Int. J. Nonlinear Anal. Appl. 14 (2023) 1, 1449–1455
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.6963

Description of the A4-graph for elements of order three in certain miscellaneous groups

Wissam Fadhel Abid[a], Zainab Hasan Msheree[b,*], Mohammed Mukheef Abed[b]

[a]*Middle Technical University, Institute of Technology, Baghdad, Iraq*

[b]*Middle Technical University, Technical Instructors Training Institute, Iraq*

(Communicated by Javad Vahidi)

Abstract

Suppose that G is a finite group and $X = t^G$ be a conjugacy class of an elements of order 3, $t \in G$.The A4-graph, is a simple undirected graph stand for $A_4(G, X)$, whose vertex set X and two vertices x, $y \in X$ are adjacent if they are different and satisfy $xy^{-1} = yx^{-1}$. In this article, the orbits under the action of $C_G(t)$ on X are analyzed, along with the description of the algebraic structure of the subgroup $< t,x >$ such that x is a $C_G(t)$ -orbit representative is provided.

Keywords: Finite simple groups, A4-graph, connectivity, cliques
2020 MSC: 20D05, 68R01, 05C40, 05C69

1 Introduction

Graph theory and group theory are two distinct branches of mathematics with a different sets of rules. However, when we present some groups as graphs, we can identify and analyze their properties effetely. Since the presentations are more intelligible and the difficulties are more achievable. Consider, for example, [2, 6, 9] for a list of recent work on this topic. The involution elements are essential for understanding the algebraic properties of finite simple groups. Order 3 elements, on the other hand, are equally essential in studying finite simple groups. In [12] for example, the Frobenius groups generated by two elements of order 3 are provided with a proper description. Maksimenko and Mamontov [7] also demonstrated that a group created by an element of order 3 conjugacy in which each pair produces an isomorphic subgroup to Z_3, A_4, A_5, $SL_2(3)$, or $SL_2(4)$ possesses local finiteness. Additional research on this topic may be seen in [3, 8]. Aubad [1] launched the concept of A4-graph as a simple undirected graph denoted by A_4(G,X) such that G is a finite group and X is a random conjugacy class of element of order 3 in G. Furthermore, the vertices $x,y \in$ X are adjacent if they are different and fulfill the following criterion $xy^{-1} = yx^{-1}$. The A4-graph structure of the simple group ${}^3D_4(3)$ is analyzed with full information also in [1]. The alternating group A_4 is formed by two connected vertices, which is something noteworthy about the vertices of the A4-graph. As a consequence, the alternating group A_4can be investigated as a subgroup from a group with a big size also the method in which it is created within larger groups, may be analyzed.

*Corresponding author
Email addresses: wissam@mtu.edu.iq (Wissam Fadhel Abid), zainab.hasan@mtu.edu.iq (Zainab Hasan Msheree), mohammed.mukheef@mtu.edu.iq (Mohammed Mukheef Abed)

Received: March 2022 *Accepted:* May 2022

This paper aims to analyze A_4(G,X) where G is a certain Miscellaneous group in particular G isomorphic to one of the following: 2^4.A_8, 2^5.$PSL_5(2)$ and 5^3.$PSL_3(5)$. The diameter, clique number, and girth of the A4-graph are also calculated as part of the research.

Finally, we should note that G act by conjugation on X is created A4-graph automorphisms, also this action on the graph vertices is transitive. Moreover, the Atlas [5] is being utilized for the labels of G-conjugacy classes.

2 General Results

The general findings of the A4-graph of a finite simple group are covered in this section. During this work, we assume that G is a member of the former group, with X being a conjugacy class that has representative $t \in G$ of order 3. First, we give the essential properties of the A4-graph in the next result:

Lemma 2.1. [1] The A_4(G,X) have the following properties:

1- Simple, undirected and regular graph.

2- Any adjacent vertices created the alternating group A_4 and their product of order 3.

Let $s \in X$, the i^{th} disc of s, symbolize by $\Delta_i(s)$, $(i \in \mathbb{Z}^+.)$ is identified as following:

$$\otimes_i(s) = \{wX|d(s, w) = i\}.$$

When employing the standard graph distance function, this distance function is represented by d(,). The relation between the discs of the A4-graph and the action of centralizer in G of t which is denoted by $C_G(t)$ on the class X is provided in following result.

The next result associated to the disc structures of the A4-graph and its relation with the $C_G(t)$-orbits.

Lemma 2.2. [1] The discs $\Delta_i(t)$ of The A_4(G,X) breakdown into a collection of specific $C_G(t)$-orbits of X.

From the above result one can conclude that determination the $C_G(t)$-orbits of X led to investigate the A4-graph structure.

3 Graph Structures

Let x be a representative of a random $C_G(t)$-orbits such $C_G(t)$ act on X by conjugation. Then the subgroup

$$< t,x >? < t,y >,$$

for any y in same $C_G(t)$-orbit of x, this because there is $h \in C_G(t)$ satisfy x^h=y and hence $<t,x>?<t,x>^h?<t,y>$. Furthermore, calculating the $C_G(t)$-orbits would be used to investigate the discs structure of the A4-graph, as indicated in **Lemma 2.2**. Therefore, during this paper we determine the subgroup structure for the subgroup $<t,x>$ where x is a representative of $C_G(t)$-orbits and determine the disc of the A4-graph contain such orbit. The technique we have chosen depends mostly on the computational method and for that purpose, we employed the system for computational discrete algebra GAP [10]. Besides that, the OnLine Atlas[11] plays a critical role in allocating classes of X and providing a representation to every group of the A4-graph.

In the tables that come, exponential notation is being used to illustrate the multiple of a size. For example, The first row in the $\Delta_2(t)$ part of the graph $A_4(2^4.A_8$, 3A) is

180^4	$2^4.A_5$

As we see in **Table 3.1** this means there are four C_G(t)-orbits each has size of 180 and for any random element x in these orbit the subgroup $<t,x> \cong 2^4.A_5$.

3.1 The A4-graph description of non-split extension 2^4.A$_8$

First we should note that we use permutation representation on 30 points to deal computationally with this group. In the finite group 2^4.A$_8$ there are two classes of order 3 namely 3A and 3B, in the following we give a description for the A4-graph in both cases:

Case i. $G \cong 2^4.A_8$ **and X=3A**

Let $t \in G$, such that t in 3A. Then we can see that C$_G(t)GL_2(4)$ and if we let X=t^G, Then X has size 1792 and the action of C$_G(t)$ on X by conjugation has 20-orbits. In the table we give the subgroup structure for fixed x in one of the C$_G(t)$ -orbits and t and decided in which disc of the $A_4(2^4$.A$_8$, 3A) to be in:

Table 1: Structure of the $A_4(2^4$.A$_8$, 3A)

$\Delta_i i(t)$	Orbits sizes	Subgroup structure
$\Delta_1 i(t)$	60,15	A4
$\Delta_2 i(t)$	60	A_4
	90^4	$(D_8):Z_3$
	180,60	A_5
	180^4	$(D_8):A_5$
$\Delta_3 i(t)$	20	D3
	60^2	$3.A4$
	180	$2.A_4$
	15	A_4
	1	Z_3

Then we conclude the $A_4(2^4$.A$_8$, 3A) is connected with diameter 3.

Case ii. $G \cong 2^4.A_8$ **and X=3B**

For $t \in G$ in this case, then t in 3B and $C_G(t) \cong 3.S_4$ and there action on X=t^G produce 124 C$_G(t)$-orbits with size of X is 4480. The description of the $A_4(2^4$.A$_8$, 3B) can be seen in the following table:

In this case the $A_4(2^4$.A$_8$, 3B)is connected has diameter 3.

3.2 The A4-graph description of Dempwolff group 2^5.PSL$_5$(2)

For the computational calculations inside this group we utilize the permutation representation on 7440 points. The Dempwolff group has two classes of elements of order 3 namely 3A with size 79360 and 3B with size 888832, in the next we provide information about for the A4-graph in both classes:

Case i. $G \cong 2^5$.**PSL**$_5$**(2) and X=3A**

Take t as a random elements of order 3 in 3A, and set X=t^G. Then centralizer in G of t isomorphic to 3.((2^3) . PSL(3,2)). Moreover, the number of C$_G(t)$-orbits equal to 116. After computational check we note that $A_4(2^5$.PSL$_5$(2), 3A) is disconnected with 310 connected components each with 256 nodes. We consider these connected components as individual connected graph and we denoted by $CA_4(2^5$.PSL$_5$(2), 3A). This graph has only five C$_G(t)$-orbits and the graph describe as follows:

Now as $A_4(2^5$.PSL$_5$(2), 3A) is regular then this results is true for each 310 connected components.

Case ii. $G \cong 2^5$.**PSL**$_5$**(2) and X=3B**

Let t be a fixed elements in the class 3B, the we have $C_G(t) \cong 3.SL2(5)$ and if we let X=t^G, then the action of C$_G(t)$ on X has 2620-orbits. The A4-graph in this case also as in cases X=3A disconnected. However, in this cases there are 3472 connected components each with 256 nodes. Let we assume that $CA_4(2^5$.PSL$_5$(2), 3B) as particular connected component. We note that this graph have 7-orbits and can be describe as the following:

The regularity of the $A_4(2^5$.PSL$_5$(2), 3B) ensure that for each 3472 connected components.

3.3 The A4-graph description of Non-split extension 5^3.PSL3(5)

To investigate structure of the A4-graph of 5^3.PSL3(5) we apply a permutation representation on 3875 points. We should note that the group has only one class of elements of order 3 namely 3A, this class has size 387500. Full details about the A4-graph can be seen in the following case:

Table 2: Structure of the $A_4(2^4.\mathrm{A}_8$, 3B)

$\Delta_i(t)$	Orbits sizes	Subgroup structure
$\Delta_1(t)$	$18, 9^2, 6,$ 24,3	A_4
$\Delta_2(t)$	18	A_4
	72	A_5
	72^2	A_7
	$36^{10}, 72^2$	$(Z_2^3) : (Z_7 : Z_3)$
	364	$((Z_2x((Z_2^3) : (K_4))) : Z_2) : (Z_7 : Z_3)$
	36^4	$Z_7 : Z_3$
	$36^2, 72^4$	$(K_4^2).\mathrm{PSL}_3(2)$
	24^2	$\mathrm{GL}_2(4)$
	24^2	$3.A_4$
	18^4	$(K_4.(D_8)) : Z_3$
	$9^2, 18^2$	SL2(3)
	18^2	$D_8 : Z_3$
$\Delta_3(t)$	$9^2, 6, 3, 24$	A_4
	72^2	A_7
	24_2	$3.A_4$
	8	D_3
	36^4	$Z_7 : Z_3$
	24^2	$\mathrm{GL}_2(4)$
	36^6	$\mathrm{SL}_2(7)$
	$9^2, 18^2$	$\mathrm{SL}_2(3)$
	18^2	$D_8 : C_3$
	18^4	$(K_4.(D_8)) : Z_3$
	72^2	$(D_{16}) : (D_3)$
	$36^{10}, 72^4$	$(3.D_8) : (Z_7 : Z_3)$
	36^4	$((((2.Q_8) : Z_2) : Z_2) : Z_2) : (Z_7 : Z_3)$
	$36^{12}, 72^4$	$(D_3) : (Z_7 : Z_3)$
	74^4	G
	1	Z_3

Table 3: Structure of the $CA_4(2^5.\mathrm{PSL}_5(2)$, 3A)

$\Delta_i(t)$	**Orbits sizes**	**Subgroup structure**
$\Delta_1(t)$	3,42	A_4
$\Delta_2(t)$	42,168	$\mathrm{SL}_2(3)$

Table 4: Structure of the $CA_4(2^5.\mathrm{PSL}_5(2)$, 3B)

$\boldsymbol{\Delta_i(t)}$	**Orbits sizes**	**Subgroup structure**
$\boldsymbol{\Delta_1(t)}$	**15^2,45,60**	**$\mathbf{A}_4$**
$\boldsymbol{\Delta_2(t)}$	**60^2**	**$\mathbf{SL}_2\mathbf{(3)}$**

Case i. $G \cong 5^3$**.PSL3(5) and X=3A**

Assume that $G \cong 5^3$.PSL3(5) and X= X=t^G, where $t \in 3A$. Then the centralizer of t in G isomorphic to 3 .(Z_5 : 2^3), moreover, there are 3650 $C_G(t)$ –orbits under the action of $C_G(t)$ on X by conjugation. The analyses of the $A_4(5^3$.PSL3(5), 3A) in this case given in the following table:

Then the $A_4(5^3$.PSL3(6), 3A) is connected and the diameter of the graph is 4.

The discs structure along with the diameters of A_4(G,X) for G one of $2^4.\mathrm{A}_8$, $2^5.\mathrm{PSL}_5(2)$ and $5^3.\mathrm{PSL}_3(5)$ are outlined in the following theorem:

Table 5: Structure of the $A_4(5^3$.PSL3(5), 3A)

$\Delta_i(t)$	Orbits sizes	Subgroup structure
$\Delta_1(t)$	120	A_4
$\Delta_2(t)$	120^{62}	G
	120^5	$5^3 . A_5$
	120^{16}	$5^3 : (Z_{31} : Z_3)$
	120^5	$5^3 : A_4$
	24	$5^2 : Z_3$
	120^4	$Z_{31} : Z_3$
$\Delta_3(t)$	120^{1274}	G
	120^{16}	$Z_{31} : Z_3$
	120^{64}	$5^3 : (Z_{31} : Z_3)$
	24^{32}	$((5^2) : Z_5) : SL2(3)$
	120^{40}, 24^{160}	$((5 . ((5^2) : Z_5)) : Z_5) : SL2(3)$
	120^{20}, 24^{100}	$((5 .((5^2) : Z_5)) : Z_5) : SL2(5)$
	24^5	$((5^2) : Z_5) :Z_3$
	120^3	$(5^3) : A_4$
	120	A_4
	24^8	$(5^2) : SL2(3)$
	120	$((5 . ((5^2) : Z_5)) : Z_5) : Z_3$
	120	$(5^2) : C_3$
$\Delta_4(t)$	120^{1434}	G
	120^{20}, 24^{80}	$((5 . ((5^2) : 5)) : 5) : SL2(3)$
	24^{22}	$((5^2) : Z_5) : SL2(3)$
	120^{80}	$(5^3) : (Z_{31} : Z_3)$
	120^{20}	$Z_{31} : Z_3$
	120^{32}	$(5^3) : ((K_4) : Z_3)$
	120^8	$(K_4) : Z_3$
	24^{35}	$((5^2) : 5) : Z3$
	24^{10}	$(5^2) : Z_3$
	24^{16}	$(5^2) : SL2(3)$
	120^9	$((5 . ((5^2) : Z_5)) : Z_5) : Z_3$
	120^{10}, 24^{50}	$((5 . ((5^2) : Z_5)) : Z_5) : SL2(5)$
	6	SL2(5)
	6^2	SL2(3)
	1	Z_3

Theorem 3.1. Let G stand for one of the Miscellaneous groups stated below. The A4-graph of G then has the following characteristics:

i. $A_4(2^4.A_8$,3A) is connected such that Dima($A_4(2^4.A_8$,3A))= 3. Furthermore, for $t \in 3A$ we have $|\Delta_1(t)|$=75 (spread out over 2 $C_G(t)$-orbits), $|\Delta_2(t)|$=1380 (spread out over 11 $C_G(t)$-orbits) and $|\Delta_3(t)|$=336 (spread out over 6 $C_G(t)$-orbits).

ii. $A_4(2^4.A_8$,3B) is connected such that Dima($A_4(2^4.A_8$,3A))= 3. Furthermore, for $t \in 3A$ we have $|\Delta_1(t)|$=69 (spread out over 6 $C_G(t)$-orbits), $|\Delta_2(t)|$=1644 (spread out over 44 $C_G(t)$-orbits) and $|\Delta_3(t)|$=2766 (spread out over 73 $C_G(t)$-orbits).

iii. $A_4(2^5.PSL_5(2)$,3A) is disconnected contains 310 connected components. Furthermore, if we let $CA_4(2^5.PSL_5(2)$,3A) to be any a random connected components of the $A_4(2^5.PSL_5(2)$,3A). Then the graph $CA_4(2^5.PSL_5(2)$,3A) is connected has diameter 2.Also, if x be a fixed vertex in $CA_4(2^5.PSL_5(2)$,3A) then we have $|\Delta_1(x)|$=45 (spread out over 2 $C_G(t)$-orbits), $|\Delta_2(x)|$=210 (spread out over 2 $C_G(t)$-orbits).

iv. $A_4(2^5.PSL_5(2)$,3B) is disconnected contains 3472 connected components. Furthermore, if we let $CA_4(2^5.PSL_5(2)$,3B) to be any a random connected components of the $A_4(2^5.PSL_5(2)$,3B). Then the graph $CA_4(2^5.PSL_5(2)$,3B) is connected has diameter 2.Also, if x be a fixed vertex in $CA_4(2^5.PSL_5(2)$,3B) then we have $|\Delta_1(x)|$=135 (spread out over 4 $C_G(t)$-orbits), $|\Delta_2(x)|$=120 (spread out over 2 $C_G(t)$-orbits).

v. $A_4(5^3.\mathrm{PSL}_3(5),3\mathrm{A})$ is connected such that Dima($A_4(5^3.\mathrm{PSL}_3(5),3\mathrm{A})$)= 4. Furthermore, for $t \in 3A$ we have $|\Delta_1(t)|$=120 (spread out over 1 $\mathrm{C}_G(t)$-orbits), $|\Delta_2(t)|$=11064 (spread out over 93 $\mathrm{C}_G(t)$-orbits), $|\Delta_3(t)|$=177624 (spread out over 1725 $\mathrm{C}_G(t)$-orbits) and $|\Delta_4(t)|$= 198691 (spread out over 1830 $\mathrm{C}_G(t)$-orbits).

Proof . The previous mentioned tables confirm all the results in the above theorem which was obtained computationally by using the Gap program. In addition to the many features provided by A4-graph properties such as the regularity of the A4-graph that played an essential role in the analysis of the structure of the graph. □

4 Girth and clique number of the A4-graph

The action by conjugation on the vertices of the A4-graph ensure that the girth and clique number of the A4-graph on X are isomorphic to the A4-graph on the first disc with t of the graph. The computational approach we use to calculate the girth and clique number are mainly depend on the gap package **YAGS** [4].

The next couples figures we present the $A_4(2^4.\mathrm{A}_8,\ \{\Delta_1(t),t\})$ in cases X=3A or 3B respectively. Then we utilizes to compute the girth and the cliques number of the A4-graphs.

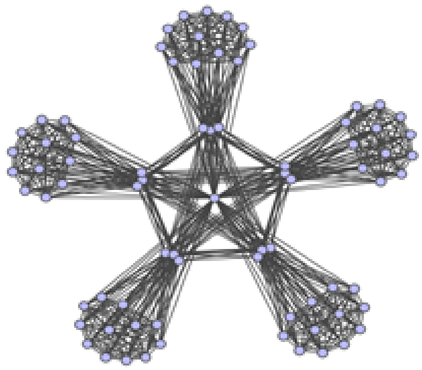

Figure 1 $A_4(\ 2^4.\mathrm{A}_8,\ \{\Delta_1(t),t\})$, X=3A

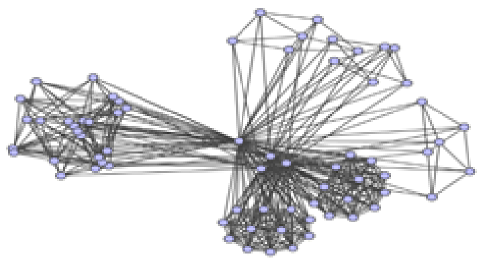

Figure 2 $A_4(2^4.\mathrm{A}_8,\{\Delta_1(t),t\}\ )$,X=3B

While in next figure the $A_4(5^3.PSL_3(5),\{\Delta_1(t),t\})$ is given below:

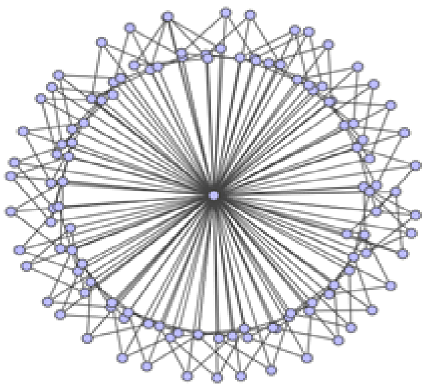

Figure 3 $A_4(5^3.PSL_3(5),\ \{\Delta_1(t),t\}\ )$

Now let A_4(G,X) be a connected A4-graph in the next theorem we provide the girth and the clique number of the graph.

Theorem 4.1. Let G be one of the groups $2^4.\mathrm{A}_8$ or 5^3. $\mathrm{PSL}_3(5)$. Then girth of their A4-graphs is 4 and the clique number are given below:

i. The clique number $A_4(2^4.\mathrm{A}_8$,X) is 16 and the girth is 3 for $X \in \{3A, 3B\}$.

ii. The clique number $A_4(5^3.\ \mathrm{PSL}_3\ (5),\mathrm{X})$ is 4 and the girth is 3 for $X \in \{3A\}$.

Proof . To calculate the girth and the clique number we only need to compute the girth and the clique number for the graphs in **Figure 4.1, Figure 4.2 and Figure 4.3** .This is executable by using gap programming with **YAGS** packages. □

The relation between the connected A4-graphs and the alternating group A_4can be seen in the next result.

Corollary 4.2 : Let G represent one of the Miscellaneous groups mentioned below. Then for the connected A4-graphs we have the following results:

i. Let $G \cong 2^4.\mathrm{A}_8$ and X=3A. Then there are 10920 subgroups the alternating group A_4in first disc of $A_4(2^4.\mathrm{A}_8$,3A) and all conjugate in G.

ii. Let $G \cong 2^4.\mathrm{A}_8$ and X=3B. Then there are 94640 subgroups the alternating group A_4in first disc of $A_4(2^4.\mathrm{A}_8$,3B) and all conjugate in G.

iii. Let $G \cong 5^3.\mathrm{PSL}_3(5)$ and X=3A. Then there are 40 subgroups the alternating group A_4in first disc of $A_4(5^3.\mathrm{PSL}_3(5)$, 3A) and all conjugate in G.

Proof . The proof follow from the fact the each connected vertices of the clique generated the alternating group A_4. Also from the information about the discs structure of the connected A4-graph in **Theorem 3.4.** □

5 Conclusions

For t be a random element of order 3 in particular Miscellaneous group Gthe structure of A_4(G,X) is studied. The study involving calculating the diameter, clique number, and girth of A_4(G,X) . Furthermore, the subgroup group structure $<t,x>$ for x is a fixed elements in the $\mathrm{C}_G(t)$ -orbit of X is offered.

References

[1] A. Aubad, *A4-graph of finite simple groups*, Iraqi J. Sci. **62** (2021), no. 1, 289–294.

[2] A. Aubad and P. Rowley,: *Commuting involution graphs for certain exceptional groups of Lie type*, Graphs Combin. **37** (2021), 1345–1355.

[3] D.A. Azeez and A. Aubad, *Investigation of commuting graphs for elements of order 3 in certain Leech lattice groups*, Iraqi J. Sci. **62** (2021), no. 8, 2640–2652.

[4] C. Cedillo, R. MacKinney-Romero, M.A. Pizaa, I.A. Robles and R. Villarroel-Flores, *Yet Another Graph System*, YAGS. Version 0.0.5. http://xamanek.izt.uam.mx/yags/ ,(2020).

[5] H. Conway, R.T. Curtis, S.P. Norton and R.A. Parker, *ATLAS of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups*, Oxford. Clarendon press, 1985.

[6] A. Kumar, L. Selvaganesh, P.J. Cameron and T.T. Chelvam, *Recent developments on the power graph of finite groups: A survey*, AKCE Int. J. Graphs Combin. **18** (2021), no 2, 65–94.

[7] A. Maksimenko and A. Mamontova, *The local finiteness of some groups generated by a conjugacy class of order 3 elements*, Siberian Math. J. **48** (2007), no. 3, 508–518.

[8] Z. Msheree, M. Abed and W. Abid, *Studying the A4-Graphs for elements of order 3 in tits group T and Mathieu group M_{20}*, Int. J. Nonlinear Anal. Appl. **12** (2021), no. 2, 1855–1860.

[9] J. Siemons and A. Zalesski, *Remarks on singular Cayley graphs and vanishing elements of simple groups*, J Algebr Comb. **50** (2019), 379–401.

[10] The GAP Group, *GAP Groups, Algorithms, and Programming, Version 4.11.1*, http://www.gap-system.org, 2021.

[11] J. Tripp, I. Suleiman, S. Rogers, R. Parker, S. Norton, S. Nickerson, S. Linton, J. Bray, A. Wilson and P. Walsh, *A world wide web atlas of group representations*, http://brauer.maths.qmul.ac.uk/Atlas/v3/.(2021).

[12] A. Zhurtov, *Frobenius groups generated by two elements of order 3*, Siberian Math. J. **42** (2001), no. 3, 450–454.