= n_1 \delta^3 \eta f r_2 - n_1 e_1 \delta^3 \eta r_2 - n_1 e_1 \delta^4 f r_2.
\end{aligned}$$

Note, it is difficult to say about the positive roots of (3.1). Let us assume that (3.1) has at least on positive u^*, then the model (2.2) has the coexisting equilibrium point $E^*(u^*, v^*, w^*)$ and also satisfies $\delta > \mu u^*$.

4 Local Stability Analysis

In this part, we study the local stability properties of the equilibrium points. For this, the Jacobian matrix is calculated at an arbitrary equilibrium point $E(u, v, w)$ and is given below

$$J(u, v, w) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \tag{4.1}$$

where

$$\begin{aligned}
&A_{11} = \frac{r_1}{1 + fv} - \frac{2 u r_1}{n_1 w (1 + fv)} - \frac{e_1 v^2}{(u+v)^2}, A_{12} = -\frac{r_1 f u}{(1+fv)^2} + \frac{r_1 u^2 f}{(1+fv)^2 n_1 w} - \frac{e_1 u^2}{(u+v)^2}, \\
&A_{13} = \frac{r_1 u^2}{(1+fv) n_1 w^2}, A_{21} = \frac{e_2 v^2}{(u+v)^2}, A_{22} = r_2 - \frac{2 r_2 v}{n_2 w} + \frac{e_2 u^2}{(u+v)^2}, A_{23} = \frac{r_2 v^2}{n_2 w^2}, \\
&A_{31} = -\mu w, A_{32} = -\eta w, A_{33} = \delta - \mu u - \eta y.
\end{aligned}$$

The following theorem is used to analyze the local stability properties of E_1, E_2, and E^*.

Theorem 4.1. For model (2.2), we have

1. The equilibrium point E_1 is unstable.
2. The equilibrium point E_2 is locally asymptotically stable, if $\frac{r_1 \eta}{\eta + f\delta} < e_1$.

3. The interior equilibrium point $E^*(u^*, v^*, w^*)$ is locally asymptotically stable, if $\epsilon_1 > 0, \epsilon_3 > 0$, and $\epsilon_1\epsilon_2 > \epsilon_3$.

Proof .

1. The Jacobian matrix at the $E_1(\frac{\delta}{\mu}, 0, \frac{\delta}{\mu n_1})$ is

$$J(E_1) = \begin{bmatrix} -r_1 & -e_1 & r_1 n_1 \\ 0 & r_2 + e_2 & 0 \\ -\frac{\delta}{n_1} & -\frac{\eta\delta}{n_1\mu} & 0 \end{bmatrix}. \tag{4.2}$$

The eigenvalues of the above Jacobian matrix at E_1 are

$$\lambda_1 = r_2 + e_2, \lambda_{2,3} = \frac{1}{2}\left(-r_1 \pm \sqrt{r_1^2 - 4r_1\delta}\right),$$

which implies, $\lambda_1 > 0$, thus E_1 is unstable.

2. The Jacobian matrix at $E_2(0, \frac{\delta}{\eta}, \frac{\delta}{\eta n_2})$ is

$$J(E_2) = \begin{bmatrix} \frac{r_1\eta}{\eta+f\delta} - e_1 & 0 & 0 \\ e_2 & -r_2 & r_2 n_2 \\ -\frac{\delta\mu}{n_2\eta} & -\frac{\delta}{n_2} & 0 \end{bmatrix}. \tag{4.3}$$

The eigenvalues of $J(E_2)$ are

$$\lambda_1 = \frac{r_1\eta}{\eta + f\delta} - e_1, \lambda_{2,3} = \frac{1}{2}\left(-r_2 \pm \sqrt{r_2^2 - 4r_2\delta}\right) < 0.$$

Then E_2 is locally asymptotically stable, if $\frac{r_1\eta}{\eta+f\delta} < e_1$.

3. The Jacobian matrix at $E^*(u^*, v^*, w^*)$ is

$$J(E^*) = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & 0 \end{bmatrix}, \tag{4.4}$$

where

$$\begin{aligned} &S_{11} = \frac{e_1 u^* v^*}{(u^*+v^*)^2} - \frac{r_1 u^*}{n_1 w^*(1+fv^*)}, S_{12} = \frac{r_1 f u^{*2}}{n_1 w^*(1+fv^*)^2} - \frac{r_1 f u^*}{(1+fv^*)^2} - \frac{e_1 u^{*2}}{(u^*+v^*)^2}, \\ &S_{13} = \frac{r_1 u^{*2}}{(1+fv^*)n_1 w^{*2}}, S_{21} = \frac{e_2 v^{*2}}{(u^*+v^*)^2}, S_{22} = -\frac{e_2 u^* v^*}{(u^*+v^*)^2} - \frac{r_2 v^*}{n_2 w^*}, S_{23} = \frac{r_2 v^{*2}}{n_2 w^{*2}}, \\ &S_{31} = -\mu w^*, S_{32} = -\eta w^*. \end{aligned}$$

The corresponding characteristic equation of (4.4) is

$$\rho^3 + \epsilon_1\rho^2 + \epsilon_2\rho + \epsilon_3 = 0, \tag{4.5}$$

where,

$$\begin{aligned} &\epsilon_1 = -(S_{11} + S_{22}), \epsilon_2 = S_{11}S_{22} - S_{12}S_{21} - S_{23}S_{32} - S_{13}S_{31}, \\ &\epsilon_3 = S_{11}S_{23}S_{32} + S_{13}S_{22}S_{31} - S_{12}S_{23}S_{31} - S_{13}S_{21}S_{32}. \end{aligned}$$

By Routh-Hurwitz criteria [34], E^* is locally asymptotically stable, if $\epsilon_1 > 0, \epsilon_3 > 0$, and $\epsilon_1\epsilon_2 > \epsilon_3$.

□

5 Hopf-bifurcation Analysis

In this part, we examine the model's bifurcation by analyzing it analytically based on the effect of fear f. The below theorem shows that the existence of Hopf-bifurcation by taking fear parameter as a varying parameter.

Theorem 5.1. The model (2.2) undergoes the Hopf-bifurcation if bifurcation parameter f (fear parameter) exceeds a critical value. The occurrence of Hopf-bifurcation conditions at $f = f^*$ as follows

1. $H(f^*) = \epsilon_1(f^*)\epsilon_2(f^*) - \epsilon_3(f^*) = 0$,
2. $\frac{d}{df}(Re(\rho(f)))\Big|_{f=f^*} \neq 0.$
   Where, ρ is the zero of characteristic equation associated with interior equilibrium point.

Proof .

For $f = f^*$, characteristic equation (4.5) must be of the form

$$(\rho^2(f^*) + \epsilon_2(f^*))(\rho(f^*) + \epsilon_1(f^*)) = 0. \tag{5.1}$$

The roots of the above equations are $\pm i\sqrt{\epsilon_2(f^*)}$ and $-\epsilon_1(f^*)$. To establish that the Hopf-bifurcation occurs at $f^* = f$, it is necessary to satisfy the following transversality condition.

$$\frac{d}{df}(Re(\rho(f)))\Big|_{f=f^*} \neq 0.$$

For all f, the roots are generally in the from

$$\begin{aligned}\rho_1(f) &= r(f) + is(f),\\ \rho_2(f) &= r(f) - is(f),\\ \rho_3(f) &= -\epsilon_1(f).\end{aligned}$$

Now, we check the condition

$$\frac{d}{df}(Re(\rho_j(f)))\Big|_{f=f^*} \neq 0, j = 1, 2.$$

Let, $\rho_1(f) = r(f) + is(f)$ in (5.1), we get

$$A(f) + iB(f) = 0,$$

where

$$\begin{aligned}A(f) &= r^3(f) + r^2(f)\epsilon_1(f) - 3r(f)s^2(f) - s^2(f)\epsilon_1(f) + \epsilon_2(f)r(f) + \epsilon_1(f)\epsilon_2(f),\\ B(f) &= \epsilon_2(f)s(f) + 2r(f)s(f)\epsilon_1(f) + 3r^2(f)s(f) - s^3(f).\end{aligned}$$

To fulfill equation (5.1), we must have $A(f) = 0$ and $B(f) = 0$. Then differentiating A and B with respect to f, we have

$$\frac{dA}{df} = \psi_1(f)r'(f) - \psi_2(f)s'(f) + \psi_3(f) = 0, \tag{5.2}$$

$$\frac{dB}{df} = \psi_2(f)r'(f) + \psi_1(f)s'(f) + \psi_4(f) = 0, \tag{5.3}$$

where

$$\begin{aligned}&\psi_1(f) = 3r^2(f) + 2r(f)\epsilon_1(f) - 3s^2(f) + \epsilon_2(f), \psi_2(f) = 6r(f)s(f) + 2s(f)\epsilon_1(f),\\ &\psi_3(f) = r^2(f)\epsilon_1'(f) - s^2(f)\epsilon_1'(f) + \epsilon_2'(f)r(f), \psi_4(f) = 2r(f)s(f)\epsilon_1'(f) + \epsilon_2'(f)s(f).\end{aligned}$$

On multiplying (5.2) and (5.3) by $\psi_1(f)$ and $\psi_2(f)$ respectively, then sum the two equations, we have

$$r'(f) = -\frac{\psi_1(f)\psi_3(f) + \psi_2(f)\psi_4(f)}{\psi_1^2(f) + \psi_2^2(f)}. \tag{5.4}$$

Substituting $r(f)=0$ and $s(f)=\sqrt{\epsilon_2(f)}$ at $f=f^*$ on $\psi_1(f),\psi_2(f),\psi_3(f)$ and $\psi_4(f)$, we obtain

$$\psi_1(f^*)=-2\epsilon_2(f^*),\psi_2(f^*)=2\epsilon_1(f^*)\sqrt{\epsilon_2(f^*)},$$
$$\psi_3(f^*)=\epsilon_3'(f^*)-\epsilon_2(f^*)\epsilon_1'(f^*),\psi_4(f^*)=\epsilon_2'(f^*)\sqrt{\epsilon_2(f^*)}.$$

The equation (5.4), implies

$$r'(f^*)=\frac{\epsilon_3'(f^*)-(\epsilon_1(f^*)\epsilon_2(f^*))'}{2\left(\epsilon_2^2(f^*)+\epsilon_1^2(f^*)\right)}. \tag{5.5}$$

If $\epsilon_3'(f^*)-(\epsilon_1(f^*)\epsilon_2(f^*))'\neq 0$, which implies that

$$\frac{d}{df}(Re(\rho_j(f)))\bigg|_{f=f^*}=r'(f^*)\neq 0, j=1,2.$$

and

$$\rho_3(f^*)=-\epsilon_1(f^*)\neq 0.$$

Therefore, the condition $\epsilon_3'(f^*)-(\epsilon_1(f^*)\epsilon_2(f^*))'\neq 0$ is guaranteed that the condition for transversality is satisfied, hence the model (2.2) has enter into Hopf-bifurcation at $f=f^*$ □

6 Bifurcation Analysis with Time Delay

In this part, we analyze the model (2.2) in the presence of time delay. It may take some time to digest u, i.e., the ratio of v to the delay in converting the food consumed into energy. So, the model (2.2) takes the following form with the effect of delay τ as

$$\begin{aligned}
\frac{du}{dt}&=\frac{r_1u}{1+fv}\left(1-\frac{u}{n_1w}\right)-\frac{e_1uv}{u+v},\\
\frac{dv}{dt}&=r_2v(1-\frac{v}{n_2w})+\frac{e_2u(t-\tau)v(t-\tau)}{u(t-\tau)+v(t-\tau)},\\
\frac{dw}{dt}&=w\left(\delta-\mu u-\eta v\right),
\end{aligned} \tag{6.1}$$

subject to the initial conditions $u_0(\theta)=\Lambda_1(\theta)>0, v_0(\theta)=\Lambda_2(\theta)>0$, and $w_0(\theta)=\Lambda_3(\theta)>0$, where $\theta\in[-\tau,0]$.

6.1 Local Stability and Hopf-bifurcation Analysis

To get the local stability of $E^*(u^*,v^*,w^*)$, use the following transformations $\bar{u}=u(t)-u^*$, $\bar{v}=v(t)-v^*$, and $\bar{w}=w(t)-w^*$ where, $\bar{u},\bar{v}$, and $\bar{w}$ are the small perturbation around u^*,v^*, and w^*, respectively. Then we get the linearized model of the form

$$\begin{pmatrix}\dot{u}\\ \dot{v}\\ \dot{w}\end{pmatrix}=\begin{pmatrix}f_{11}&f_{12}&f_{13}\\0&f_{22}&f_{23}\\f_{31}&f_{32}&f_{33}\end{pmatrix}\begin{pmatrix}u^*\\v^*\\w^*\end{pmatrix}+\begin{pmatrix}0&0&0\\f_{21}&f_{24}&0\\0&0&0\end{pmatrix}\begin{pmatrix}u^*(t-\tau)\\v^*(t-\tau)\\w^*(t-\tau)\end{pmatrix}, \tag{6.2}$$

where

$$\begin{aligned}
f_{11}&=\frac{r_1}{1+fv^*}-\frac{2u^*r_1}{n_1w^*(1+fv^*)}-\frac{e_1v^{*2}}{(u^*+v^*)^2}, f_{12}=-\frac{r_1fu^*}{(1+fv^*)^2}+\frac{fr_1u^{*2}}{n_1w^*(1+fv^*)^2}-\frac{e_1u^{*2}}{(u^*+v^*)^2},\\
f_{13}&=\frac{r_1u^{*2}}{n_1w^{*2}(1+fv^*)}, f_{22}=r_2-\frac{2r_2v^*}{n_2w^*}, f_{23}=\frac{r_2v^{*2}}{n_2w^{*2}}, f_{21}=\frac{e_2v^{*2}}{(u^*+v^*)^2},\\
f_{24}&=\frac{e_2u^{*2}}{(u^*+v^*)^2}, f_{31}=-\delta w^*, f_{32}=-\eta w^*, f_{33}=\delta-\mu u^*-\eta v^*.
\end{aligned}$$

Characteristic equation of (6.2) is

$$\begin{vmatrix}f_{11}-\lambda & f_{12} & f_{13}\\ f_{21}e^{-\lambda\tau} & f_{22}+f_{24}e^{-\lambda\tau}-\lambda & f_{23}\\ f_{31} & f_{32} & f_{33}-\lambda\end{vmatrix}=0,$$

which implies that

$$\lambda^3 + (K_1\lambda^2 + K_2\lambda + K_3) + (K_4\lambda^2 + K_5\lambda + K_6)e^{-\lambda\tau} = 0, \tag{6.3}$$

where

$$\begin{aligned}
&K_1 = -(f_{11} + f_{22} + f_{33}), K_2 = f_{11}f_{22} + f_{11}f_{33} + f_{22}f_{33} - f_{23}f_{32} - f_{13}f_{31},\\
&K_3 = f_{11}f_{23}f_{32} + f_{13}f_{22}f_{31} - f_{11}f_{22}f_{33} - f_{12}f_{22}f_{31}, K_4 = -f_{22},\\
&K_5 = f_{11}f_{22} + f_{24}f_{33} - f_{12}f_{21}, K_6 = f_{12}f_{21}f_{33} + f_{13}f_{24}f_{31} - f_{11}f_{24}f_{33} - f_{13}f_{21}f_{32}.
\end{aligned}$$

If $\lambda = i\gamma(\gamma > 0)$ is a zero of (6.3), we have

$$-i\gamma^3 - \gamma^2K_1 + i\gamma K_2 + K_3 + (-\gamma K_4 + i\gamma K_5 + K_6)e^{-i\gamma\tau} = 0.$$

On simplification, we have

$$\gamma^2K_1 - K_3 = (K_6 - \gamma^2K_4)cos\gamma\tau + \gamma K_5 sin\gamma\tau, \tag{6.4}$$

$$\gamma K_2 - \gamma^3 = -\gamma K_5 cos\gamma\tau + (K_6 - \gamma^2K_4)sin\gamma\tau. \tag{6.5}$$

From (6.4) and (6.5), we have

$$\gamma^6 + (K_1^2 - K_4^2 - 2K_2)\gamma^4 + (K_2^2 - 2K_1K_3 + 2K_4K_6 - K_5^2)\gamma^2 + (K_3^2 - K_6^2) = 0, \tag{6.6}$$

substituting $\tau = \hat{\tau}$ and $\gamma = \hat{\gamma}$ in above equation, which implies that

$$\hat{\gamma}^6 + (K_1^2 - K_4^2 - 2K_2)\hat{\gamma}^4 + (K_2^2 - 2K_1K_3 + 2K_4K_6 - K_5^2)\hat{\gamma}^2 + (K_3^2 - K_6^2) = 0. \tag{6.7}$$

Let, $\hat{\gamma}^2 = m$, we get

$$m^3 + \sigma_1m^2 + \sigma_2m + \sigma_3 = 0, \tag{6.8}$$

where

$$\begin{aligned}
\sigma_1 &= K_1^2 - K_4^2 - 2K_2,\\
\sigma_2 &= K_2^2 - 2K_1K_3 + 2K_4K_6 - K_5^2,\\
\sigma_3 &= K_3^2 - K_6^2.
\end{aligned}$$

By the Descartes' identity rule [34], if σ_1 and σ_3 are opposite signs, then (6.7) has at least one positive root. Then the discussion on the the local stability and bifurcation behavior of model (6.1) near E^* on the basis of parameter τ is summarized in the following theorem.

Theorem 6.1. *Let as assume that E^* exists and is locally asymptotically stable when $\tau = 0$. Also, assume the equation (6.7) has the positive root $\hat{\gamma}^2 = m$.*

i. *Then, there is $\tau = \tau^*$, i.e. the model (6.1) at E^* is locally asymptotically stable for $\tau \in [0, \tau^*)$ and if, $\tau > \tau^*$ the model (6.1) becomes unstable.*

ii. *Also, when $\tau = \tau^*$, the model (6.1) has enters into a Hopf-bifurcation at E^* ,
if $2\gamma^6 + (K_1^2 - 2K_2 - K_4^2)\gamma^4 - K_5^2 + K_6^2 \neq 0$*

Proof . Since, $\hat{\gamma_i}$ be the zero of (6.8) and the equation (6.3) has couple of purely imaginary roots $\pm i\gamma$. From (6.4) and (6.5), we have τ_i^* as as function of $\hat{\gamma_i}$ for $i = 0, 1, 2, \cdots$, which gives

$$\tau_i^* = \frac{1}{\hat{\gamma_i}}\left\{\arccos\left[\frac{(\hat{\gamma_i^2}K_1 - K_3)(K_6 - K_4\hat{\gamma_i^2}) - \hat{\gamma_i}K_5(\hat{\gamma}K_2 - \hat{\gamma^3})}{(K_6 - K_4\hat{\gamma_i^2})^2 + (K_5\hat{\gamma_i^2})^2}\right] + 2i\pi\right\}. \tag{6.9}$$

By Butlers lemma [35], E^* is stable if $\tau < \tau^*$ and unstable if $\tau \geq \tau^*$, which is given the following condition is met.

To get that condition, differentiate (6.4), we have

$$\begin{aligned}
\frac{d\lambda}{d\tau} &= \frac{\lambda e^{-\lambda\tau}(K_4\lambda^2 + K_5\lambda + K_6)}{3\lambda^2 + 2K_1\lambda - \tau(K_4\lambda^2 + K_5\lambda + K_6)e^{-\lambda\tau}},\\
\left(\frac{d\lambda}{d\tau}\right)^{-1} &= \frac{(3\lambda^2 + 2K_1\lambda + K_2)e^{-\lambda\tau} + 2K_4\lambda + K_5}{\lambda(K_4\lambda^2 + K_5\lambda + K_6)} - \frac{\tau}{\lambda},\\
Re\left\{\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\} &= \frac{2\gamma^6 + (K_1^2 - 2K_2 - K_4^2)\gamma^4 - K_5^2 + K_6^2}{\gamma^2(K_6 - K_4\gamma^2)^2 + \gamma^4K_5^2} \neq 0.
\end{aligned}$$

It is guaranteed, if $2\gamma^6 + (K_1^2 - 2K_2 - K_4^2)\gamma^4 - K_5^2 + K_6^2 \neq 0$. Hence the model (6.1) undergo a Hopf-bifurcation at $\tau = \tau^*$. $\square$

7 Numerical Simulations

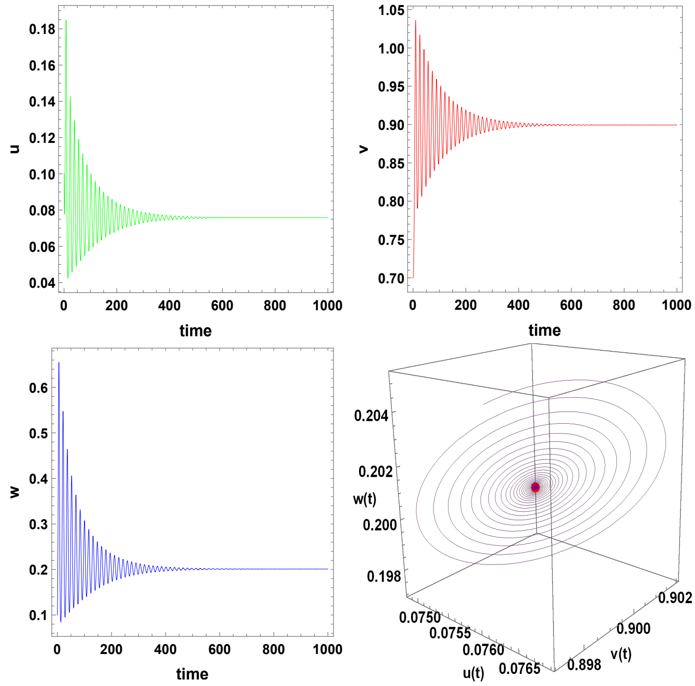

Figure 1: The time evaluation of intraguild prey, intraguild predator and biotic resource and phase portrait for the model (2.2) when $f = 0.2$, and the red point represents $E^*(0.075821, 0.899442, 0.200972)$.

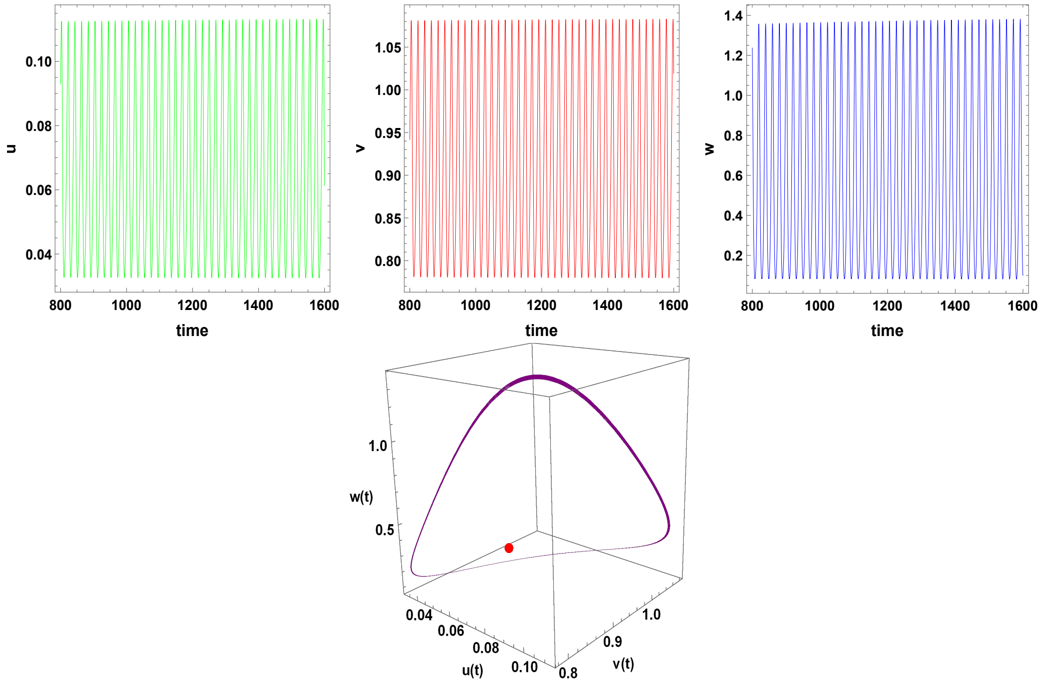

Figure 2: The oscillating time evaluation and phase portrait for the model (2.2) with $f = 0.5$, and the red point represents $E^*(0.0568578, 0.914628, 0.264678)$.

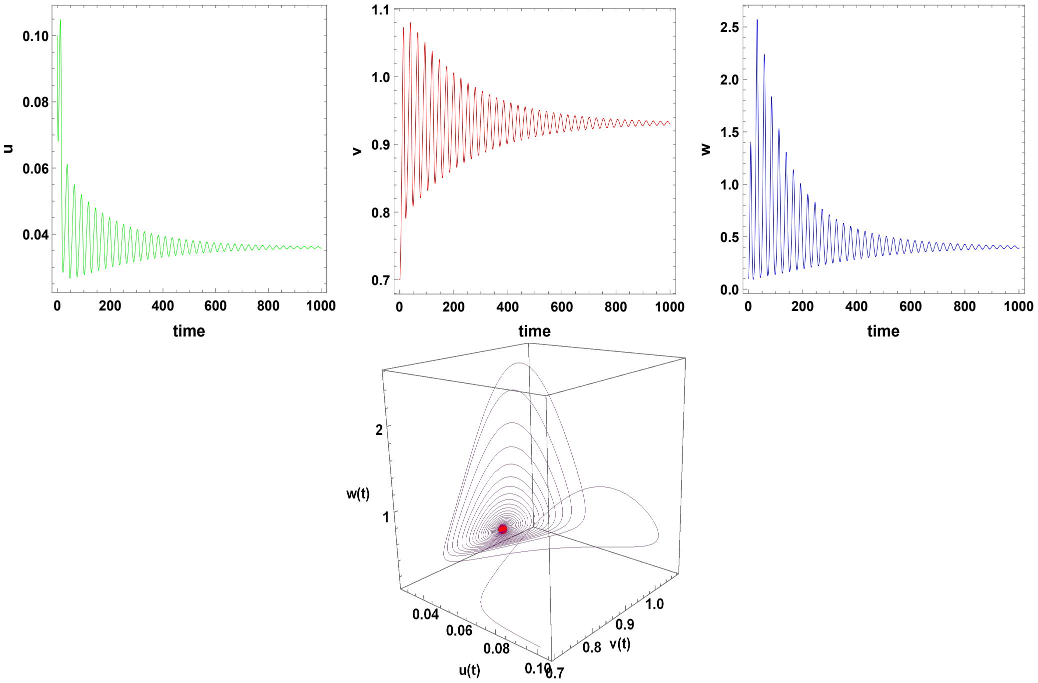

Figure 3: The time evaluation of intraguild prey, intraguild predator and biotic resource and phase portrait for the model (2.2) when $f = 0.7$, and the red point represents $E^*(0.0359866, 0.931341, 0.40073)$.

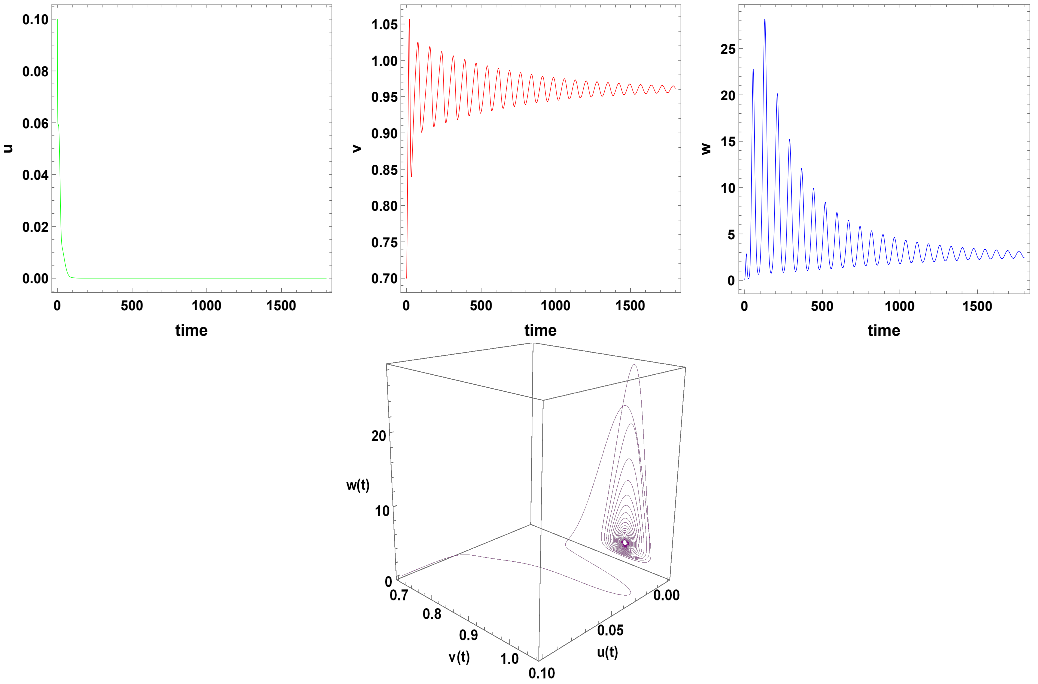

Figure 4: Represents the extinction of the intraguild prey in time-series and phase portrait for the model (2.2) with $f = 1$.

In this part, we give numerical simulations based on our analytic conclusions to highlight the impact of fear on the non-delayed model (2.2) and to illustrate our theoretical conclusions. The simulations are performed using MATLAB's ode45 and dde23 solvers for 1800-time steps. The fixed parameter values of the model (2.2) are as $r_1 = 1.01$, $r_2 = 0.0031$, $\delta = 2.41$, $\mu = 2.01$, $\eta = 2.51$, $n_1 = 1$, $n_2 = 0.35$, $e_2 = 0.47$, $e_1 = 0.578$, and varying $f \in (0, 1]$. By choosing $f = 0.2$, we have the predator free equilibrium $E_1(1.199, 0, 1.199)$, the prey free equilibrium point $E_2(0, 0.960159, 2.74331)$, and the coexisting equilibrium point $E^*(0.075821, 0.899442, 0.200972)$ are exists. Then from Theorem 4.1 the equilibria E_1 and E_2 are

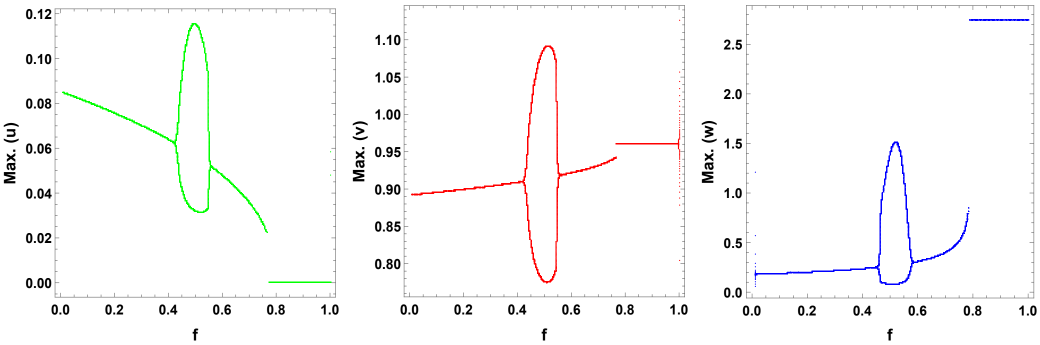

Figure 5: The bifurcation diagram for the model (2.2) with $f \in (0, 1]$, the Hopf-bifurcation occurs at $f = 0.41$ and $f = 0.57$.

unstable according to the eigenvalues of the Jacobian matrices (4.2) and (4.3). Again, the Jacobian matrix at the positive equilibrium E^* has complex conjugate eigenvalues with negative real part as the inequalities $\epsilon_1 > 0, \epsilon_3 > 0$ and $\epsilon_1\epsilon_2 > \epsilon_3$ are hold. Hence, the interior equilibrium E^* is locally asymptotically stable, we can say that the all three populations prey, predator, and biotic resource coexists and survive long time for the smaller value of fear parameter $f = 0.2$. Thus, the locally asympotically stable time series and phase portrait is shown in Figure 1. On increasing the fear parameter f gradually, we have $E^*(0.0568578, 0.914628, 0.264678)$ for $f = 0.5$ which is unstable and the near by trajectories shows oscillating behavior i.e., limit cycle exist. Thus, the existence of limit cycle is clearly depicted in Figure 2. This stability loss and the birth of periodic oscillations is due to the Hopf-bifurcation when the fear criterion passes the critical value $f = f^* = 0.41$. From the results derived in the section 5, for $f = f^* = 0.41$ we obtain $\epsilon_1(f^*)\epsilon_2(f^*) - \epsilon_3(f^*) = 0.000184258$, $\epsilon_3'(f^*) - (\epsilon_1(f^*)\epsilon_2(f^*))' = 0.0397351 \neq 0.$, and also, Theorem 5.1 ensures the presence of Hopf-bifurcation for the model (2.2). This oscillations reduces on further increasing the cost of fear criterion. At $f = 0.7$, we have locally asymptotically stable equilibrium $E^*(0.0359866, 0.931341, 0.40073)$, see Figure 3. Thus the unstable E^* becomes stable via the Hopf-bifurcation, when f crosses the critical value $f = 0.57$. For larger value of fear $f = 1$, the prey population eventually reduces and goes extinct, only the predator and biotic resources exists, see Figure 4. Hence, the fear coefficient should be less than some critical value, such that all three species coexist and survive for long time. For the clear representation, the changes in dynamics of the model (2.2) with the effect of fear parameter f, the bifurcation diagram is depicted in Figure 5. It is shown that E^* of the model (2.2) is locally asymptotically stable for $f \in (0, 0.41)$, exhibits periodic oscillations for $f \in (0.41, 0.57)$, locally asymptotically stable for $f \in (0.57, 0.755)$, and prey goes extinction for $f \in (0.755, 1.0)$. Thus, the Hopf bifurcation occurs at two critical points $f = 0.41$ and $f = 0.57$ of the fear coefficient near E^* of the model (2.2).

In order to validate the analytical findings derived for delayed model (6.1), we consider the same set of values of parameters given above. The introduction of gestation delay does not affect the equilibrium point. Therefore $E^*(0.075821, 0.899442, 0.200972)$ is an interior equilibrium point of system (6.1). By taking τ as a bifurcation parameter. Here we fix the fear parameter $f = 0.2$ and varying the time delay in the range $\tau \in (0, 1]$. We have the coexisting equilibrium point $E^*(0.075821, 0.899442, 0.200972)$ for $f = 0.2$. The E^* is locally asymptotically stable for $\tau \in (0, 0.64942]$, losses its stability and undergoes Hopf-bifurcation when the time delay reaches the critical value $\tau = 0.64942$. Also, the transversality condition $Re\left\{\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\} = 21.9154$ holds form Theorem 6.1. The locally locally asymptotically stable E^* for $\tau = 0.3$ is depicted in Figure 6 and occurrence of periodic oscillations for $\tau = 0.75$ is depicted in Figure 7. Finally, in order to show the existence of Hopf-bifurcation, for the better visualization of the dynamical changes of the system (6.1), the bifurcation diagram is plotted in Figure 8 for $k = 0.2$ and $\tau \in [0.6, 1]$. It is shown that for the delayed model (6.1), E^* is locally asymptotically stable for $\tau \in (0, 0.64942)$, undergoes Hopf bifurcation at $\tau = \tau^* = 0.64942$, and exhibits periodic oscillations for $\tau \in (0.64942, 1)$.

8 Conclusion

In this current work, we have modified and studied the intraguild predation model discussed by Putra et al. [32] with a ratio-dependent functional response. We considered that, while the rate of fear of predators increases, the intrinsic rate of growth of the prey population decreases. Also, the conversion of food into predator is not attained immediately; there is a time lag in the predator's gestation process. As a result, in order to make the model more realistic, we included a time-delay in the predator interaction term. The primary goal of this work is to investigate the impact of the proposed model on the cost of fear in the prey population with and without time delay. To accomplish our goals,

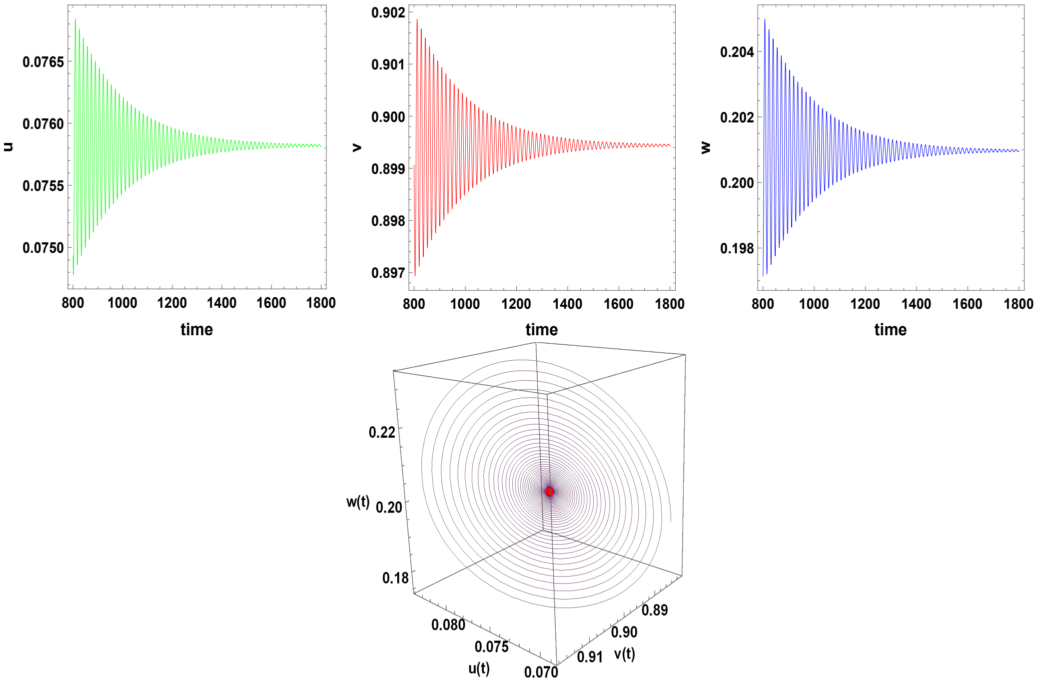

Figure 6: The time evaluation of intraguild prey, intraguild predator and biotic resource and phase portrait for the model (6.1) when $f = 0.2$ and $\tau = 0.3$.

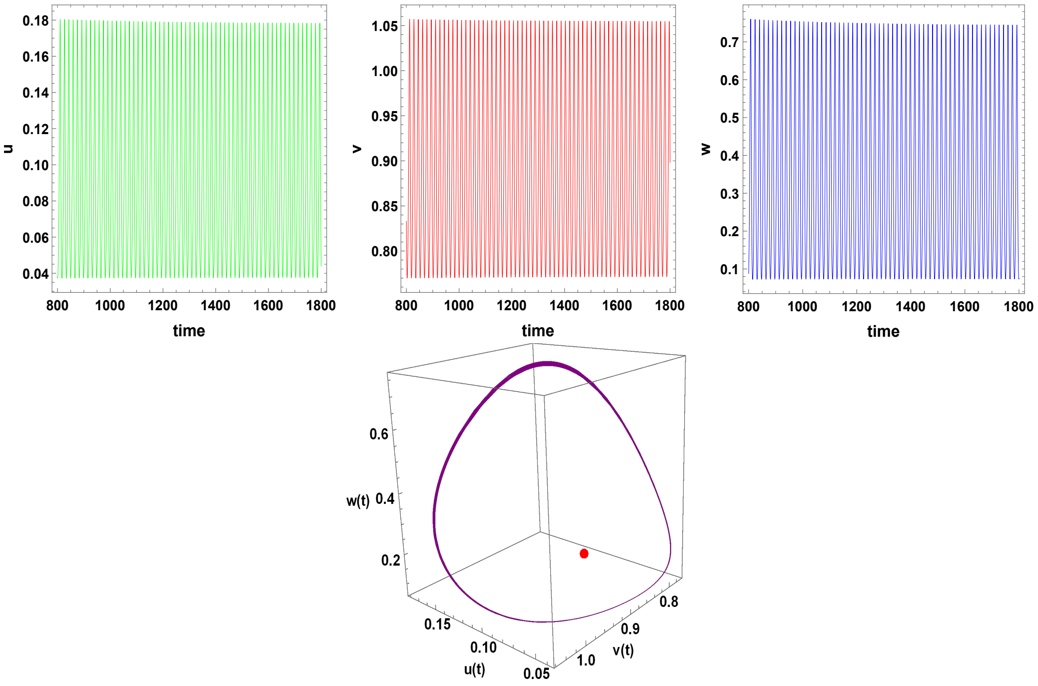

Figure 7: The oscillating time evaluation and phase portrait for the model (6.1) with $f = 0.2$ and $\tau = 0.75$.

we explored our proposed model both analytically and numerically. Based on that, we linked the fear of predators with the development of their corresponding prey species. First, we demonstrated the existence of positive equilibrium points analytically. Furthermore, we have analyzed the local stability and the appearance of Hopf-bifurcation for the proposed model (2.2), and we discovered that changing the cost of fear f affects the stability of the proposed model (2.2) without delay. Moreover, we performed local stability and Hopf-bifurcation analyses of the proposed model with some time delay. For larger values of the delay parameter, this causes Hopf-bifurcation around E^*. The developed analytical discussions are confined with the help simulation results. We conclude that the presence of fear and delay

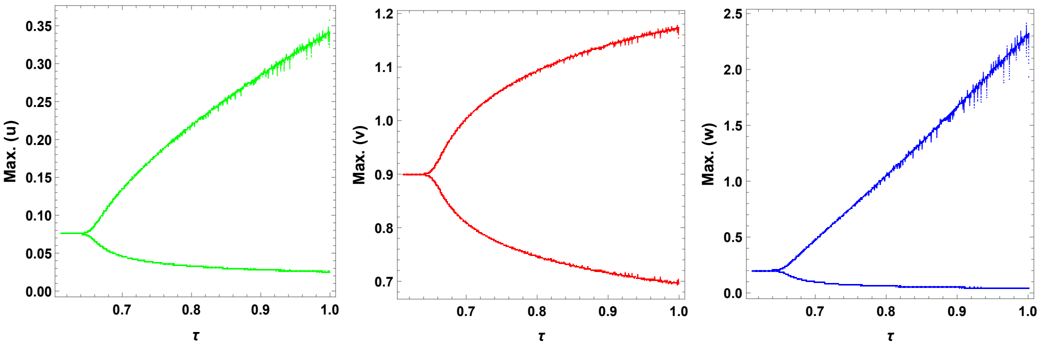

Figure 8: The bifurcation diagram for the model (6.1) with $f = 0.2$ and $\tau \in [0.6, 1]$, the Hopf-bifurcation occurs at $\tau = 0.64942$.

has a greater impact on stability switches via the Hopf-bifurcation in the proposed models. Finally, we draw the time series, phase portraits, and bifurcation diagrams for the non-delayed and delayed models with respect to fear effect and time delay.

In this work, we only explored the ratio-dependent intraguild predation model (2.2) with effect to fear and time delay. Other functional responses may provide some intriguing extensions to our model (2.2) with different biological situations. It could be significant from the standpoint of biological diversity. We will leave it for future work.

References

[1] C.S. Holling, *The functional response of predators to prey density and its role in mimicry and population dynamics*, Mem. Entomol. Soc. **97** (1965), no. 45, 560.

[2] S. Vinoth, R. Sivasamy, K. Sathiyanathan, B. Unyong, R. Vadivel and N. Gunasekaran, *A novel discrete-time Leslie–Gower model with the impact of Allee effect in predator population*, Complexity, **2022** (2022), 6931354.

[3] S. Magudeeswaran, S. Vinoth, K. Sathiyanathan and M. Sivabalan, *Impact of fear on delayed three species food-web model with Holling type-II functional response*, Int. J. Biomath. **15** (2022), no. 4, 2250014.

[4] P.H. Crowley, *Functional responses and interference within and between year classes of a dragonfly population*, J. N. Am. Benthol. Soc. **8** (1989), no. 3, 211–221.

[5] J.R. Beddington, *Mutual interference between parasites or predators and its effect on scarcity efficiency*, J. Anim. Ecol. **97** (1975), no. 45, 560.

[6] R. Arditi and L.R. Ginzburg, *Coupling in predator-prey dynamics: Ratio-dependence*, J. Theor. Biol. **139** (1989), 311–326.

[7] C. Jost, O. Arino and R. Arditi, *About deterministic extinction in ratio-dependent predator-prey models*, Bull. Math. Biol. **61** (1999), no. 1, 19–32.

[8] F. Berezovskaya, G. Karev and R. Arditi, *Parametric analysis of the ratio-dependent predator-prey model*, J. Math. Biol. **43** (2001), no. 3, 221–246.

[9] M. Agarwal and V. Singh, *Persistence and stability for the three species ratio-dependent food-chain model*, Int. J. Eng. Sci. Technol. **3** (2011), no. 1, 305–322.

[10] J.D. Flores and E. Gonzalez-Olivares, *A modified Leslie-Gower predator-prey model with ratio-dependent functional response and alternative food for the predator*, Math. Methods Appl. Sci. **40** (2017), no. 7, 2313–2328.

[11] X. Jiang, Z. She, Z. Feng and X. Zheng, *Bifurcation analysis of a predator-prey system with ratio-dependent functional response*, Int. J. Bifurc. Chaos **27** (2017), no. 14, 1750222.

[12] S. Khajanchi and S. Banerjee, *Role of constant prey refuge on stage structure predator-prey model with ratio dependent functional response*, Appl. Math. Comput. **314** (2017), 193–198.

[13] R. Sivasamy, M. Sivakumar, K. Balachandran and K. Sathiyanathan, S*patial pattern of ratio-dependent predator-prey model with prey harvesting and cross-diffusion*, Int. J. Bifurc. Chaos **29** (2019), no. 3, 1950036.

[14] J. Roy, D. Barman and S. Alam, *Role of fear in a predator-prey system with ratio-dependent functional response in deterministic and stochastic environment*, Biosyst. **197** (2020), 104176.

[15] W.B. Cannon, *Bodily changes in pain, hunger, fear, and rage*, D. Appleton and company, New York and London, 1915.

[16] S. Pal, S. Majhi, S. Mandal and N. Pal, *Role of fear in a predator-prey model with Beddington-DeAngelis functional response*, Z. Naturforsch. **74** (2019), no. 7, 581–595.

[17] S. Vinoth, R. Sivasamy, K. Sathiyanathan, B. Unyong, G. Rajchakit, R. Vadivel and N. Gunasekaran, *The dynamics of a Leslie type predator-prey model with fear and Allee effect*, Adv. Differ. Equ. **338** (2021), 1–22.

[18] M. Hossain, N. Pal, S. Samanta and J. Chattopadhyay, *Fear Induced Stabilization in an intraguild predation model*, Int. J. Bifurc. Chaos **30** (2020), no. 4, 2050053.

[19] V. Kumar and N. Kumari, *Controlling chaos in three species food chain model with fear effect*, AIMS Math., **5** (2020), no. 2, 828–842.

[20] P. Panday, S. Samanta, N. Pal and J. Chattopadhyay, *Delay induced multiple stability switch and chaos in a predator-prey model with fear effect*, Math. Comput. Simul. **172** (2020), 134–158.

[21] Y. Takeuchi, W. Ma and E. Beretta, *Global asymptotic properties of a delay SIR epidemic model with finite incubation times*, Nonlinear Anal. Real World Appl. **42** (2000), no. 6, 931–947.

[22] S.A. Gourley and Y. Kuang, *A stage structured predator-prey model and its dependence on maturation delay and death rate*, J. Math. Biol. **49** (2004), no. 2, 188–200.

[23] A. Maiti, A.K. Pal and G.P. Samanta, *Effect of time-delay on a food chain model*, Appl. Math. Comput. **200** (2008), no. 1, 189–203.

[24] N. Pal, S. Samanta, S. Biswas, M. Alquran, K. Al-Khaled and J. Chattopadhyay, *Stability and bifurcation analysis of a three-species food chain model with delay*, Int. J. Bifurc. Chaos **25** (2015), no. 9, 1550123.

[25] X.P. Yan and C.H. Zhang, *Hopf bifurcation in a delayed Lokta-Volterra predator-prey system*, Nonlinear Anal. Real World Appl. **9** (2008), no. 1, 114–127.

[26] S. Vinoth, R. Sivasamy, K. Sathiyanathan, G. Rajchakit, P. Hammachukiattikul, R. Vadivel and N. Gunasekaran, *Dynamical analysis of a delayed food chain model with additive Allee effect*, Adv. Differ. Equ. **54** (2021), 1–20.

[27] C. Xu and M. Liao, *Bifurcation behaviours in a delayed three-species food-chain model with Holling type-II functional response*, Appl. Anal. **92** (2013), no. 12, 2468–2486.

[28] Y. Song, T. Yin and H. Shu, *Dynamics of a ratio-dependent stage-structured predator-prey model with delay*, Math. Methods Appl. Sci. **40** (2017), no. 8, 6451–6467.

[29] S. Magudeeswaran, K. Sathiyanathan, R. Sivasamy, S. Vinoth and M. Sivabalan, *Analysis on dynamics of delayed intraguild predation model with ratio-dependent functional response*, Discontin. Nonlinearity Complex. **10** (2021), no. 3, 381–396.

[30] H.M. Safuan and S.B. Musa, *Food chain model with competition interaction in environment of a biotic resource*, AIP Conf. Proc. **1750** (2016), 030015.

[31] S.N. Abdullah, H.M. Safuan and M.E. Nor, *Harvesting effects on population model of competitive interaction with shared biotic resource*, AIP Conf. Proc. **1974** (2018), 020078.

[32] M. Putra, A.J. Dwi, A. Suryanto and N. Shofianah, *Stability analysis and Hopf bifurcation of intraguild predation model with Holling type II functional response*, Int. J. Ecol. Dev. **32** (2017), no. 4, 110–118.

[33] L.Y. Zanette, A.F. White, M.C. Allen and M. Clinchy, *Perceived predation risk reduces the number of offspring songbirds produce per year*, Science **334** (2011), 1398–1401.

[34] J.D. Murray, *Mathematical biology I: An introduction*, Springer, 2002.

[35] H.I. Freedman and V.S.H. Rao, *The trade-off between mutual interference and time lags in predator-prey systems*, Bull. Math. Biol. **45** (1983), 991–1004.

[36] W. Cresswell, *Predation in bird populations*, J. Ornithol. **152** (2011), no. 1, 251—263.