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Application of a fractional-order financial system with disturbance in encryption and decryption

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Abstract

This paper focuses on the anti-synchronization of two identical and non-identical chaotic fractional-order financial systems with disturbance observe (FOFSDO), such that the anti-synchronization is discussed with new parameters and disturbance in the slave system by using the nonlinear active control technique. The stability of the scheme is proved by applying the Lyapunov stability method for the error system. The result of anti-synchronization with disturbance is applied in cryptography. Numerical examples and simulation analysis indicate the application and validity of the scheme and considered system.

Keywords: Chaos, Anti-synchronization, Financial fractional-order system, Disturbance observer, Secure Communication
2020 MSC: Primary 34D06, 34H10; Secondary 34A08, 91G45, 68P25

1 Introduction

In recent years, the study of dynamic systems of economic models has become particularly important, especially due to the adaptation of accurate economic models to financial systems of the fractional-order of these systems has received more attention [13, 47, 38]. Recently the application of dynamic systems is rapidly increasing in various sciences including electromagnetic waves, quantitative finance, engineering-biology, dielectric polarization, etc [19, 24, 17, 18, 23].

In the last two decades, helpful research has been done on fractional-order financial systems (FOFS), some of which we will mention. A study of the stable dynamics of a fractional-order chaotic financial system by changing parameters is presented by Marius-F. Danca et al. [11]. Sara Dadras, Hamidreza Momeni [10] have investigated the control of a fractional-order economical system with a sliding mode scheme. Zhen Wang and Xia Huang in [40] presented synchronization of a chaotic fractional-order economic system with active control and they presented in [41] control of an uncertain fractional-order economic system via the adaptive sliding mode method. Baogui Xin and Jinyi Zhang [42] have studied finite-time stabilizing a fractional-order chaotic financial system with market confidence. Ayub Khan and Arti Tyagi [21] have designed disturbance observer-based adaptive sliding mode hybrid projective synchronization of identical fractional-order financial systems. Norely Aguila-Camacho et al. [1] have investigated Lyapunov functions for fractional-order systems. In 2016, analysis and circuit simulation of a novel nonlinear fractional incommensurate

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order financial system was reported [15]. Tacha et al. [35] have presented the determination of chaotic behavior in a fractional-order finance system with negative parameters. Zhe Zhang et al. [48] have recommended a novel stability criterion of time-varying delay fractional-order financial systems based on a new functional transformation Lemma.

Research on the anti-synchronization of dynamic systems using various methods has been done by many researchers, which we will mention below. In 2011, Diyi Chen et al. [6] have investigated chaotic synchronization and anti-synchronization for a class of multiple chaotic systems via a sliding mode control scheme. Waffa Jawaada et al [20] have studied robust active sliding mode anti-synchronization of hyperchaotic systems with uncertainties and disturbances. Anti-synchronization of uncertain chaotic systems with adaptive terminal sliding mode control was reported in [14], and intermittent anti-synchronization of two identical hyperchaotic Chua systems via impulsive control scheme was presented by Hong-LiLi et al. [26]. Al-Sawalla and Noorani [3, 2] have studied chaos and adaptive reduced-order for anti-synchronization of uncertain chaotic systems with unknown parameters. Also, some other schemes such as projections, active control, phase, anti-phase, and reduced-order method are used for the anti-synchronization of fractional-order chaotic systems [36, 33, 34, 44, 29].

Recently, chaotic systems have been used as a practical method of transmitting information so that the transmitter information is integrated with a chaotic signal. Sending information signals through a public channel is retrieved by a chaotic receiver.

The most popular methods for transmitting and retrieving information are chaos masking, chaos modulation, and chaos shift keying. In chaos masking, the signal is added directly to the transmitter; in chaos modulation, it is injected into the transmitter, and in chaos shift keying, it draws the chaotic signal to the transmitter as a binary signal. In these three cases, the information signal is retrieved by the receiver using synchronization or anti-synchronization between the chaotic transmitter and the receiver [22, 5, 32]. Development of a chaos masking approach, chaotic shift keying, and modulation method can be found in [9, 45, 12, 30, 27, 31].

Recently, several fractional or integer-order chaotic systems have been introduced. Synchronization and anti-synchronization of these systems through methods such as adaptive control, sliding-mode, and, feedback control have been discussed so that these systems are without disturbance. Also some manuscripts used these systems for secure communication [37, 39, 46, 16, 30, 43].

In this paper, anti-synchronization between FOFS is investigated using the nonlinear active control method in the presence of new parameters, different initial conditions, and disturbance observers. By using Lyapunov stability, sufficient conditions are obtained to achieve anti-synchronization of the chaotic FOFSDO through active control. Disturbance can play an essential role in anti-synchronization and its applications. One of the anti-synchronization applications is encryption and decryption. In most of the researchers' previous work, systems without disturbance have been used for encryption and decryption. We use the slave system with disturbance for encryption and decryption and show the results by numerical simulation.

This paper is organized as follows: Section 2 contains the fundamental definitions, lemma, and theorem of fractional calculations, and a description of the system used. In Section 3, we explain the FOFSDO anti-synchronization of the appropriate order by the active control method. Section 4 discusses secure communication based on anti-synchronization of the fractional-order system with disturbance, and finally, concluding remarks are presented in Section 5.

2 Preliminaries

In this section, we review some of the basic definitions of fractional accounting. We also present the stability theorems, the characteristics of fractional-order dynamic systems, and the system used to order the deficit.

2.1 Fractional calculus

Definition 2.1. [4] The qth fractional-order Caputo derivative of function $G(t)$ is as follow:

$$^{c}D_t^q G(t) = D^{-(m-q)} D^{(m)} G(t) = \frac{1}{\Gamma(m-q)} \int_0^t (t-\zeta)^{m-q-1} G^{(m)}(\zeta) d\zeta, \tag{2.1}$$

where $m-1 < q \leqslant m, m \in N, q \in \mathbb{R}^+, \Gamma(q) = \int_0^\infty t^{q-1} e^{-t} dt$.

Some properties of fractional-order differential equations are:

- The linear characteristic of the Caputo fractional-order derivative

$$ {}^{c}D_t^q[c_1G_1(t)+c_2G_2(t)] = c_1{}^{c}D_t^qG_1(t)+c_2{}^{c}D_t^qG_2(t), \tag{2.2}$$

  where c_1, c_2 are constants and G_1, G_2 are functions of t variable[25].

- If $G(t)$ is a constant function then

$$ {}^{c}D_t^qG(t)=0, \tag{2.3}$$

  where $0 < q \leq 1$ [25].

Lemma 2.2. [1] Assume that $G(t) \in \mathbb{R}$ is a continuous and differentiable function, then we have

$$\frac{1}{2}({}^{c}D_t^qG^2(t)) \leqslant G(t){}^{c}D_t^qG(t), \tag{2.4}$$

where $0 < q < 1$.

Theorem 2.3. [28] Autonomous system ${}^{c}D_t^qx(t) = Ax(t)$, $x(0) = x_0$ is asymptotically stable if the following condition is satisfied

$$|arg(\lambda(A))| > \frac{q\pi}{2},$$

where $0 < q < 1$ and $\lambda(A)$ is the eigenvalue of matrix A. Also, the system ${}^{c}D_t^qx(t) = Ax(t)$ is stable if and only if $|arg(\lambda(A))| \geq \frac{q\pi}{2}$, and those critical eigenvalues that satisfy $|arg(\lambda(A))| = \frac{q\pi}{2}$, have geometric multiplicity of one.

2.2 Description of system

Consider the following FOFS [15]:

$$\begin{cases} D^{q_1}x = z + (y-a)x \\ D^{q_2}y = 1 - by - |x| \\ D^{q_3}z = -x - cz, \end{cases} \tag{2.5}$$

where x, y and z are the rate, investment demand, and price index, respectively. The constant parameters $a > 0$, $b > 0$ and $c > 0$ are the saving amount, cost per investment and elasticity of demand, respectively, and $0 < q_i \leq 1 (i = 1, 2, 3)$ is the fractional derivative order financial system. Figure 1 shows that the lowest value of $q_i(1 = 1, 2, 3)$ for which the system remains chaotic is commensurate order $q_1 = q_2 = q_3 = 0.79$ of the FOFS (2.5). Consider the new parameters as a $a = 0.7$, $b = 0.1$, $c = 0.9$ and the different initial condition $(x(0), y(0), z(0)) = (3, -3.5, 1.5)$ [35].

3 Anti-synchronization of two fractional-order financial systems with disturbance

In this section, we discuss the anti-synchronization of two fractional-order financial systems with disturbance in two identical and non-identical cases. Then we simulate the analytical results.

3.1 Anti-synchronization of two identical fractional-order finance systems

The FOFS [7] is considered as the master system

$$\begin{cases} D^{q_1}x = z + (y-a)x, \\ D^{q_2}y = 1 - by - x^2, \\ D^{q_3}z = -x - cz, \end{cases} \tag{3.1}$$

and the slave system

$$\begin{cases} D^{q_1}x_1 = z_1 + (y_1-a)x_1 + u_1(t) + d_1(t), \\ D^{q_2}y_1 = 1 - by_1 - x_1^2 + u_2(t) + d_2(t), \\ D^{q_3}z_1 = -x_1 - cz_1 + u_3(t) + d_3(t), \end{cases} \tag{3.2}$$

where $u_1(t)$, $u_2(t)$ and $u_3(t)$ are the controllers, and $d_1(t)$, $d_2(t)$, $d_3(t)$ with $(d_i \geq 0, i = 1, 2, 3)$ are unknown bounded disturbances. In this paper, the finding active controllers $u_1(t), u_2(t)$ and $u_3(t)$ so as to regulate the states x, y, and

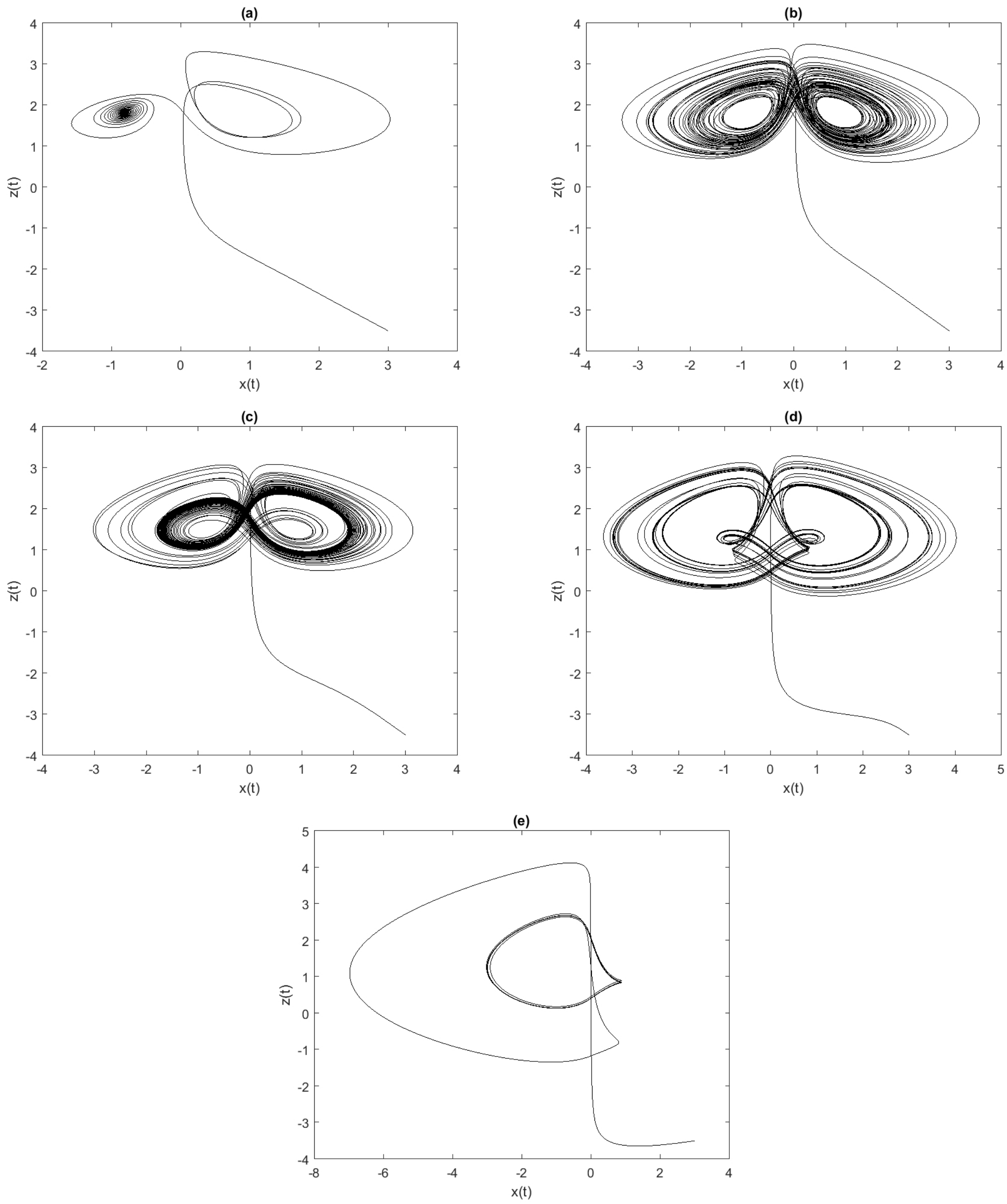

Figure 1: The phase portrait of fractional-order finance system (2.5) for the commensurate orders at $q_1 = q_2 = q_3 = q$. (a)q=0.78, (b)q=0.79, (c)q=0.86, (d)q=0.95, (e)q=1.

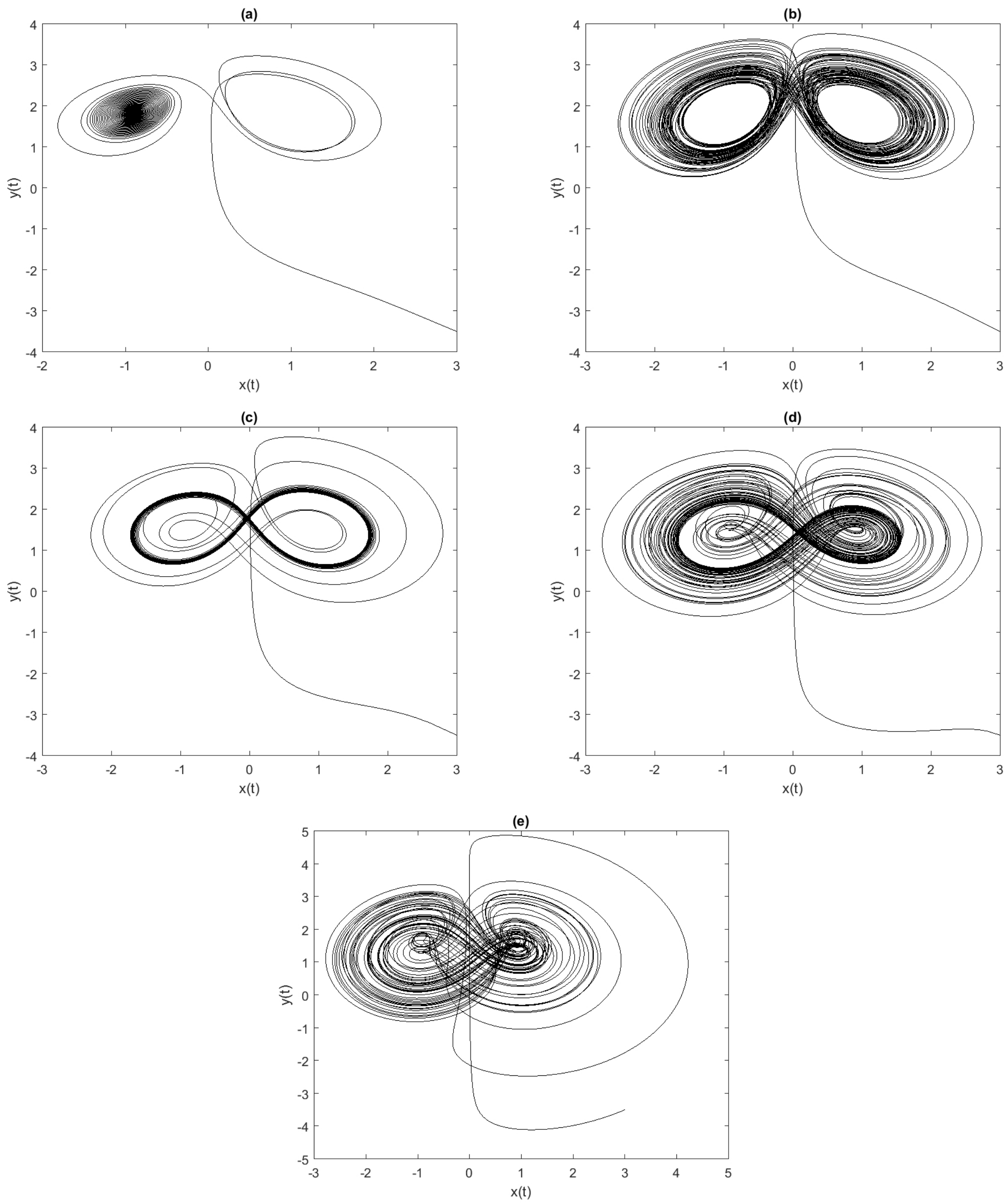

Figure 2: The phase portrait of fractional-order finance system (3.1) for the commensurate orders at $q_1 = q_2 = q_3 = q$. (a)$q = 0.82$, (b)$q = 0.83$, (c)$q = 0.9$, (d)$q = 0.96$, (e) $q = 1$.

z of the system (3.2) to desired constant values a, b, and c respectively, is considered. Figure 2 shows that the lowest value of $q_i (i = 1, 2, 3)$ for which the system remains chaotic is commensurate order $q_1 = q_2 = q_3 = 0.83$ of the FOFS (3.1). Define the error of anti-synchronization between(3.1) and (3.2) as follows:

$$\begin{cases} e_1(t) = x_1(t) + x(t), \\ e_2(t) = y_1(t) + y(t), \\ e_3(t) = z_1(t) + z(t). \end{cases} \tag{3.3}$$

We obtain the error of dynamics system by adding master system (3.1) and the slave system (3.2) is

$$\begin{cases} {}^cD_t^{q_1} e_1(t) = e_3 + y_1x_1 + yx - ae_1 + u_1(t) + d_1(t), \\ {}^cD_t^{q_2} e_2(t) = 2 - be_2 - x^2 - x_1^2 + u_2(t) + d_2(t), \\ {}^cD_t^{q_3} e_3(t) = -e_1 - ce_3 + u_3(t) + d_3(t). \end{cases} \tag{3.4}$$

And we consider the active control method with the following controls

$$\begin{cases} u_1(t) = -y_1x_1 - yx + w_1(t) - d_1(t), \\ u_2(t) = -2 + x^2 - x_1^2 - w_2(t) - d_2(t), \\ u_3(t) = w_3(t) - d_3(t). \end{cases} \tag{3.5}$$

By substituting of the system (3.5) in (3.4), we obtain the system error as follows

$$\begin{cases} {}^cD_t^{q_1} e_1(t) = e_3 - ae_1 + w_1(t), \\ {}^cD_t^{q_2} e_2(t) = -be_2 + w_2(t), \\ {}^cD_t^{q_3} e_3(t) = -e_1 - ce_3 + w_3(t), \end{cases} \tag{3.6}$$

where $w_1(t)$, $w_2(t)$ and $w_3(t)$ are the linear control inputs. We choose $w_1(t)$, $w_2(t)$ and $w_3(t)$ so that the system (3.4) become stable. We consider

$$\begin{pmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{pmatrix} = k \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \tag{3.7}$$

where

$$k = \begin{pmatrix} a-1 & 0 & -1 \\ 0 & b-1 & 0 \\ 1 & 0 & c-1 \end{pmatrix}. \tag{3.8}$$

The value of k can satisfy all eigenvalues λ_i of Jacobi matrix of equation (3.4) are -1. By the theorem(2.3), the master and slave system can achieve chaotic anti-synchronization. Now, we define the Lyapunov function as follows

$$V(e) = \frac{1}{2} \sum_{i=1}^{3} e_i^2, \tag{3.9}$$

the using Lemma (2.2), we get

$$D^{q_i} V(t) \leq \sum_{i=1}^{3} e_i(t) D^{q_i} e_i(t), \tag{3.10}$$

substituting the values of $D_t^{q_i} e_i, i = 1, 2, 3$ from (3.4) in (3.10), we get

$$\begin{aligned} D^{q_i} V(t) &\leq e_1(e_3 - ae_1 + y_1x_1 + yx + u_1(t) + d_1(t)) \\ &+ e_2(2 - be_2 - x^2 - x_1^2 + u_2(t) + d_2(t)) \\ &+ e_3(-e_1 - ce_3 + u_3(t) + d_3(t)), \end{aligned} \tag{3.11}$$

now, substituting (3.5) in (3.11), is written as

$$D^{q_i}V(e) \leqslant -e_1^2 - e_2^2 - e_3^2 < 0. \tag{3.12}$$

Then anti-synchronization error between master and slave systems is stable. Also, anti-synchronization error between master(3.1) and slave (3.2) systems is reduced to

$$\begin{cases} D^{q_1}e_1 = -e_1 \\ D^{q_2}e_2 = -e_2 \\ D^{q_3}e_3 = -e_3. \end{cases} \tag{3.13}$$

Here all the eigenvalue of the error system (3.13) are -1, and hence satisfy the stability condition. Therefore, the system error converges to zero when $t \longrightarrow \infty$.

3.2 Anti-synchronization of two non-identical fractional-order finance systems

In this section, anti-synchronization between two different FOFSDO is studied. We assume that FOFS is described as a master system as in (2.5) and FOFSDO is described as a slave as (3.2). Then we have the fractional-order master system as

$$\begin{cases} D^{q_1}x &= z + (y-a)x, \\ D^{q_2}y &= 1 - by - |x|, \\ D^{q_3}z &= -x - cz, \end{cases} \tag{3.14}$$

and slave system as

$$\begin{cases} D^{q_1}x_1 = z_1 + (y_1 - a)x_1 + u_1(t) + d_1(t), \\ D^{q_2}y_1 = 1 - by_1 - x_1^2 + u_2(t) + d_2(t), \\ D^{q_3}z_1 = -x_1 - cz_1 + u_3(t) + d_3(t), \end{cases} \tag{3.15}$$

where $u(t) = (u_1, u_2, u_3)^T$ is the controller and d_i $(i = 1, 2, 3)$ are the unknown disturbance observers. So anti-synchronization error dynamic system between (3.15) and (3.14) is as follows

$$\begin{cases} {}^cD_t^{q_1}e_1(t) = e_3 + y_1x_1 + yx - ae_1 + u_1(t) + d_1(t) \\ {}^cD_t^{q_2}e_2(t) = 2 - be_2 - x_1^2 - |x| + u_2(t) + d_2(t), \\ {}^cD_t^{q_3}e_3(t) = -e_1 - ce_3 + u_3(t) + d_3(t). \end{cases} \tag{3.16}$$

The controllers u_1, u_2 and u_3 are as follows:

$$\begin{cases} u_1(t) = -y_1x_1 - yx + w_1(t) - d_1(t), \\ u_2(t) = -2 + x_1^2 - |x,| + w_2(t) - d_2(t), \\ u_3(t) = w_3(t) - d_3(t), \end{cases} \tag{3.17}$$

where w_1, w_2 and w_3 are the control inputs as

$$\begin{cases} w_1(t) = (a-1)e_1 - e_3, \\ w_2(t) = (b-1)e_2, \\ w_3(t) = e_1 + (c-1)e_3. \end{cases} \tag{3.18}$$

By substituting (3.18) in (3.17) and then substituting (3.17) in (3.16), we get the error system as

$$\begin{cases} D^{q_1}e_1 = -e_1, \\ D^{q_2}e_2 = -e_2, \\ D^{q_3}e_3 = -e_3. \end{cases} \tag{3.19}$$

By define the Lyapunov function $V(e)$ as follows

$$V(e) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2), \tag{3.20}$$

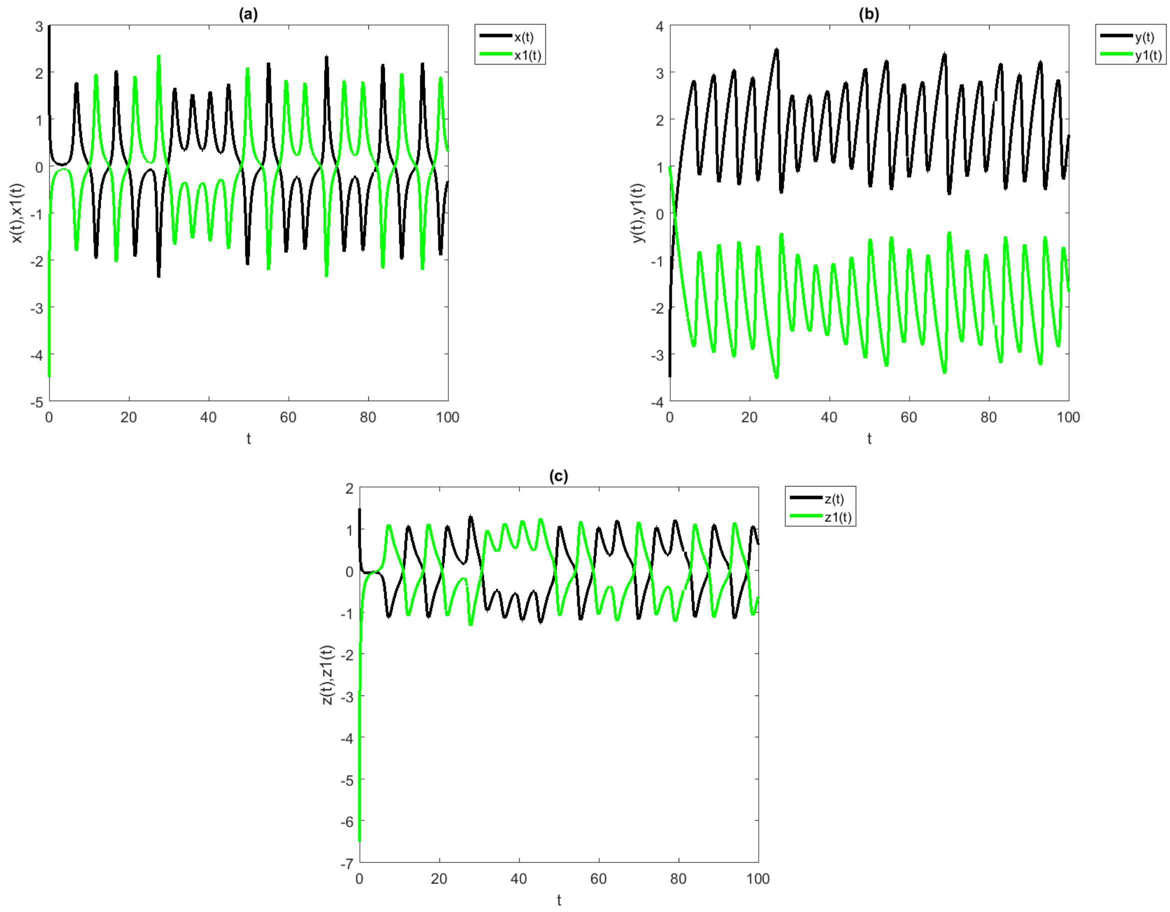

Figure 3: Depicts the phase portraits of anti-synchronization of system (3.1) and (3.2).

according Lemma (2.2) and property (2.2), we have

$$D^{q_i}V(e) \leqslant e_1 D^{q_1} e_1 + e_2 D^{q_2} e_2 + e_3 D^{q_3} e_3. \tag{3.21}$$

By substituting (3.19) in (3.21), we get

$$D^{q_i}V(e) \leqslant -e_1^2 - e_2^2 - e_3^2 < 0. \tag{3.22}$$

The error system (3.19) is asymptotically stable. Therefore anti-synchronization is archived between (3.14) and (3.15) systems.

3.3 Numerical simulation

In this subsection, we provide numerical simulation for illustrating the proposed method. The Numerical solution method is used to solve the systems. In the numerical simulation, we choose the new parameters as $a = 0.7, b = 0.1, c = 0.9$ and the initial conditions of the master and slave systems are taken as $(x(0), y(0), z(0)) = (3, -3.5, 1.5)$, $(x_1(0), y_1(0), z_1(0)) = (-4.5, 1, -6.5)$ respectively. Thus the initial error are $(-1.5, -2.5, 5)$. Disturbance observer is considered as $d_1 = 0.1\sin(200t)$, $d_2 = 0.2\cos(200t), d_3 = 0.3\cos(300t)$. Figure 3 (a-c) shows anti-synchronization between the master system (3.1) and the slave system (3.2) at $q_1 = q_2 = q_3 = 0.83$. Figure 4 (a-d) shows anti-synchronization error functions e_1, e_2 and e_3 for the commensurate orders $q_1 = q_2 = q_3 = q = 0.83, q = 0.9, q = 0.96, q = 1$, respectively. As shown in figure 4, the error converges to zero at approximately $t = 4$. Figure 5 shows anti-synchronization between the master system (3.14) and slave system (3.15) at $q_1 = q_2 = q_3 = 0.79$. Anti-synchronization error functions e_1, e_2 and e_3 are shown in Figure 6 (a-d). Also, Figure 6(a-d) shows that the error converges to zero at approximately $t = 4$.

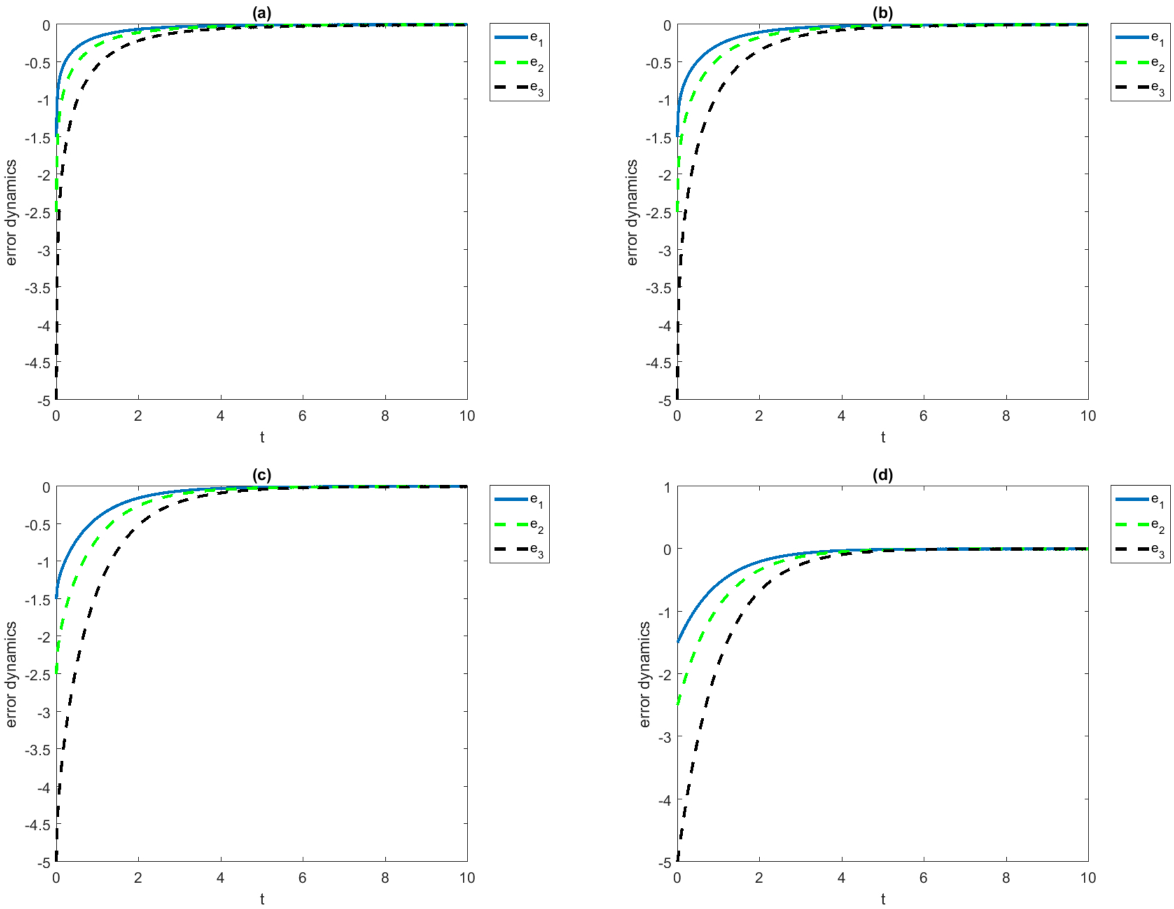

Figure 4: anti-synchronization error for the commensurate orders, (a) $q_1 = q_2 = q_3 = q = 0.83$, (b) $q = 0.9$, (c) $q = 0.96$, (d) $q = 1$.

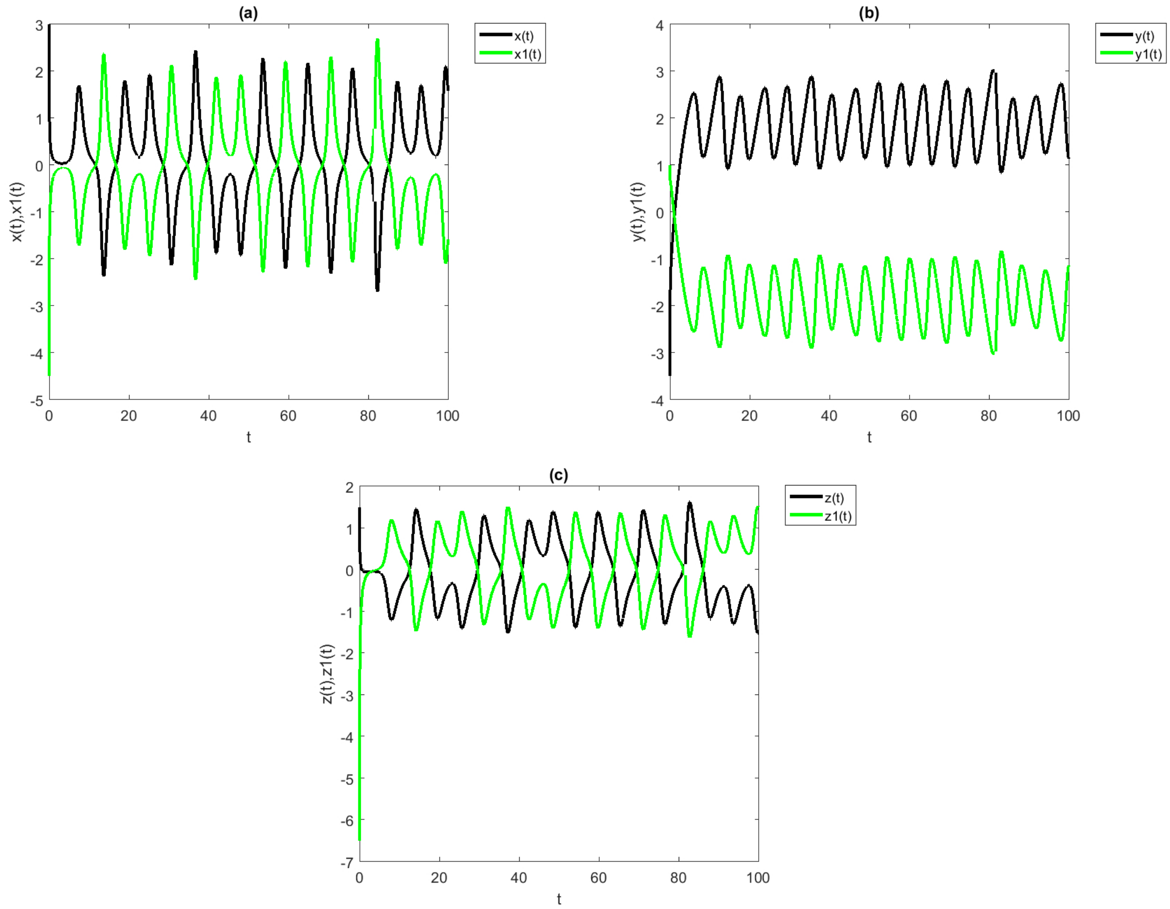

Figure 5: Depicts the phase portraits of anti-synchronization of the master system (3.14) and the slave system (3.15).

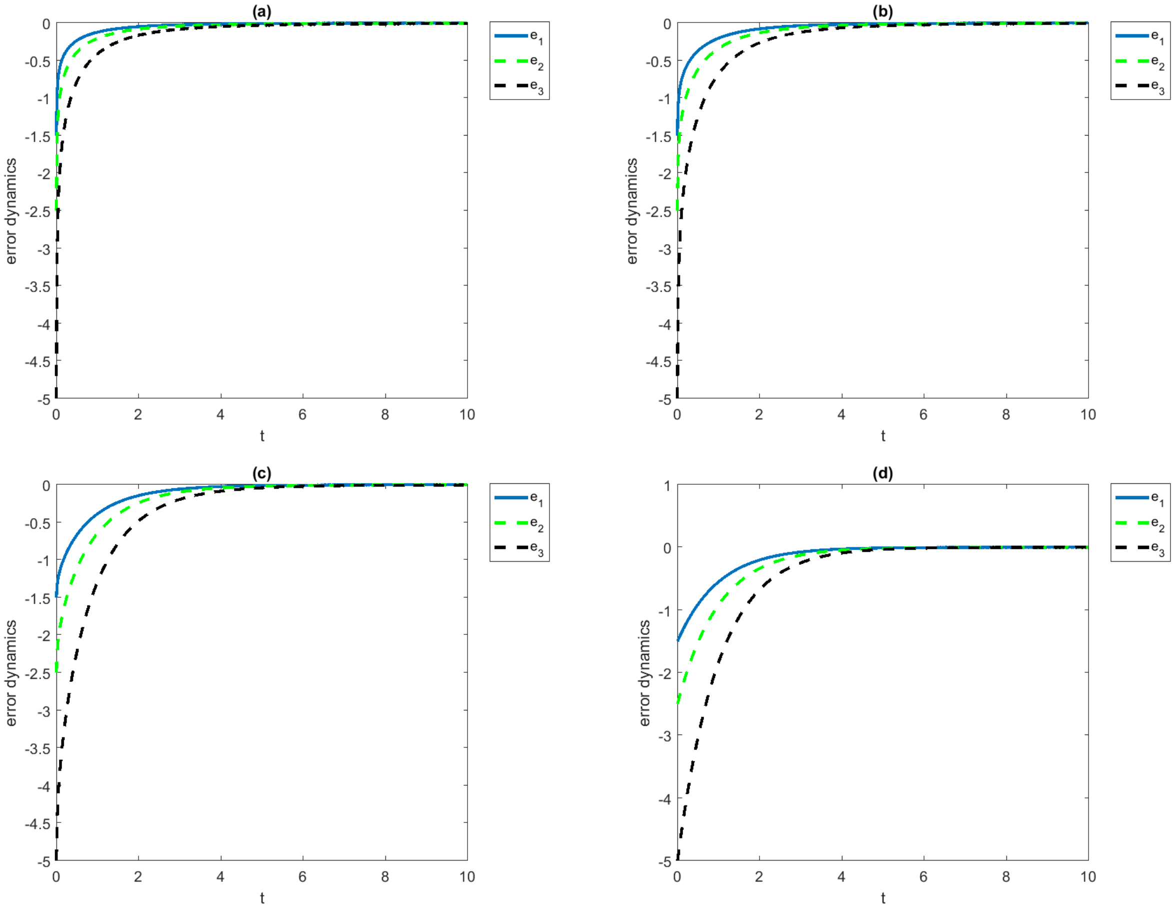

Figure 6: Anti-synchronization error for the commensurate orders, (a) $q_1 = q_2 = q_3 = q = 0.79$, (b) $q = 0.86$, (c) $q = 0.95$, (d) $q = 1$.

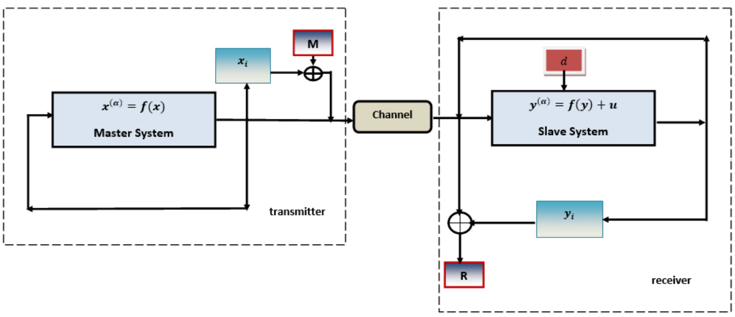

Figure 7: Diagram of secure communication based on anti-synchronization.

4 Application of chaotic financial system with disturbance in cryptography

In this section, we investigate the masking secure communication scheme based on the anti-synchronization of two fractional-order chaotic financial systems so that the slave system is considered with disturbance. The diagram of secure communication methods by two fractional-order financial chaotic systems is shown in figure 7. The systems used on the transmitter side are the (3.1) and (3.14) systems, and on the receiver side are the (3.2) and (3.15) systems.

On the transmitter side, the original message $M(t)$ is masked by the chaotic signal. The masked message is denoted by $T(t)$ and is defined as follows

$$T(t) = M(t) + hx(t). \tag{4.1}$$

$M(t)$ must be selected to successfully masked by $hx(t)$. Otherwise, the original message $M(t)$ multiplied by a scaling factor [8] is used to resize the original message. The resulting $T(t)$ signal is sent from the transmitter to the receiver using a public channel. With the results of section 3, anti-synchronization is achieved by the designed controllers. If T_c is greater than T_s (T_s is the synchronization and anti-synchronization time), it will be suitable for transfer and recovery. The signal received by the receiver can be recovered with the following equation of anti-synchronization:

$$R(t) = T(t) + hx_1 \cong M(t).$$

Because according to the anti-synchronization concept, we have the following:

$$R(t) = M(t) + hx(t) + hx_1(t) = M(t) + h(x(t) + x_1(t)) = M(t) + he_1(t) \cong M(t).$$

According to the results of section 3, the proposed secure communication scheme is established by two chaotic fractional-order financial systems, as shown in figures 7.

5 Conclusions

In this paper, we used the active control method to anti-synchronization the FOFSDO with new parameters and the different initial conditions. The stability between two FOFSDOs was investigated using the appropriate Lyapunov function. The obtained results of anti-synchronization of systems with disturbance were used for secure communication through the masking method. The results show that the designed controllers for anti-synchronization and secure communication with disturbance in the slave system are effective. Numerical simulations confirm the theoretical result and the proposed method.

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