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# Second-order evolution problems by time and state-dependent maximal monotone operators and set-valued perturbations

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**Abstract**

The present work is concerned with a class of perturbed second-order differential inclusions driven by state-dependent maximal monotone operators, in the context of an infinite-dimensional framework. To prove our existence theorem, we make crucial use of a compactness type assumption on the domain of the operators and succeed in adapting to the second-order setting a discretization approach. An application of our main result to the theory of quasi-variational evolution inequalities is provided.

Keywords: Differential inclusion, second-order, maximal monotone operator, pseudo-distance, quasi-variational inequality
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## 1 Introduction

First-order evolution problems governed by maximal monotone operators have been developed by several authors in the scientific literature, see for instance [10], [11], [12], [13], [19], [26], [28], [29], [33], [36], [43], [48]. They have found many contributions such as sweeping processes [3], [38], [39], [40], Skorokhod problems [50], hysteresis operators [32], and Lur'e dynamical systems [51].

In the vein of various works involving an assumption on the operators using the pseudo-distance in [52] (see (2.2) below), we introduce a new class of second-order differential inclusions described by

$$\begin{cases} -\dot{u}(t) \in A(t,x(t))u(t) + F(t,x(t),u(t)) \quad \text{a.e. } t \in I := [0,T], \\ x(t) = x_0 + \int_0^t u(s)ds, \quad t \in I \\ u(t) \in D(A(t,x(t))), \quad t \in I \\ u(0) = u_0 \in D(A(0,x_0)),\ x(0) = x_0 \in H. \end{cases} \tag{1.1}$$

For each $(t,y) \in I \times H$, $A(t,y)$ is a maximal monotone operator of domain $D(A(t,y))$ in a real separable Hilbert space $H$. The variation of these operators is absolutely continuous on time, and Lipschitz continuous on state, in the sense of the pseudo-distance (see $(H_1)$ below). In addition, a compactness assumption on $D(A(t,y))$ is required to perform a discrete schema ensuring the existence of the solution to (1.1), also imposed in Theorem 3.1 [46] dealing with a perturbation containing a finite delay. The latter has been replaced by an anti-monotone condition on $D(A(t,y))$ in

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a recent work (see Theorem 6 [45]). While all these aforementioned works rely on a discretisation approach, a fixed point theorem has been used in Theorem 2 [23].
The perturbation force is supposed to be scalarly upper semi-continuous with convex closed values and only needs to satisfy a linear growth condition: for some non-negative function $\kappa(\cdot) \in L^2_{\mathbb{R}}(I)$, one has

$$F(t,x,y) \subset \kappa(t)(1 + \|x\| + \|y\|)\overline{B}_H \text{ for all } (t,x,y) \in I \times H \times H,$$

where $\overline{B}_H$ denotes the closed unit ball of $H$. Let us point out that the related results established under a pseudo-distance assumption on $A(t)$ or $A(t,x)$ involve a constant instead of $\kappa(\cdot)$ in the growth condition above (even when $F$ is a single-valued map), see e.g., [11], [12], [13], [23], [45], [46], [48].

Among contributions on second-order evolution problems driven by maximal monotone or subdifferential operators, see [1], [2], [9], [14], [23], [25], [42], [45], [46], [47] and the references therein.

Much attention has been devoted to second-order sweeping process which is an important class of differential inclusions expressed with $A(t,x) = N_{C(t,x)}$, where $N_{C(t,x)}$ is the normal cone of the subset $C(t,x)$ of $H$. This problem has been studied for the first time by Castaing [21], where $F = 0$ and $C(\cdot)$ which depend on the state variable are convex compact subsets of a finite dimensional Hilbert space $H$. Later, it has been extensively considered in the scientific literature (by adding a set-valued map $F$ to the normal cone), under different hypothesis on $C$ and $F$. For some achievements in the finite or infinite dimensional Hilbert setting, or Banach spaces, see e.g., [4], [5], [6], [7], [8], [16], [17], [18], [22], [24], [30], [35], [37], [41], among others.

The main area of research in sweeping processes and differential inclusions theory are optimization, equilibrium problems, ect. The link between second-order differential inclusions and dry friction has been discussed in numerous papers e.g., [37], [40]. We refer the reader to [20] for more details on formalism and applications of such problems.

The paper is organized as follows. We collect in section 2, some basic notations, definitions and useful results needed in the next section. We establish our main existence theorem regarding (1.1), in section 3. This result is a new contribution in the current framework of infinite-dimensional second-order state-dependent maximal monotone operators. Its proof is difficult since we deal with a linear growth condition that involves a square integrable function $\kappa(\cdot)$. The last section is dedicated to study a quasi-variational inequality.

## 2 Notation and preliminaries

We consider in all the paper the following notations and definitions: $I := [0,T]$ $(T > 0)$ is an interval of $\mathbb{R}$, and $H$ is a real separable Hilbert space whose inner product is denoted by $\langle \cdot,\cdot \rangle$ and the associated norm by $\|\cdot\|$. Denote by $\overline{B}_H$ the closed unit ball of $H$.
On the space $\mathcal{C}_H(I)$ of continuous maps $x : I \to H$, we consider the norm of uniform convergence on $I$, $\|x\|_\infty = \sup\limits_{t\in I} \|x(t)\|$.
By $L^p_H(I)$ for $p \in [1,+\infty[$ (resp. $p = +\infty$), we denote the space of measurable maps $x : I \to H$ such that $\int_I \|x(t)\|^p dt < +\infty$ (resp. which are essentially bounded) endowed with the usual norm $\|x\|_{L^p_H(I)} = (\int_I \|x(t)\|^p dt)^{\frac{1}{p}}$, $1 \le p < +\infty$ (resp. endowed with the usual essential supremum norm $\|\cdot\|_{L^\infty_H(I)}$). By $W^{1,2}(I,H)$, we denote the space of absolutely continuous functions from $I$ to $H$ with derivatives in $L^2_H(I)$. By $W^{2,2}(I,H)$, we denote the space of absolutely continuous functions $u$ from $I$ to $H$, with absolutely continuous derivatives $w$ such that the derivatives of $w$ are in $L^2_H(I)$.

Let us recall some properties of maximal monotone operators, see [15], [19], [53].
Let $A : D(A) \subset H \rightrightarrows H$ be a set-valued operator whose domain, range and graph are defined by

$$\begin{aligned} D(A) &= \{x \in H : \ Ax \neq \emptyset\}, \\ R(A) &= \{y \in H : \ \exists\, x \in D(A),\ y \in Ax\} = \cup\{Ax : \ x \in D(A)\}, \\ Gr(A) &= \{(x,y) \in H \times H : \ x \in D(A),\ y \in Ax\}. \end{aligned}$$

The operator $A : D(A) \subset H \rightrightarrows H$ is said to be monotone, if $\langle y_1 - y_2, x_1 - x_2\rangle \ge 0$ whenever $(x_i,y_i) \in Gr(A)$, $i = 1,2$.
It is maximal monotone, if its graph could not be contained strictly in the graph of any other monotone operator, in this case, for all $\lambda > 0$, $R(I_H + \lambda A) = H$, where $I_H$ denotes the identity map of $H$.
If $A$ is a maximal monotone operator then, for every $x \in D(A)$, $Ax$ is nonempty, closed and convex. Then, the projection of the origin into $Ax$, $A^0(x)$, exists and is unique.
For $\lambda > 0$, we define the resolvent and the Yosida approximation of A respectively by

$J_\lambda^A = (I_H + \lambda A)^{-1}$ and $A_\lambda = \frac{1}{\lambda}\left(I_H - J_\lambda^A\right)$. These operators are both single-valued and defined on the whole space $H$, and we have

$$J_\lambda^A x \in D(A) \text{ and } A_\lambda(x) \in A(J_\lambda^A x), \text{ for every } x \in H, \tag{2.1}$$
$$\|A_\lambda(x)\| \leq \|A^0(x)\| \text{ for every } x \in D(A).$$

Let $A : D(A) \subset H \rightrightarrows H$ and $B : D(B) \subset H \rightrightarrows H$ be two maximal monotone operators, then we denote by $dis(A,B)$ (see [52]) the pseudo-distance between $A$ and $B$ defined by

$$dis(A,B) = \sup\left\{\frac{\langle y - y', x' - x\rangle}{1 + \|y\| + \|y'\|} : (x,y) \in Gr(A),\ (x',y') \in Gr(B)\right\}. \tag{2.2}$$

Clearly, $dis(A,B) \in [0,+\infty]$, $dis(A,B) = dis(B,A)$ and $dis(A,B) = 0$ iff $A = B$.

To prove our main results, we need the following lemmas (see [33]).

**Lemma 2.1.** Let $A$ be a maximal monotone operator of $H$. If $x \in \overline{D(A)}$ and $y \in H$ are such that

$$\langle A^0(z) - y, z - x\rangle \geq 0 \ \ \forall z \in D(A),$$

then $x \in D(A)$ and $y \in A(x)$.

**Lemma 2.2.** Let $A_n$ $(n \in \mathbb{N})$, $A$ be maximal monotone operators of $H$ such that $dis(A_n, A) \to 0$. Suppose also that $x_n \in D(A_n)$ with $x_n \to x$ and $y_n \in A_n(x_n)$ with $y_n \to y$ weakly for some $x, y \in H$. Then $x \in D(A)$ and $y \in A(x)$.

**Lemma 2.3.** Let $A$, $B$ be maximal monotone operators of $H$. Then
(1) for $\lambda > 0$ and $x \in D(A)$

$$\|x - J_\lambda^B(x)\| \leq \lambda\|A^0(x)\| + dis(A,B) + \sqrt{\lambda\big(1 + \|A^0(x)\|\big)dis(A,B)}.$$

(2) For $\lambda > 0$ and $x, x' \in H$

$$\|J_\lambda^A(x) - J_\lambda^A(x')\| \leq \|x - x'\|.$$

**Lemma 2.4.** Let $A_n$ $(n \in \mathbb{N})$, $A$ be maximal monotone operators of $H$ such that $dis(A_n, A) \to 0$ and $\|A_n^0(x)\| \leq c(1 + \|x\|)$ for some $c > 0$, all $n \in \mathbb{N}$ and $x \in D(A_n)$. Then for every $z \in D(A)$ there exists a sequence $(z_n)$ such that

$$z_n \in D(A_n), \quad z_n \to z \text{ and } A_n^0(z_n) \to A^0(z).$$

## 3 Main result

In this section, we are interested in the existence of absolutely continuous solutions for the problem (1.1). To prove our main theorem, we develop some ideas used in [7], [45].

**Theorem 3.1.** Assume that for any $(t,y) \in I \times H$, $A(t,y) : D(A(t,y)) \subset H \rightrightarrows H$ is a maximal monotone operator satisfying
$(H_1)$ There exist a non-negative real constant $\lambda$, and a function $\beta \in W^{1,2}(I,\mathbb{R})$ which is non-negative on $[0,T[$ and non-decreasing with $\beta(T) < \infty$ and $\beta(0) = 0$ such that

$$dis(A(t,y), A(s,z)) \leq |\beta(t) - \beta(s)| + \lambda\|y - z\|, \ \forall t,s \in I, \ \forall y,z \in H.$$

$(H_2)$ There exists a non-negative real number $c$ such that

$$\|A^0(t,y)z\| \leq c(1 + \|y\| + \|z\|) \text{ for } t \in I, \ y \in H, \ z \in D(A(t,y)).$$

$(H_3)$ For any bounded subset $B$ of $H$, the set $D(A(I \times B))$ is relatively ball-compact.
Let $F : I \times H \times H \rightrightarrows H$ be a set-valued map with nonempty closed convex values such that

(*i*) $F(\cdot,\cdot,\cdot)$ is globally scalarly upper semi-continuous on $I \times H \times H$;

$(ii)$ for some non-negative function $\kappa(\cdot) \in L^2_{\mathbb{R}}(I)$, one has the growth type condition

$$F(t,x,y) \subset \kappa(t)(1+\|x\|+\|y\|)\overline{B}_H \text{ for all } (t,x,y) \in I \times H \times H.$$

Then, for any $(u_0, x_0) \in D(A(0,x_0)) \times H$, the evolution problem (1.1) has an absolutely continuous solution $(u,x) : I \to H \times H$.
More precisely, the second-order differential inclusion

$$\begin{cases} -\ddot{x}(t) \in A(t,x(t))\dot{x}(t) + F(t,x(t),\dot{x}(t)) \quad \text{a.e. } t \in I, \\ \dot{x}(t) \in D(A(t,x(t))), \quad t \in I \\ x(0) = x_0, \; \dot{x}(0) = u_0; \end{cases}$$

has at least one $W^{2,2}(I,H)$-solution $x(\cdot)$.
In other words, there exist an absolutely continuous map $x(\cdot) : I \to H$ whose derivative is absolutely continuous, and an integrable map $y(\cdot) : I \to H$ such that $x(0) = x_0$, $\dot{x}(0) = u_0$, $\dot{x}(t) \in D(A(t,x(t)))$ for all $t \in I$, and for almost all $t \in I$, $y(t) \in F(t,x(t),\dot{x}(t))$ and $-\ddot{x}(t) - y(t) \in A(t,x(t))\dot{x}(t)$.

**Proof .** Suppose that

$$1 - \left((1+T)\int_0^T (\kappa(s)+1)\,ds + \frac{3\lambda T}{2} + \frac{3cT}{2}(1+T)\right) > 0. \tag{3.1}$$

**(I)** Let us construct the sequences $(u_n)$ and $(x_n)$.
For any $n \geq 1$, define a partition of $I := [0,T]$ with

$$0 = t_0^n < t_1^n < \cdots < t_i^n < t_{i+1}^n < \cdots < t_n^n = T.$$

For any $n \geq 1$ and $i = 0, 1, \cdots, n-1$, set

$$h_{i+1}^n = t_{i+1}^n - t_i^n, \quad \beta_{i+1}^n = \beta(t_{i+1}^n) - \beta(t_i^n), \tag{3.2}$$

and suppose that

$$h_i^n \leq h_{i+1}^n, \quad \beta_i^n \leq \beta_{i+1}^n. \tag{3.3}$$

Let us define the function $\gamma$ by $\gamma(t) = t + \beta(t)$, $t \in I$. We can choose this partition such that for any $n \geq 1$ and $i = 0, 1, \cdots, n-1$,

$$\sigma_{i+1}^n = h_{i+1}^n + \beta_{i+1}^n \leq \eta_n = \frac{\gamma(T)}{n}, \tag{3.4}$$

since $\beta \in W^{1,2}(I,\mathbb{R})$ and $\beta(0) = 0$ by assumption. It is clear that $\eta_n \to 0$ as $n \to \infty$.
Let us consider for $i \in \{0, \cdots, n-1\}$, $\delta_i^n \in [t_i^n, t_{i+1}^n[$ such that

$$\kappa(\delta_i^n) \leq \inf_{t \in [t_i^n, t_{i+1}^n[} \kappa(t) + 1. \tag{3.5}$$

Then, fix any $n \geq 1$. Put $x_0^n = x_0$, $u_0^n = u_0 \in D(A(0,x_0))$. For $i \in \{0, \cdots, n-1\}$ choose $y_i^n \in F(\delta_i^n, x_i^n, u_i^n)$, and set

$$x_{i+1}^n = x_i^n + h_{i+1}^n u_i^n, \tag{3.6}$$

$$u_{i+1}^n = J_{i+1}^n\left(u_i^n - \int_{t_i^n}^{t_{i+1}^n} y_i^n\,ds\right), \tag{3.7}$$

where

$$J_{i+1}^n = J_{h_{i+1}^n}^{A(t_{i+1}^n, x_{i+1}^n)} = \left(I_H + h_{i+1}^n A(t_{i+1}^n, x_{i+1}^n)\right)^{-1}.$$

This along with (2.1) yields

$$u_{i+1}^n \in D(A(t_{i+1}^n, x_{i+1}^n)), \tag{3.8}$$

and

$$u_i^n - \int_{t_i^n}^{t_{i+1}^n} y_i^n\,ds \in u_{i+1}^n + h_{i+1}^n A(t_{i+1}^n, x_{i+1}^n) u_{i+1}^n.$$

This inclusion may be rewritten as follows

$$-\frac{u_{i+1}^n - u_i^n}{h_{i+1}^n} \in A(t_{i+1}^n, x_{i+1}^n)u_{i+1}^n + \frac{1}{h_{i+1}^n}\int_{t_i^n}^{t_{i+1}^n} y_i^n ds. \tag{3.9}$$

Note that by construction (see (3.6)), one has

$$\begin{aligned}\|x_i^n - x_{i-1}^n\| &= \|x_{i-1}^n + h_i^n u_{i-1}^n - x_{i-1}^n\| \\ &= h_i^n \|u_{i-1}^n\|,\end{aligned}$$

and then

$$\begin{aligned}\|x_i^n\| = \|x_{i-1}^n + h_i^n u_{i-1}^n\| &= \|x_0 + h_1^n u_0^n + h_2^n u_1^n + \cdots + h_i^n u_{i-1}^n\| \\ &\le \|x_0\| + h_1^n\|u_0^n\| + \cdots + h_i^n\|u_{i-1}^n\|. \end{aligned}\tag{3.10}$$

It results after simplification (keeping in mind (3.4))

$$\|x_i^n\| \le \|x_0\| + \eta_n(\|u_0^n\| + \|u_1^n\| + \cdots + \|u_{i-1}^n\|). \tag{3.11}$$

In view of Lemma 2.3, one gets

$$\begin{aligned}\|u_{i+1}^n - u_i^n\| &= \left\|J_{i+1}^n\left(u_i^n - \int_{t_i^n}^{t_{i+1}^n} y_i^n ds\right) - u_i^n\right\| \\ &\le \left\|J_{i+1}^n\left(u_i^n - \int_{t_i^n}^{t_{i+1}^n} y_i^n ds\right) - J_{i+1}^n(u_i^n)\right\| + \|J_{i+1}^n(u_i^n) - u_i^n\| \\ &\le \int_{t_i^n}^{t_{i+1}^n} \|y_i^n\| ds + h_{i+1}^n\|A^0(t_i^n, x_i^n)u_i^n\| + dis(A(t_i^n, x_i^n), A(t_{i+1}^n, x_{i+1}^n)) \\ &\quad + \left(h_{i+1}^n(1 + \|A^0(t_i^n, x_i^n)u_i^n\|)dis(A(t_i^n, x_i^n), A(t_{i+1}^n, x_{i+1}^n))\right)^{\frac{1}{2}}.\end{aligned}$$

This along with the fact that $(ef)^{\frac{1}{2}} \le \frac{1}{2}e + \frac{1}{2}f$ for positive real constants $e$, $f$, yields

$$\begin{aligned}\|u_{i+1}^n - u_i^n\| \le &\int_{t_i^n}^{t_{i+1}^n} \|y_i^n\| ds + h_{i+1}^n\|A^0(t_i^n, x_i^n)u_i^n\| \\ &+ \frac{h_{i+1}^n}{2}(1 + \|A^0(t_i^n, x_i^n)u_i^n\|) + \frac{3}{2}dis(A(t_i^n, x_i^n), A(t_{i+1}^n, x_{i+1}^n)).\end{aligned}\tag{3.12}$$

By construction of $y_i^n$, along with $(ii)$, for $i = 0, 1, \cdots, n-1$, for $t \in [t_i^n, t_{i+1}^n[$, one writes

$$\|y_i^n\| \le \kappa(\delta_i^n)(1 + \|x_i^n\| + \|u_i^n\|), \tag{3.13}$$

which yields (by (3.5))

$$\|y_i^n\| \le (\kappa(t) + 1)(1 + \|x_i^n\| + \|u_i^n\|). \tag{3.14}$$

Then, it follows

$$\int_{t_i^n}^{t_{i+1}^n} \|y_i^n\| ds \le d_i^n(1 + \|x_i^n\| + \|u_i^n\|), \tag{3.15}$$

where

$$d_i^n = \int_{t_i^n}^{t_{i+1}^n} (\kappa(s) + 1) ds. \tag{3.16}$$

Coming back to (3.12), using $(H_1)$, $(H_2)$, (3.2), (3.15), (3.16), one has for any $n \ge 1$ and $i = 0, 1, \cdots, n-1$

$$\begin{aligned}\|u_{i+1}^n - u_i^n\| \le &d_i^n(1 + \|x_i^n\| + \|u_i^n\|) + h_{i+1}^n c(1 + \|x_i^n\| + \|u_i^n\|) \\ &+ \frac{3}{2}\beta_{i+1}^n + \frac{3\lambda}{2}\|x_{i+1}^n - x_i^n\| + \frac{h_{i+1}^n}{2}\Big(1 + c(1 + \|x_i^n\| + \|u_i^n\|)\Big).\end{aligned}\tag{3.17}$$

Simplifying (using (3.4) and (3.6)) yields

$$\|u_{i+1}^n - u_i^n\| \leq \left(d_i^n + (\frac{3\lambda}{2} + \frac{3c}{2})h_{i+1}^n\right)\|u_i^n\| + (d_i^n + \frac{3c}{2}h_{i+1}^n)\|x_i^n\| + d_i^n + 2\sigma_{i+1}^n + \frac{3c}{2}\sigma_{i+1}^n, \tag{3.18}$$

then,

$$\begin{aligned}
\|u_{i+1}^n\| &\leq \left(1 + d_i^n + (\frac{3\lambda}{2} + \frac{3c}{2})h_{i+1}^n\right)\|u_i^n\| + (d_i^n + \frac{3c}{2}h_{i+1}^n)\|x_i^n\| + d_i^n + (2 + \frac{3c}{2})\sigma_{i+1}^n \\
&\leq \|u_0\| + \sum_{j=0}^{i}\left(d_j^n + (\frac{3\lambda}{2} + \frac{3c}{2})h_{j+1}^n\right)\|u_j^n\| + \sum_{j=0}^{i}(d_j^n + \frac{3c}{2}h_{j+1}^n)\|x_j^n\| \\
&\quad + \sum_{j=0}^{i}\left(d_j^n + (2 + \frac{3c}{2})\sigma_{j+1}^n\right).
\end{aligned}$$

Note that by (3.10)

$$\begin{aligned}
\|x_j^n\| &\leq \|x_0\| + \sum_{k=0}^{j-1} h_{k+1}^n \|u_k^n\| \\
&\leq \|x_0\| + \max_{0\leq k\leq n}\|u_k^n\| \sum_{k=0}^{j-1} h_{k+1}^n \\
&\leq \|x_0\| + T\max_{0\leq k\leq n}\|u_k^n\|.
\end{aligned} \tag{3.19}$$

Hence, one gets

$$\begin{aligned}
\|u_{i+1}^n\| \leq &\|u_0\| + \sum_{j=0}^{i}\left(d_j^n + (\frac{3\lambda}{2} + \frac{3c}{2})h_{j+1}^n\right)\max_{0\leq j\leq n}\|u_j^n\| \\
&+ \sum_{j=0}^{i}(d_j^n + \frac{3c}{2}h_{j+1}^n)\|x_0\| + \sum_{j=0}^{i}(d_j^n + \frac{3c}{2}h_{j+1}^n)T\max_{0\leq j\leq n}\|u_j^n\| + d + (2 + \frac{3c}{2})\gamma(T),
\end{aligned}$$

where

$$d = \sum_{i=0}^{n-1} d_i^n = \sum_{i=0}^{n-1}\int_{t_i^n}^{t_{i+1}^n}(\kappa(s) + 1)ds = \int_0^T(\kappa(s) + 1)ds < +\infty,$$

since by $(ii)$, $\kappa \in L^2_{\mathbb{R}}(I)$. It results that

$$\begin{aligned}
\|u_{i+1}^n\| &\leq \|u_0\| + \left(d + (\frac{3\lambda}{2} + \frac{3c}{2})T\right)\max_{0\leq j\leq n}\|u_j^n\| + (d + \frac{3c}{2}T)\|x_0\| \\
&\quad + (d + \frac{3c}{2}T)T\max_{0\leq j\leq n}\|u_j^n\| + d + (2 + \frac{3c}{2})\gamma(T) \\
&\leq \left((1+T)d + (\lambda + c(1+T))\frac{3T}{2}\right)\max_{0\leq j\leq n}\|u_j^n\| \\
&\quad + \|u_0\| + (d + \frac{3c}{2}T)\|x_0\| + d + (2 + \frac{3c}{2})\gamma(T).
\end{aligned}$$

Thanks to (3.1), that is, $(1+T)d + \frac{3\lambda T}{2} + \frac{3cT}{2}(1+T) < 1$, one deduces that

$$\|u_i^n\| \leq M, \tag{3.20}$$

where

$$M := \frac{\|u_0\| + (d + \frac{3c}{2}T)\|x_0\| + d + (2 + \frac{3c}{2})\gamma(T)}{1 - \left((1+T)d + (\lambda + c(1+T))\frac{3T}{2}\right)}.$$

Thanks to (3.19), one has
$$\|x_i^n\| \leq L, \tag{3.21}$$
where $L := \|x_0\| + TM$.
For any $n \geq 1$, define the following sequences $u_n$, $x_n$ for all $t \in [t_i^n, t_{i+1}^n[$, $i \in \{0, \cdots, n-1\}$ by
$$u_n(t) = u_i^n + \frac{t - t_i^n}{t_{i+1}^n - t_i^n}\left(u_{i+1}^n - u_i^n + \int_{t_i^n}^{t_{i+1}^n} y_n(s)ds\right) - \int_{t_i^n}^{t} y_n(s)ds,\ u_n(T) = u_n^n. \tag{3.22}$$
and
$$x_n(t) = x_i^n + (t - t_i^n)\frac{x_{i+1}^n - x_i^n}{t_{i+1}^n - t_i^n},\ x_n(T) = x_n^n; \tag{3.23}$$
where
$$y_n(t) = \begin{cases} y_i^n & \text{if } t \in [t_i^n, t_{i+1}^n[ \text{ for some } i \in \{0, \cdots, n-1\} \\ y_{n-1}^n & \text{if } t = T; \end{cases}$$
and $u_n(t_{i+1}^n) = u_{i+1}^n$ and $x_n(t_{i+1}^n) = x_{i+1}^n$.
Observe that by (3.14), (3.20), (3.21), for any $t \in I$
$$\|y_n(t)\| \leq (\kappa(t) + 1)(1 + M + L), \tag{3.24}$$
and then $(y_n)$ is bounded in $L_H^2(I)$. So, up to a subsequence that we do not relabel, $(y_n)$ weakly converges in $L_H^2(I)$ to some map $y \in L_H^2(I)$, that is,
$$y_n \to y \text{ weakly in } L_H^2(I). \tag{3.25}$$
Note that the functions $u_n, x_n : I \to H$ are absolutely continuous on $I$. Moreover, for all $t \in ]t_i^n, t_{i+1}^n[$
$$\dot{u}_n(t) = \frac{1}{t_{i+1}^n - t_i^n}\left(u_{i+1}^n - u_i^n + \int_{t_i^n}^{t_{i+1}^n} y_n(s)ds\right) - y_n(t), \tag{3.26}$$
and
$$\dot{x}_n(t) = \frac{x_{i+1}^n - x_i^n}{t_{i+1}^n - t_i^n} = u_i^n. \tag{3.27}$$
The inclusion (3.9) takes the form
$$-\dot{u}_n(t) - y_n(t) \in A(t_{i+1}^n, x_{i+1}^n)u_{i+1}^n. \tag{3.28}$$
Combining (3.12) and (3.13), with the reasoning above, it results
$$\begin{aligned}\|u_{i+1}^n - u_i^n\| \quad \leq \quad & h_i^n\kappa(\delta_i^n)(1 + \|x_i^n\| + \|u_i^n\|) + h_{i+1}^n c(1 + \|x_i^n\| + \|u_i^n\|) \\ & + \frac{3}{2}\beta_{i+1}^n + \frac{3\lambda}{2}h_{i+1}^n\|u_i^n\| + \frac{h_{i+1}^n}{2}\Big(1 + c(1 + \|x_i^n\| + \|u_i^n\|)\Big),\end{aligned}$$
and then
$$\begin{aligned}\frac{\|u_{i+1}^n - u_i^n\|}{h_{i+1}^n} &\leq (\kappa(\delta_i^n) + \frac{3c}{2})(1 + \|x_i^n\| + \|u_i^n\|) + \frac{3\lambda}{2}\|u_i^n\| + \frac{1}{2} + \frac{3}{2}\left(\frac{\beta(t_{i+1}^n) - \beta(t_i^n)}{t_{i+1}^n - t_i^n}\right), \\ &\leq (\kappa(\delta_i^n) + \frac{3c}{2})(1 + M + L) + \frac{3\lambda}{2}M + \frac{1}{2} + \frac{3}{2}\left(\frac{\beta(t_{i+1}^n) - \beta(t_i^n)}{t_{i+1}^n - t_i^n}\right),\end{aligned} \tag{3.29}$$
using (3.20) and (3.21).
In view of (3.4), (3.13), (3.20), (3.21), (3.26) and (3.29) one has for all $t \in [t_i^n, t_{i+1}^n[$
$$\begin{aligned}\|\dot{u}_n(t)\| &\leq \frac{1}{h_{i+1}^n}\|u_{i+1}^n - u_i^n\| + 2\kappa(\delta_i^n)(1 + L + M) \\ &\leq (3\kappa(\delta_i^n) + \frac{3c}{2})(1 + M + L) + \frac{3\lambda}{2}M + \frac{1}{2} + \frac{3}{2}\left(\frac{\beta(t_{i+1}^n) - \beta(t_i^n)}{t_{i+1}^n - t_i^n}\right),\end{aligned} \tag{3.30}$$
which yields using (3.5)
$$\|\dot{u}_n(t)\| \leq \left(3(\kappa(t) + 1) + \frac{3c}{2}\right)(1 + M + L) + \frac{3\lambda}{2}M + \frac{1}{2} + \frac{3}{2}\left(\frac{\beta(t_{i+1}^n) - \beta(t_i^n)}{t_{i+1}^n - t_i^n}\right). \tag{3.31}$$

The absolute continuity of $\beta$ implies that for a.e. $t \in ]t_i^n, t_{i+1}^n[$: $\dot{\beta}(t) = \lim\limits_{n\to\infty} \dfrac{\beta(t_{i+1}^n) - \beta(t_i^n)}{t_{i+1}^n - t_i^n}$. Consequently, there is a null-Lebesgue measure subset $J \subset I$, such that for every $t \in I \setminus J$, we can ensure the existence of $a_t < +\infty$ such that

$$\|\dot{u}_n(t)\| \leq a_t. \tag{3.32}$$

Note that by (3.16), for any $i \in \{0, \cdots, n-1\}$ one has

$$d_i^n \leq \sqrt{h_{i+1}^n}\sqrt{c_{i+1}^n},$$

using the Cauchy-Schwarz inequality, where $c_{i+1}^n = \int_{t_i^n}^{t_{i+1}^n} (\kappa(t)+1)^2 dt$.
The last estimate along with (3.17) yields

$$\begin{aligned}
\|u_{i+1}^n - u_i^n\| \quad &\leq \quad \sqrt{h_{i+1}^n}\sqrt{c_{i+1}^n}(1 + \|x_i^n\| + \|u_i^n\|) + \frac{3h_{i+1}^n c}{2}(1 + \|x_i^n\| + \|u_i^n\|) \\
&\quad + \frac{3}{2}\beta_{i+1}^n + \frac{3\lambda}{2}h_{i+1}^n\|u_i^n\| + \frac{h_{i+1}^n}{2}, \\
&= \quad \sqrt{h_{i+1}^n}\left(\sqrt{c_{i+1}^n} + \frac{3c}{2}\sqrt{h_{i+1}^n} + \frac{3\lambda}{2}\sqrt{h_{i+1}^n}\right)\|u_i^n\| \\
&+ \sqrt{h_{i+1}^n}\left(\sqrt{c_{i+1}^n} + \frac{3c}{2}\sqrt{h_{i+1}^n}\right)\|x_i^n\| + \sqrt{h_{i+1}^n}\left(\sqrt{c_{i+1}^n} + \frac{3c}{2}\sqrt{h_{i+1}^n} + \frac{\sqrt{h_{i+1}^n}}{2}\right) + \frac{3}{2}\beta_{i+1}^n,
\end{aligned}$$

then, by (3.2)

$$\begin{aligned}
\frac{\|u_{i+1}^n - u_i^n\|}{t_{i+1}^n - t_i^n} \quad &\leq \quad \frac{1}{\sqrt{h_{i+1}^n}}\left(\sqrt{c_{i+1}^n} + \frac{3c}{2}\sqrt{h_{i+1}^n} + \frac{3\lambda}{2}\sqrt{h_{i+1}^n}\right)\|u_i^n\| + \frac{3}{2}\left(\frac{\beta(t_{i+1}^n) - \beta(t_i^n)}{t_{i+1}^n - t_i^n}\right) \\
&+ \frac{1}{\sqrt{h_{i+1}^n}}\left(\sqrt{c_{i+1}^n} + \frac{3c}{2}\sqrt{h_{i+1}^n}\right)\|x_i^n\| + \frac{1}{\sqrt{h_{i+1}^n}}\left(\sqrt{c_{i+1}^n} + \frac{3c}{2}\sqrt{h_{i+1}^n} + \frac{\sqrt{h_{i+1}^n}}{2}\right).
\end{aligned}$$

Using (3.20), (3.21), it follows

$$\begin{aligned}
\frac{\|u_{i+1}^n - u_i^n\|}{t_{i+1}^n - t_i^n} &\leq \frac{1}{\sqrt{h_{i+1}^n}}\left(\sqrt{c_{i+1}^n} + \frac{3c}{2}\sqrt{h_{i+1}^n} + \frac{3\lambda}{2}\sqrt{h_{i+1}^n}\right)M + \frac{3}{2}\left(\frac{\beta(t_{i+1}^n) - \beta(t_i^n)}{t_{i+1}^n - t_i^n}\right) \\
&+ \frac{1}{\sqrt{h_{i+1}^n}}\left(\sqrt{c_{i+1}^n} + \frac{3c}{2}\sqrt{h_{i+1}^n} + \frac{\sqrt{h_{i+1}^n}}{2}\right) + \frac{1}{\sqrt{h_{i+1}^n}}\left(\sqrt{c_{i+1}^n} + \frac{3c}{2}\sqrt{h_{i+1}^n}\right)L \\
&= \frac{1}{\sqrt{h_{i+1}^n}}\left((M+L+1)\sqrt{c_{i+1}^n} + \alpha_{i+1}^n\right) + \frac{3}{2}\left(\frac{\beta(t_{i+1}^n) - \beta(t_i^n)}{t_{i+1}^n - t_i^n}\right),
\end{aligned}$$

where $\alpha_{i+1}^n := \sqrt{h_{i+1}^n}\left(\frac{3c}{2}(M+L+1) + \frac{1}{2} + \frac{3}{2}\lambda M\right)$.
Taking into account the fact that $(a+b)^2 \leq 2a^2 + 2b^2$ for any $a, b \in \mathbb{R}$, one writes

$$\begin{aligned}
\left(\frac{\|u_{i+1}^n - u_i^n\|}{t_{i+1}^n - t_i^n}\right)^2 \quad &\leq \quad \frac{2}{t_{i+1}^n - t_i^n}\left((M+L+1)\sqrt{c_{i+1}^n} + \alpha_{i+1}^n\right)^2 + \frac{9}{2}\left(\frac{\beta(t_{i+1}^n) - \beta(t_i^n)}{t_{i+1}^n - t_i^n}\right)^2 \\
&\leq \quad \frac{4}{t_{i+1}^n - t_i^n}\left((M+L+1)^2 c_{i+1}^n + (\alpha_{i+1}^n)^2\right) + \frac{9}{2}\left(\frac{\beta(t_{i+1}^n) - \beta(t_i^n)}{t_{i+1}^n - t_i^n}\right)^2.
\end{aligned}$$

Hence, one obtains

$$\frac{\|u_{i+1}^n - u_i^n\|^2}{t_{i+1}^n - t_i^n} \leq 4(M+L+1)^2 c_{i+1}^n + 4(\alpha_{i+1}^n)^2 + \frac{9}{2}\frac{(\beta(t_{i+1}^n) - \beta(t_i^n))^2}{t_{i+1}^n - t_i^n},$$

summing (since these inequalities hold true for $i = 0, \cdots, n-1$), one gets

$$\begin{aligned}\sum_{i=0}^{n-1}\left(\frac{\|u_{i+1}^n - u_i^n\|^2}{t_{i+1}^n - t_i^n}\right) &\leq 4(M+L+1)^2\sum_{i=0}^{n-1} c_{i+1}^n + 4\sum_{i=0}^{n-1}(\alpha_{i+1}^n)^2 + \frac{9}{2}\sum_{i=0}^{n-1}\left(\frac{(\beta(t_{i+1}^n) - \beta(t_i^n))^2}{t_{i+1}^n - t_i^n}\right)\\ &= 4(M+L+1)^2 M_1 + 4T\alpha + \frac{9}{2}\sum_{i=0}^{n-1}\left(\frac{(\beta(t_{i+1}^n) - \beta(t_i^n))^2}{t_{i+1}^n - t_i^n}\right),\end{aligned}$$

where $M_1 = \sum_{i=0}^{n-1} c_{i+1}^n = \int_0^T (\kappa(t)+1)^2 dt$, $\sum_{i=0}^{n-1}(\alpha_{i+1}^n)^2 = \alpha T$, $\alpha = \left(\frac{3c}{2}(M+L+1) + \frac{1}{2} + \frac{3}{2}\lambda M\right)^2$.
Observe that by Cauchy-Schwarz (since $\beta \in W^{1,2}(I, \mathbb{R})$)

$$(\beta(t_{i+1}^n) - \beta(t_i^n))^2 = (\int_{t_i^n}^{t_{i+1}^n} \dot{\beta}(t)dt)^2 \leq (t_{i+1}^n - t_i^n)(\int_{t_i^n}^{t_{i+1}^n} \dot{\beta}^2(t)dt).$$

Coming back to the sum above, it follows

$$\sum_{i=0}^{n-1}\left(\frac{\|u_{i+1}^n - u_i^n\|^2}{t_{i+1}^n - t_i^n}\right) \leq L_1 < +\infty, \tag{3.33}$$

where $L_1 := 4(M+L+1)^2 M_1 + 4T\alpha + \frac{9}{2}\int_0^T \dot{\beta}^2(t)dt$.
Using (3.30) and (3.5) gives

$$\begin{aligned}\|\dot{u}_n\|^2_{L^2_H(I)} &= \sum_{i=0}^{n-1}\int_{t_i^n}^{t_{i+1}^n} \|\dot{u}_n(t)\|^2 dt\\ &\leq \sum_{i=0}^{n-1}\int_{t_i^n}^{t_{i+1}^n}\left(\frac{\|u_{i+1}^n - u_i^n\|}{h_{i+1}^n} + 2\kappa(\delta_i^n)(1+L+M)\right)^2 dt\\ &\leq 2\sum_{i=0}^{n-1}\int_{t_i^n}^{t_{i+1}^n}\left(\left(\frac{\|u_{i+1}^n - u_i^n\|}{t_{i+1}^n - t_i^n}\right)^2 + 4(\kappa(t)+1)^2(1+L+M)^2\right)dt,\end{aligned}$$

$$\begin{aligned}&\leq 2\sum_{i=0}^{n-1}\left(\left(\frac{\|u_{i+1}^n - u_i^n\|}{t_{i+1}^n - t_i^n}\right)^2 (t_{i+1}^n - t_i^n) + 4(1+L+M)^2\int_{t_i^n}^{t_{i+1}^n}(\kappa(t)+1)^2 dt\right),\\ &= 2\sum_{i=0}^{n-1}\left(\left(\frac{\|u_{i+1}^n - u_i^n\|^2}{t_{i+1}^n - t_i^n}\right) + 4(1+L+M)^2\int_{t_i^n}^{t_{i+1}^n}(\kappa(t)+1)^2 dt\right),\end{aligned}$$

along with (3.33), it is readily seen that

$$\|\dot{u}_n\|^2_{L^2_H(I)} \leq 2\left(L_1 + 4(1+L+M)^2\int_0^T(\kappa(t)+1)^2 dt\right).$$

Hence,

$$\|\dot{u}_n\|_{L^2_H(I)} \leq \mathcal{S} = \left(2L_1 + 8(1+L+M)^2 M_1\right)^{\frac{1}{2}} < +\infty. \tag{3.34}$$

Put for any $n \geq 1$

$$\theta_n(t) = \begin{cases} 0 & \text{if } t = 0\\ t_i^n & \text{if } t \in ]t_i^n, t_{i+1}^n] \text{ for some } i \in \{0, \cdots, n-1\},\end{cases}$$

$$\delta_n(t) = \begin{cases} 0 & \text{if } t = 0\\ t_{i+1}^n & \text{if } t \in ]t_i^n, t_{i+1}^n] \text{ for some } i \in \{0, \cdots, n-1\};\end{cases}$$

and

$$\Delta_n(t) = \begin{cases} \delta_i^n & \text{if } t \in [t_i^n, t_{i+1}^n[ \text{ for some } i \in \{0, \cdots, n-1\} \\ \delta_{n-1}^n & \text{if } t = T \end{cases}$$

Recall that by construction (see (3.4)) for any $t \in I$, one has

$$\lim_{n\to\infty} \theta_n(t) = t, \ \lim_{n\to\infty} \delta_n(t) = t \text{ and } \lim_{n\to\infty} \Delta_n(t) = t. \tag{3.35}$$

Now, set for any $t \in I$

$$\phi_n(t) = x_0 + \int_0^t u_n(\theta_n(s))ds, \tag{3.36}$$

which is piecewise affine.
By (3.6), one writes

$$\begin{aligned} \phi_n(t) &= x_0 + \int_0^{t_1^n} u_n(\theta_n(s))ds + \int_{t_1^n}^{t_2^n} u_n(\theta_n(s))ds + \cdots + \int_{t_i^n}^{t} u_n(\theta_n(s))ds \\ &= x_0 + h_1^n u_0^n + h_2^n u_1^n + \cdots + (t - t_i^n)u_i^n \\ &= x_1^n + h_2^n u_1^n + \cdots + (t - t_i^n)u_i^n = x_i^n + (t - t_i^n)u_i^n \\ &= x_i^n + (t - t_i^n)\frac{x_{i+1}^n - x_i^n}{h_{i+1}^n} = x_n(t). \end{aligned}$$

In view of (3.8), (3.28) and (3.36), one obtains the following

$$-\dot{u}_n(t) \in A(\delta_n(t), x_n(\delta_n(t)))u_n(\delta_n(t)) + y_n(t) \text{ a.e } t \in I, \tag{3.37}$$

$$x_n(t) = x_0 + \int_0^t u_n(\theta_n(s))ds, \ t \in I \tag{3.38}$$

$$u_n(\delta_n(t)) \in D(A(\delta_n(t), x_n(\delta_n(t)))), \ t \in I. \tag{3.39}$$

**(II)** Let us prove the convergence of the sequences $(u_n)$ and $(x_n)$.
Thanks to (3.39) and (3.21), one gets for all $t \in I$

$$(u_n(\delta_n(t))) \subset D(A(I \times L\overline{B}_H)).$$

Moreover by (3.20), one has $(u_n(\delta_n(t))) \subset M\overline{B}_H$, for all $t \in I$. These inclusions along with $(H_3)$ entail that the set $\{u_n(\delta_n(t)) : n \in \mathbb{N}^*\}$ is relatively compact in $H$.
In view of (3.34), and the absolute continuity of $u_n$ for any $n$, one has

$$\|u_n(\delta_n(t)) - u_n(t)\| = \left\| \int_t^{\delta_n(t)} \dot{u}_n(r)dr \right\| \leq \int_t^{\delta_n(t)} \|\dot{u}_n(r)\|dr \leq |\delta_n(t) - t|^{\frac{1}{2}} \mathcal{S}.$$

Recall that by construction $|\delta_n(t) - t| \to 0$ as $n$ tends to $\infty$. Then, it follows that $\lim_{n\to\infty} \|u_n(\delta_n(t)) - u_n(t)\| = 0$. Hence, the set $\{u_n(t) : n \in \mathbb{N}^*\}$ is relatively compact in $H$.
Observe that by (3.34) for any $s, t \in I$, $t \leq s$

$$\|u_n(s) - u_n(t)\| = \left\| \int_t^s \dot{u}_n(\tau)d\tau \right\| \leq \int_t^s \|\dot{u}_n(\tau)\|d\tau \leq (s - t)^{\frac{1}{2}} \mathcal{S}, \tag{3.40}$$

that is, $\{u_n(\cdot) : n \in \mathbb{N}^*\}$ is equicontinuous. By Ascoli's theorem, $(u_n(\cdot))_n$ is relatively compact in $\mathcal{C}_H(I)$. So, we can extract a subsequence of $(u_n(\cdot))_n$ (not relabeled) that uniformly converges on $I$ to some map $u(\cdot) \in \mathcal{C}_H(I)$ and satisfying $u(0) = u_0$.
Since $(\dot{u}_n(\cdot))$ is bounded in $L^2_H(I)$ (see (3.34)), up to a subsequence that we do not relabel, $(\dot{u}_n(\cdot))$ weakly converges in $L^2_H(I)$ to some element $z(\cdot) \in L^2_H(I)$.
For any integer $n$ and any $e \in H$ and for $0 \leq s \leq t \leq T$, relying on the absolute continuity of $(u_n(\cdot))_n$, we can write

$$\int_0^T \left\langle e\mathbf{1}_{[s,t]}(\tau), \dot{u}_n(\tau) \right\rangle d\tau = \langle e, u_n(t) - u_n(s) \rangle,$$

where $\mathbf{1}_{[s,t]}$ denotes the characteristic function of the interval $[s,t]$.
Next, passing to the limit in the equality yields

$$\left\langle e, \int_s^t z(\tau)d\tau \right\rangle = \langle e, u(t)-u(s)\rangle.$$

Hence, given any $s,t \in [0,T]$ with $s \le t$, we get $\int_s^t z(\tau)d\tau = u(t)-u(s)$, and then $u(\cdot)$ is absolutely continuous and $z(\cdot)$ coincides almost everywhere in $[0,T]$ with $\dot{u}(\cdot)$. Moreover, it results

$$\dot{u}_n \to \dot{u} \text{ weakly in } L^2_H(I). \tag{3.41}$$

Further, observing that

$$\|u_n(\theta_n(t)) - u(t)\| \le \|u_n(\theta_n(t)) - u_n(t)\| + \|u_n(t) - u(t)\|,$$

we deduce that

$$\lim_{n\to\infty} \|u_n(\theta_n(t)) - u(t)\| = 0 \text{ for any } t \in I. \tag{3.42}$$

In the same vein, one gets

$$\lim_{n\to\infty} \|u_n(\delta_n(t)) - u(t)\| = 0 \text{ for any } t \in I. \tag{3.43}$$

Thanks to (3.38), (3.20) and (3.42), the Lebesgue dominated convergence theorem yields for all $t \in I$

$$\lim_{n\to\infty} x_n(t) = x_0 + \int_0^t u(s)ds = x(t). \tag{3.44}$$

We conclude that $\dot{x}(\cdot) \equiv u(\cdot)$ a.e., $x(0) = x_0$ and $x(\cdot)$ is absolutely continuous.
Thanks to (3.20), (3.38) and (3.44), one gets for all $t \in I$

$$\begin{aligned}
\lim_{n\to\infty} \|x_n(\theta_n(t)) - x(t)\| &\le \lim_{n\to\infty} \Big( \|x_n(\theta_n(t)) - x_n(t)\| + \|x_n(t) - x(t)\| \Big) \\
&\le \lim_{n\to\infty} \left( \int_{\theta_n(t)}^t \|u_n(\theta_n(s))\|ds + \|x_n(t) - x(t)\| \right) \\
&\le \lim_{n\to\infty} \Big( M|t - \theta_n(t)| + \|x_n(t) - x(t)\| \Big),
\end{aligned}$$

which yields

$$\lim_{n\to\infty} \|x_n(\theta_n(t)) - x(t)\| = 0. \tag{3.45}$$

In the same vein, one obtains

$$\lim_{n\to\infty} \|x_n(\delta_n(t)) - x(t)\| = 0. \tag{3.46}$$

**(III)** Now we are going to show that

$$-\dot{u}(t) \in A(t, x(t))u(t) + F(t, x(t), u(t)) \quad \text{a.e. } t \in I, \tag{3.47}$$

$$\begin{gathered}
x(t) = x_0 + \int_0^t u(s)ds, \quad t \in I, \\
u(t) \in D(A(t, x(t))), \quad t \in I, \\
u(0) = u_0 \in D(A(0, x_0)),\ x(0) = x_0 \in H.
\end{gathered} \tag{3.48}$$

We show first (3.48). Recall that $u_n(\delta_n(t)) \in D(A(\delta_n(t), x_n(\delta_n(t))))$ for all $t \in I$ (see (3.39)). Combining $(H_1)$, (3.2), (3.4), and (3.46) yields

$$\begin{aligned}
dis(A(\delta_n(t), x_n(\delta_n(t))), A(t, x(t))) &\le |\beta(\delta_n(t)) - \beta(t)| + \lambda\|x_n(\delta_n(t)) - x(t)\| \\
&\le \eta_n + \lambda\|x_n(\delta_n(t)) - x(t)\| \to 0 \text{ as } n \to \infty.
\end{aligned} \tag{3.49}$$

Remark that in view of $(H_2)$, (3.20) and (3.21), $(w_n) = (A^0(\delta_n(t), x_n(\delta_n(t)))u_n(\delta_n(t)))$ is bounded. Then, we may extract from $(w_n)$ a subsequence that weakly converges to $w \in H$. Since the sequence $(u_n(\delta_n(t)))$ converges to $u(t)$ in $H$ (see (3.43)), applying Lemma 2.2, one concludes that $u(t) \in D(A(t, x(t)))$, $t \in I$.

Now, let us state (3.47). Recall that by (3.41) and (3.25), one deduces that $(\dot{u}_n + y_n)$ weakly converges to $\dot{u}(\cdot) + y(\cdot)$ in $L^2_H(I)$. Then, there exists a sequence $(\zeta_j)$ such that for each $j \in \mathbb{N}$, $\zeta_j \in co\{\dot{u}_k + y_k,\ k \geq j\}$ and $(\zeta_j)$ strongly converges to $\dot{u}(\cdot) + y(\cdot)$ in $L^2_H(I)$. Then, we may extract from $(\zeta_j)$ a subsequence that converges a.e. to $\dot{u}(\cdot) + y(\cdot)$. In other words, there exists a subset $K$ of $I$ with null-Lebesgue measure and a subsequence $(j_p)$ of $\mathbb{N}$ such that for all $t \in I \setminus K$, $(\zeta_{j_p}(t))$ converges to $\dot{u}(t) + y(t)$. Hence, for $t \in I \setminus K$

$$\dot{u}(t) + y(t) \in \bigcap_{p \in \mathbb{N}} \overline{co}\{\dot{u}_k(t) + y_k(t),\ k \geq j_p\},$$

which means that for $t \in I \setminus K$ and any $w \in H$

$$\langle \dot{u}(t) + y(t), w\rangle \leq \limsup_{n\to\infty} \langle \dot{u}_n(t) + y_n(t), w\rangle. \tag{3.50}$$

Recall that $u(t) \in D(A(t, x(t)))$, $t \in I$. To prove that $-\dot{u}(t) \in A(t, x(t))u(t) + y(t)$ a.e. $t \in I$, it suffices to show that

$$\langle \dot{u}(t) + y(t), u(t) - z\rangle \leq \langle A^0(t, x(t))z, z - u(t)\rangle \quad \text{a.e. } t \in I,$$

for all $z \in D(A(t, x(t)))$, by Lemma 2.1. We will apply Lemma 2.4 to maximal monotone operators $A(\delta_n(t), x_n(\delta_n(t)))$ and $A(t, x(t))$ which satisfy (3.49), to ensure the existence of a sequence $(z_n)$ such that $z_n \in D(A(\delta_n(t), x_n(\delta_n(t))))$

$$z_n \to z \text{ and } A^0(\delta_n(t), x_n(\delta_n(t)))z_n \to A^0(t, x(t))z. \tag{3.51}$$

For $n \geq 1$, let $I \setminus K_n$ denote the set on which (3.37) holds. Since $A(t, y)$ is monotone for any $(t, y) \in I \times H$, one obtains for $t \in I \setminus K_n$

$$\langle \dot{u}_n(t) + y_n(t), u_n(\delta_n(t)) - z_n\rangle \leq \langle A^0(\delta_n(t), x_n(\delta_n(t)))z_n, z_n - u_n(\delta_n(t))\rangle. \tag{3.52}$$

Combining (3.32), (3.52), (3.24), it results that for $t \in I \setminus (\bigcup_{n\in\mathbb{N}} K_n \cup K \cup J)$

$$\begin{aligned}
\langle \dot{u}_n(t) + y_n(t), u(t) - z\rangle &= \langle \dot{u}_n(t) + y_n(t), u_n(\delta_n(t)) - z_n\rangle \\
&\quad + \langle \dot{u}_n(t) + y_n(t), (u(t) - u_n(\delta_n(t))) - (z - z_n)\rangle \\
&\leq \langle A^0(\delta_n(t), x_n(\delta_n(t)))z_n, z_n - u_n(\delta_n(t))\rangle \\
&\quad + \Big(a_t + (\kappa(t) + 1)(1 + M + L)\Big)\Big(\|u_n(\delta_n(t)) - u(t)\| + \|z_n - z\|\Big).
\end{aligned}$$

Taking (3.43) and (3.51) into account entail that

$$\limsup_{n\to\infty} \langle \dot{u}_n(t) + y_n(t), u(t) - z\rangle \leq \langle A^0(t, x(t))z, z - u(t)\rangle.$$

The latter inequality along with (3.50) gives

$$\langle \dot{u}(t) + y(t), u(t) - z\rangle \leq \langle A^0(t, x(t))z, z - u(t)\rangle \quad \text{a.e. } t \in I.$$

The inclusion

$$-\dot{u}(t) \in A(t, x(t))u(t) + y(t) \quad \text{a.e. } t \in I, \tag{3.53}$$

therefore holds true. It remains to show that

$$y(t) \in F(t, x(t), u(t)) \text{ a.e } t \in I.$$

By construction of $(y_n)$, one has $y_n(t) \in F(\Delta_n(t), x_n(\theta_n(t)), u_n(\theta_n(t)))$, for all $t \in I$ and all $n \geq 1$. Since

$$(\Delta_n(t), x_n(\theta_n(t)), u_n(\theta_n(t)))$$

pointwise converges to $(t, x(t), u(t))$ for all $t \in I$ (see (3.35), (3.42), (3.45)) and $(y_n)_n$ weakly converges in $L^2_H(I)$ to $y$ (see (3.25)), by $(i)$ $F(\cdot, \cdot, \cdot)$ is scalarly upper semi-continuous on $I \times H \times H$, invoking the closure theorem (see Theorem $VI$-4 [27]), the required inclusion holds true. Combining this with (3.53), we conclude that (3.47) is satisfied. Thus, the problem (1.1) has at least one absolutely continuous solution $(u, x) : I \to H$.

Now, suppose that

$$1-\left((1+T)\int_0^T(\kappa(s)+1)\,ds+\frac{3\lambda T}{2}+\frac{3cT}{2}(1+T)\right)\le 0,$$

then, there exists a finite subdivision $T_0=0<T_1<T_2<\cdots<T_k=T$ such that for each $j=0,\cdots,k-1$ one has

$$1-\left((1+(T_{j+1}-T_j))\int_{T_j}^{T_{j+1}}(\kappa(s)+1)\,ds+\frac{3\lambda(T_{j+1}-T_j)}{2}+\frac{3c(T_{j+1}-T_j)}{2}(1+(T_{j+1}-T_j))\right)>0. \tag{3.54}$$

The analysis above yields for $j=0,\cdots,k-1$, absolutely continuous maps $u_j(\cdot),x_j(\cdot):[T_j,T_{j+1}]\to H$ and $L^2_H(I)$-integrable maps $\varphi_j(\cdot):[T_j,T_{j+1}]\to H$ such that

$$u_{j-1}(T_j)=u_j(T_j) \text{ and } x_{j-1}(T_j)=x_j(T_j)$$

$$u_j(t)\in D(A(t,x_j(t))),\ t\in[T_j,T_{j+1}]$$

$$u_j(t)=\dot{x}_j(t) \text{ and } \varphi_j(t)\in F(t,x_j(t),u_j(t)) \text{ a.e. } t\in[T_j,T_{j+1}]$$

and

$$-\dot{u}_j(t)-\varphi_j(t)\in A(t,x_j(t))u_j(t) \text{ a.e. } t\in[T_j,T_{j+1}].$$

Putting $u(t)=u_j(t)$ and $x(t)=x_j(t)$, $\varphi(t)=\varphi_j(t)$ if $t\in[T_j,T_{j+1}]$ $j=0,\cdots,k-1$, we see that $(u(\cdot),x(\cdot))$ is an absolutely continuous solution of (1.1) on the whole interval $I:=[0,T]$. This ends the proof of the theorem. □

The normal cone to a non-empty convex closed moving set $C(t,x)$, namely $N_{C(t,x)}$, being a time and state-dependent maximal monotone operator, we derive from Theorem 3.1, a related result to the particular case of the sweeping process.

**Corollary 3.2.** Let $C:I\times H\rightrightarrows H$ be a set-valued mapping satisfying
$(H_1')$ For each $(t,y)\in I\times H$, $C(t,y)$ is a non-empty closed convex subset of $H$.
$(H_2')$ There exist a non-negative real constant $\lambda$, and a function $\beta\in W^{1,2}(I,\mathbb{R})$ which is non-negative on $[0,T[$ and non-decreasing with $\beta(T)<\infty$ and $\beta(0)=0$ such that

$$|d(x,C(t,u))-d(x,C(s,v))|\le|\beta(t)-\beta(s)|+\lambda\|v-u\|\ \forall t,s\in I,\ \ \forall x,v,u\in H.$$

$(H_3')$ For any bounded subset $B$ of $H$, the set $C(I\times B)$ is relatively ball-compact.
Let $F:I\times H\times H\rightrightarrows H$ be a set-valued map satisfying assumptions of Theorem 3.1.
Then, for any $(u_0,x_0)\in C(0,x_0)\times H$, the problem

$$\begin{cases}-\dot{u}(t)\in N_{C(t,x(t))}u(t)+F(t,x(t),u(t)) & \text{a.e. } t\in I,\\ x(t)=x_0+\int_0^t u(s)ds, \quad t\in I\\ u(t)\in C(t,x(t)), \quad t\in I\\ u(0)=u_0\in C(0,x_0),\ x(0)=x_0\in H;\end{cases}$$

has an absolutely continuous solution $(u,x):I\to H\times H$.
More precisely, the perturbed second-order sweeping process

$$\begin{cases}-\ddot{x}(t)\in N_{C(t,x(t))}\dot{x}(t)+F(t,x(t),\dot{x}(t)) & \text{a.e. } t\in I,\\ \dot{x}(t)\in C(t,x(t)), \quad t\in I\\ x(0)=x_0,\ \dot{x}(0)=u_0;\end{cases} \tag{3.55}$$

admits at least one $W^{2,2}(I,H)$-solution $x(\cdot)$.

**Proof .** The proof is omitted since it is clear that $A_{(t,x)}=N_{C(t,x)}$ fulfills the assumptions of Theorem 3.1 (see Corollary 8 [45]). □

## 4 Second-order evolution quasi-variational inequality

We end the paper by an investigation in the theory of evolution quasi-variational inequalities. For related examples and other applications, the reader is referred to see e.g., [7], [20], [31], [35], [49], and the references therein.

Let $C : I \times H \rightrightarrows H$ be a set-valued mapping satisfying $(H_1')$-$(H_2')$-$(H_3')$ of Corollary 3.2. Let $g : H \times H \to H$ and $h : I \to H$ be two maps. We are looking for a map $x : I \to H$ which satisfies the second-order evolution quasi-variational inequality of the form

$$\langle \ddot{x}(t) + g(x(t), \dot{x}(t)), w - \dot{x}(t)\rangle \geq \langle h(t), w - \dot{x}(t)\rangle, \tag{4.1}$$

for all $w \in C(t, x(t))$. This amounts to find a solution $x : I \to H$ which is absolutely continuous on $I$; and an absolutely continuous $u : I \to H$ such that $u(t) \in C(t, x(t))$ for all $t \in I$, with $u(\cdot) = \dot{x}(\cdot)$ a.e., and for a.e. $t \in I$, for all $w \in C(t, x(t))$, one has

$$\langle \dot{u}(t) + g(x(t), u(t)), w - u(t)\rangle \geq \langle h(t), w - u(t)\rangle.$$

Corollary 3.2 allows us to state the following existence result.

**Proposition 4.1.** Let $C : I \times H \rightrightarrows H$ be a set-valued mapping satisfying $(H_1')$-$(H_2')$-$(H_3')$ of Corollary 3.2. Let $g : H \times H \to H$ be defined by $g(x, y) = B_1x + B_2y$ where $B_1, B_2 : H \to H$ are linear continuous maps. Let $h : I \to H$ be a bounded continuous map. Then, the second-order evolution quasi-variational inequality (4.1) has at least a solution.

**Proof .** Define the set-valued map $F : I \times H \times H \rightrightarrows H$ by

$$F(t, x, y) := \{g(x, y) - h(t)\},\ (t, x, y) \in I \times H \times H.$$

Remark that

$$F(t, x, y) \subset \kappa(t)(1 + \|x\| + \|y\|)\overline{B}_H,\ (t, x, y) \in I \times H \times H,$$

for a map $\kappa(\cdot) \in L^2_{\mathbb{R}_+}(I)$ defined by $\kappa(t) := \max(\|B_1\| + \|B_2\| + \|h(t)\|)$ for any $t \in I$. Then, it is easily seen that $F$ satisfies assumptions of Corollary 3.2. As a consequence, for any $(u_0, x_0) \in H \times H$ such that $u_0 \in C(0, x_0)$, Corollary 3.2 ensures that the problem

$$\begin{cases} -\ddot{x}(t) \in N_{C(t,x(t))}\dot{x}(t) + F(t, x(t), \dot{x}(t)) \quad \text{a.e. } t \in I,\\ \dot{x}(t) \in C(t, x(t)), \quad t \in I\\ x(0) = x_0,\ \dot{x}(0) = u_0, \end{cases}$$

has at least one $W^{2,2}(I, H)$-solution $x(\cdot)$.
Note that the normal cone of a convex subset $S$ of $H$ at $y \in H$ is given by

$$N_S(y) = \{w \in H : \langle w, z - y\rangle \leq 0,\ \forall z \in S\},$$

we deduce that $x(\cdot)$ is a solution to the second-order evolution quasi-variational inequality (4.1). This finishes the proof of the proposition. □

We close this section by the following examples.
**Examples. 1)** In [34] and Example 3 [35], the following parabolic quasi-variational inequality

$$\begin{aligned} &\text{Find } v(t) \in \Gamma(v(t)) :\ \langle \dot{v}(t) + f(t), w - v(t)\rangle \geq 0,\ \forall w \in \Gamma(v(t)),\\ &v(0) = v_0 \in \Gamma(v_0), \end{aligned}$$

has been considered where the set $\Gamma(\cdot)$ satisfies assumptions of Corollary 3.2.
A related time-dependent problem has been introduced in [44] to model the evolution of sandpile.
**2)** Another example related to (4.1) has been occurred in Problem 5.5 [31] where $C(t)$ is a subset of the Sobolev space $H^1(\Omega)$, and $\Omega \subset \mathbb{R}^n$.

**Comments and open problems.** The current work deals with a perturbed second-order evolution problem governed by time and state-dependent maximal monotone operators. Our methodology is based on convex and variational analysis. We show existence of $W^{2,2}(I, H)$-solutions to the differential inclusion under consideration, in a real separable Hilbert space. Our existence result is applied to the theory of quasi-variational evolution inequalities. In our development, the convexity assumption on the values of $F$ is crucial in order to ensure the existence of a selection satisfying a linear growth condition. We also require a compactness assumption on the operators to guarantee our main theorem.

There remain **open problems** that would be the subject of further investigations:

$\mathbf{P}_1$) To study properties of the solution set (compactness, ...) or the dependence on the initial data and parameters of the problem is an interesting topic.

$\mathbf{P}_2$) To show the relationship between the obtained notion of the solution and other, weaker, notions known from the literature may be the subject of future works.

$\mathbf{P}_3$) To consider a non-convex set-valued map $F$ is not an easy task.

$\mathbf{P}_4$) To find applications in control theory is an open field of research.

$\mathbf{P}_5$) To impose new assumptions on the operators, in order to compensate for the lack of compactness. $\mathbf{P}_6$) To generalize the results of the paper to Banach spaces is open and waiting for new ideas.

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