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An approximation approach towards a class of integro-differential equation with pure delay

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Abstract

In this article, we study a new numerical approach to solve some particular class of delay integro-differential equations. The considered problem is a singularly perturbed Volterra integro-differential equation with a pure delay term. To solve such equations numerically we adopt the standard Adomian decomposition method followed by a first-order truncated Taylor approximation. The most appealing advantage of the present method is that it provides an adequate result for a wide scale of values to the perturbation parameter. The efficiency of the proposed method is illustrated with an example. Moreover, a vivid realization of the treatment is described by the theoretical study related to error analysis. Under some relevant assumptions boundedness of solution, and stability analysis are also established in the agreement of the current method. To strengthen our findings, a comparative study between the proposed technique and the well-renowned spline method is presented in the manuscript. Moreover, outcomes suggest the prior efficiency of the method which is also supported by the theoretical results.

Keywords: Volterra integro-differential equation, Adomian Decomposition, Hyer-Ulam-Rassias stability
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1 Introduction

Consider the integro-differential equation

$$\epsilon \frac{dy(t)}{dt} = \alpha(t)y(t-\tau) + \int_0^t \mathcal{X}(t,s)h(y(s))ds + g(t), \tag{1.1}$$

$$y(0) = \mu \tag{1.2}$$

where $t \in [0,T] = \mathcal{A}, \epsilon \in \mathbb{R}^+ = (0,\infty)$ is the perturbation parameter and τ is the pure delay term which is independent of ϵ. The domain $\mathcal{A}$ is taken to be bounded with $0 < T < T^*$ for some positive constant T^*. $\mathcal{X}(t,s) : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$,

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$g(t)$, $\alpha(t)$ are sufficiently smooth functions and $y(t)$ is considered as continuously differentiable function from $\mathcal{A}$ to $\mathbb{R}$, μ is a given constant, $h(y(t))$ is a smooth non linear function of $y(t)$. Moreover, the parameters are such that $\epsilon + \alpha(t)\tau \neq 0$. (1.1) and (1.2) together construct a special type of integro-differential equation, generally known as singularly perturbed Volterra integro-differential equations (SPVIDEs) with pure delay term and they are commonly used to model many physical processes like fluid flow at high Reynold number, flow in porous media, biological problems and other physical phenomenons [2, 3, 19, 35, 36]. For $\epsilon << 1$, equation (1.1) behaves like a standard integral equation. Preciously, for $\epsilon = \tau = 0$ we obtain second kind of Volterra integral equation

$$\alpha(t)y(t) + \int_0^t \mathcal{X}(t,s)y(s)ds + g(t) = 0 \tag{1.3}$$

which is a special case of (1.1) during no perturbation parameter and without delay term [46]. Numerous applications of SPVIDEs in real-world problems are the source of its study in both theoretical and applied points of view [2, 3, 29, 26]. Volterra delay-integro differential equations occurs vigorously in various areas such as biology, ecology, medicine and physics [11, 12, 15, 23, 31]. This class of equations plays an important role in modelling diverse problems of engineering and natural science, and hence has lead researchers to develop a theory and numerical analysis for the concerned class of problems. Differential equations with a small parameter ϵ multiplying the highest order derivative terms are said to be singularly perturbed and normally boundary layers occur in their solutions. These equations play an important role in today's advanced scientific computations. Many mathematical models starting from fluid dynamics to the problems in mathematical biology are modelled by singularly perturbed problems. Typical examples include high Reynold's number flow in fluid dynamics, heat transport problem, etc. For more details on singularly perturbation, one can refer the books [19, 17, 32, 4, 6] and references cited therein. It is well known that, for small values of ϵ, standard numerical methods for solving such problems are unstable and do not give accurate results. Therefore it is important to develop suitable method for solving such problems, whose does not depend on the parameter value ϵ, i.e. methods that are convergent $\epsilon-$ uniformly. However, for (1.1) and (1.2) due to rapid oscillations of the solution, it is not convenient to use classical numerical methods (like- finite difference, finite element, finite volume) and the difficulty of various approximate numerical methods are available in the literature [26, 5, 13, 14, 18, 8, 20, 22, 42, 48, 49, 40].

For a survey of earlier results in theoretical analysis of singularly perturbed Volterra integro-differential equations and the implementation of various numerical techniques for these problems we refer to the text [26]. Various approximating aspects for singularly perturbed Volterra integro-differential equations have also been investigated in [25, 33, 38, 9, 10, 27, 30, 37, 39, 41].

Recently, there has been a growing interest in the numerical solution of Volterra integro-differential equations. For example, Koto [28] studied stability of Runge-Kutta method for Volterra integro-differential equations with a constant delay. The qualitative behaviour of numerical approximations to a non linear Volterra integro-differential equation with unbounded delay is investigated in [43]. Zang and Vandewalle [48] gave a numerical approximation based on the combination of general linear methods with compound quadrature rules. Gan [20] studied the analytic and numerical dissipativity of $\theta-$ methods. The adaptation of linear multistep methods for Volterra integro-differential equation has been discussed in [22]. Shakourifar and Enright [42] considered standard software based on the collocation method for solving considered class of equations. The numerical stability of linear Volterra integro-differential equation with real coefficient has been discussed by Zhao *et al.* [50]. Bellour and Bousselsal [8] used the Taylor polynomial method for approximating Volterra integro-differential equations.

The above mentioned papers, related with Volterra integro-differential equations were only concerned with regular cases. Also, singularly perturbed Volterra integro-differential equations frequently arise in many scientific applications. Wu and co researchers [47] investigated error behaviour of linear multistep method for singularly perturbed problem. He and Xu [21] discussed the exponential stability of impulsive problems. Amiraliyev *et al.* gave an exponetially fitted difference method on a uniform mesh except for a delay term in functional part and shown that the method is first order convergent uniformly in ϵ.

In the present paper, a simple but effective numerical method is proposed to approximate the solution. The method is based on Taylor series expansion and Adomian decomposition method, which can be easily adopted for the computational purpose of such problems. Along with the methodology, we have studied an error analysis for numerical treatment. Moreover, the boundedness of solution and stability analysis are also described.

The arrangement of this article is as follows, section 2 deals with basic concepts related to the work, in section 3 main results are discussed, in section 4 an example is illustrated for the verification of suggested scheme and some closing remarks are there in section 5.

2 Basic concepts

Adomian decomposition Method [46] The Adomian decomposition method (ADM) for a nonlinear problem is described as follows

$$L\Big(y(t)\Big) + N\Big(y(t)\Big) = f(t) \tag{2.1}$$

where f, y are system input and output respectively. L and N are linear and non-linear operators and L^{-1} stands for the inverse of L. The choice of the linear operator is not unique in a numerical scheme that involves ADM [1, 45**?**]. Although commonly we consider $L^{\xi} = \frac{d^{\xi}}{dt^{\xi}}(.)$ for ξ-th order differential operator and its inverse becomes ξ-fold definite integral operator and is denoted by $L^{-\xi}$. For $[0, T]$, the operator L^{-1} and L^{-2} are used to represent single and double integrals respectively. For the considered problem $\xi = 1$.
Applying L^{-1} to equation (2.1) we have

$$y = L^{-1}\Big(f(t)\Big) - L^{-1}\Big(N\Big(y(t)\Big)\Big). \tag{2.2}$$

Consider $N\Big(y(t)\Big)$ into a series of standard Adomian polynomials as

$$N\Big(y(t)\Big) = \sum_{k=0}^{\infty} A_k \tag{2.3}$$

and the solution is given by

$$y(t) = \sum_{k=0}^{\infty} y_k(t) \tag{2.4}$$

where A_k's depend on $y_0(t), y_1(t), y_2(t), y_3(t), ...$ and for $N\Big(y(t)\Big)$ we have the following standard relation

$$A_k = \frac{1}{k!}\frac{\partial^k}{\partial\lambda^k}\left[N\left(\sum_{s=0}^{\infty}\lambda^s y_s(t)\right)\right]_{\lambda=0}, k = 0, 1, 2, 3, ... \tag{2.5}$$

$N\Big(y(t)\Big)$ stands for the nonlinear function which is to be approximated by Adomian polynomials. Here we enlist first six Adomian polynomials

$$\begin{aligned}
A_0 &= f(y_0(t))\\
A_1 &= f'(y_0(t))y_1(t)\\
A_2 &= f'(y_0(t))y_2(t) + f''(y_0(t))\tfrac{y_1(t)^2}{2!}\\
A_3 &= f'(y_0(t))y_3(t) + f''(y_0(t))y_1(t)y_2(t) + f'''(y_0(t))\tfrac{y_1(t)^3}{3!}\\
A_4 &= f'(y_0(t))y_4(t) + f''(y_0(t))\Big(\tfrac{y_2(t)^2}{2!} + y_1(t)y_3(t)\Big) + f'''(y_0(t))\tfrac{y_1(t)^2 y_2(t)}{2!} + f^{iv}(y_0(t))\tfrac{y_1(t)^4}{4!}\\
A_5 &= f'(y_0(t))y_5(t) + f''(y_0(t))(y_2(t)y_3(t) + y_1(t)y_4(t)) + f'''(y_0(t))(\tfrac{y_1(t)y_2(t)^2}{2!} + \tfrac{y_1(t)^2 y_3(t)}{2!}) +\\
&\qquad f^{iv}(y_0(t))\tfrac{y_1(t)^3 y_2(t)}{3!} + f^{v}(y_0(t))\tfrac{y_1(t)^5}{5!}.
\end{aligned}$$

Substituting Adomian polynomials for $N\Big(y(t)\Big)$ in equation (2.2)

$$\sum_{k=0}^{\infty} y_k(t) = L^{-1}\Big(f(t)\Big) - L^{-1}\Big(\sum_{k=0}^{\infty} A_k\Big). \tag{2.6}$$

Finally, we have the classical recursion scheme

$$\begin{aligned}
y_0(t) &= L^{-1}\Big(f(t)\Big),\\
y_{k+1}(t) &= -L^{-1}\Big(A_k\Big).
\end{aligned} \tag{2.7}$$

Therefore, k-th approximation is given by

$$\sum_{i=0}^{k-1} y_i(t). \tag{2.8}$$

where the approximate solution is given by (2.4).

Definition 2.1. [23] Consider a non linear Volterra integro-differential equation

$$V^{'}(t) = G(t, V(t), V(\gamma(t))) + \int_0^t X(t, x, V(x), V(\gamma(x)))dx \tag{2.9}$$

where $t \in [0, T]$.
$G(t, V(t), V(\gamma(t))) : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $X(t, x, V(x), V(\gamma(x))) : [0, T] \times [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ both continuous function with respect to t, V and t, x, V respectively, $\gamma(t) : [0, T] \to [0, T]$ is delay function. If for a solution $V(t)$ of equation (2.9) satisfying

$$|V^{'}(t) - G(t, V(t), V(\gamma(t))) - \int_0^t X(t, x, V(x), V(\gamma(x)))dx| \leq \Phi(t), \tag{2.10}$$

for some $\Phi : [0, T] \to (0, \infty)$, there exist V_0 as a solution of (12) and a positive constant $\mathcal{M}$ such that

$$|V(t) - V_0(t)| \leq \mathcal{M}\Phi(t), \tag{2.11}$$

for all $t \in [0, T]$, then equation (2.9) is said to be Hyers-Ulam- Rassias stable on $[0, T]$. If $\Phi(t)$ is constant function, then (2.9) has Hyers-Ulam stability on $[0, T]$.

Definition 2.2. [24] A function $\tilde{d} : \tilde{Y} \to \tilde{Y}$ is said to be generalized metric on some non empty set $\tilde{Y}$ if and only if the following conditions are satisfied,
(I) $\tilde{d}(\tilde{u}, \tilde{v}) = 0 \Longleftrightarrow \tilde{u} = \tilde{v}$
(II) $\tilde{d}(\tilde{u}, \tilde{v}) = \tilde{d}(\tilde{v}, \tilde{u})$
(III) $\tilde{d}(\tilde{u}, \tilde{w}) \leq \tilde{d}(\tilde{u}, \tilde{v}) + \tilde{d}(\tilde{v}, \tilde{w})$
for all $\tilde{u}$, $\tilde{v}$, $\tilde{w} \in \tilde{Y}$.

Theorem 2.3. [24] Consider a complete metric space $(\tilde{Y}, \tilde{d})$ with a strictly contractive operator $\mathcal{F} : \tilde{Y} \to \tilde{Y}$. If there exist a natural number n such that $\tilde{d}(\mathcal{F}^{n+1}\tilde{u}, \mathcal{F}^n\tilde{u}) < \infty$ for some $\tilde{u} \in \tilde{Y}$, then the following hold:
(I) The sequence $\mathcal{F}^n\tilde{u}$ tends to a fixed point $\tilde{u}_0$ of $\mathcal{F}$,
(II) $\tilde{u}_0$ is the unique fixed point of $\mathcal{F}$ in $\mathcal{Y}_0 = \{\tilde{v} \in \tilde{Y} | \tilde{d}(\mathcal{F}^n\tilde{u}, \tilde{v}) < \infty\}$,
(III) If $\tilde{v} \in \mathcal{Y}_0$, then $\tilde{d}(\tilde{v}, \tilde{u}_0) \leq \frac{1}{1-\mathcal{L}}\tilde{d}(\mathcal{F}\tilde{v}, \tilde{v})$, where $\mathcal{L} < 1$ being a Lipschitz constant.

3 Main results

3.1. Numerical scheme

The numerical treatment deals with nonlinear delay integro-differential equations as a sequence of iterates. Due to the presence of delay term, perturbation parameter, and nonlinearity, more complexity enters into the problem. With the motivation to overcome this scenario, we present the current scheme which is simple to implement and less time consuming for larger calculations.
With the help of linearization through truncated Taylor series, equation (1.1) takes the form

$$\Big(\epsilon + \alpha(t)\tau\Big)\frac{dy(t)}{dt} = \alpha(t)y(t) + \int_0^t \mathcal{X}(t, s)h(y(s))ds + g(t) + \alpha(t)R_n(\tau, \mathcal{O}(y^{'}(t))) \tag{3.1}$$

where $R_n(\tau, \mathcal{O}(y^{'}(t)))$ represents the remainder term after truncation.

We consider the linear and nonlinear operators as

$$L\Big(y(t)\Big) = \tfrac{d}{dt}\Big\{(\epsilon + \alpha(t)\tau)y(t)\Big\}$$

and

$$N\Big(y(t)\Big) = -y(t)\Big(\tau\tfrac{d\alpha(t)}{dt} + \alpha(t)\Big) - \mathcal{I},$$

where, $\mathcal{I} = \int_0^t \mathcal{X}(t,s)h(y(s))ds + \alpha(t)R_n(\tau, \mathcal{O}(y^{'}(t))$.

Then (3.1) takes the form $L\Big(y(t)\Big) + N\Big(y(t)\Big) = g(t)$.
Now implementing ADM as described in section 2, we get the standard iteration scheme as follows

$$y_0(t) = L^{-1}\Big(\frac{g(t)}{\epsilon + \alpha(t)\tau}\Big), \tag{3.2}$$

$$y_{k+1}(t) = -L^{-1}\Big(\frac{A_k}{\epsilon + \alpha(t)\tau}\Big), k = 0, 1, 2, 3, ... \tag{3.3}$$

3.2. Error analysis

This section is entirely devoted to the convergence and error analysis. In this context two theorems are established. The following assumptions are taken into consideration for the theoretical study presented in this section.

(A1) $h : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous function. That is for all $y_1(t), y_2(t) \in \mathbb{R}$ there exists some constant $\mathcal{K}_2^* > 0$ such that

$$|h(y_1(t)) - h(y_2(t))| \leq \mathcal{K}_2^* |y_1(t) - y_2(t)|, \text{ where } t \in \mathcal{A}.$$

(A2) $\mathcal{X} : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ and $g : \mathcal{A} \to \mathbb{R}$ are continuous functions such that

$$|\mathcal{X}(t,s)| \leq \mathcal{X}^*, |g(t)| \leq \mathcal{G}$$

where $\mathcal{X}^* = \sup_{(t,s)\in\mathcal{A}\times\mathcal{A}} |\mathcal{X}(t,s)|$ and $\mathcal{G} = \sup_{t\in\mathcal{A}} |g(t)|$.

Theorem 3.2.1 The series solution of the proposed scheme converges whenever $0 < \omega^* < 1$ and $|y_1| < \infty$, where $\omega^* = lT + \frac{1}{2}\mathcal{X}^*\mathcal{K}_2^* T^2$, $\;l = \max\Big\{|\tau\frac{d\alpha(t)}{dt} + \alpha(t)| : t \in \mathcal{A}\Big\}$.

Proof. We consider two arbitrary partial sums S_p and S_q with $p \geq q$ of the series solution (2.4). To prove the convergence, it is enough to deduce that $\{S_p\}$ is a Cauchy sequence in the related Banach space.

$$\begin{aligned}
||S_p - S_q|| &= \max\Big\{|S_p - S_q| : t \in \mathcal{A}\Big\} \\
&= \max\Big\{|\textstyle\sum_{i=q+1}^{p} y_i(t)| : t \in \mathcal{A}\Big\} \\
&= \max\Big\{| - \textstyle\sum_{i=q+1}^{p} L^{-1}\Big(\frac{A_{i-1}}{\epsilon+\alpha(t)\tau}\Big)| : t \in \mathcal{A}\Big\} \\
&\leq \max\Big\{|L^{-1}\Big(\textstyle\sum_{i=q+1}^{p} A_{i-1}\Big)| : t \in \mathcal{A}\Big\} \\
&= \max\Big\{|L^{-1}\Big(\textstyle\sum_{i=q}^{p-1} A_i\Big)| : t \in \mathcal{A}\Big\}
\end{aligned}$$

Thus we have

$$||S_p - S_q|| \leq \max\ \Big\{|L^{-1}\Big(\sum_{i=q}^{p-1} A_i\Big)| : t \in \mathcal{A}\Big\}. \tag{3.4}$$

Adomian decomposition of the non linear term provides $A_n = N(S_n) - \sum_{i=0}^{n-1} A_i$ and consequently $\sum_{i=q}^{p-1} A_i = N(S_{p-1}) - N(S_{q-1})$. From the construction we obtain L^{-1} as a definite integral operator on $[0, T]$.
Therefore (3.4) leads to the following form

$$||S_p - S_q|| \leq \max\Big\{|L^{-1}\Big(N(S_{p-1}) - N(S_{q-1})\Big)| : t \in \mathcal{A}\Big\}. \tag{3.5}$$

From the choice of non linear operator, we have

$$N(S_p) = -S_p\Big(\tau\frac{d\alpha(t)}{dt} + \alpha(t)\Big) - \mathcal{I},$$

and consequently the truncated series leads to the following inequality

$$\Big(N(S_{p-1}) - N(S_{q-1})\Big) \leq -\Big(\tau\frac{d\alpha(t)}{dt} + \alpha(t)\Big)\Big(S_{p-1} - S_{q-1}\Big) - \int_0^t \mathcal{X}(t,s) \Big(h(S_{p-1}) - h(S_{q-1})\Big)ds. \tag{3.6}$$

Implementing L^{-1} on (3.6) we have the following estimates

$$\int_0^T \Big(N(S_{p-1}) - N(S_{q-1})\Big)dt \leq -\int_0^T \Big(\tau\frac{d\alpha(t)}{dt} + \alpha(t)\Big)\Big(S_{p-1} - S_{q-1}\Big)dt - \int_0^T \int_0^t \mathcal{X}(t,s)\Big(h(S_{p-1}) - h(S_{q-1})\Big)dsdt$$

thus we obtain

$$\begin{aligned} |\int_0^T \Big(N(S_{p-1}) - N(S_{q-1})\Big)dt| &\leq |\int_0^T \Big(\tau\frac{d\alpha(t)}{dt} + \alpha(t)\Big)\Big(S_{p-1} - S_{q-1}\Big)dt| + \\ &\quad |\int_0^T \int_0^t \mathcal{X}(t,s)\Big(h(S_{p-1}) - h(S_{q-1})\Big)dsdt| \\ &\leq \int_0^T |\tau\frac{d\alpha(t)}{dt} + \alpha(t)|\ |S_{p-1} - S_{q-1}|dt + \\ &\quad \int_0^T \int_0^t |\mathcal{X}(t,s)|\ |h(S_{p-1}) - h(S_{q-1})|dsdt \end{aligned}$$

by assumption $(A1)$, $(A2)$ and $l = \max\Big\{|\tau\frac{d\alpha(t)}{dt} + \alpha(t)| : t \in \mathcal{A}\Big\}$ we get

$$\begin{aligned} \max\Big\{|L^{-1}\Big(N(S_{p-1}) - N(S_{q-1})\Big)| : t \in \mathcal{A}\Big\} &\leq l\ \max\Big\{|L^{-1}\Big(S_{p-1} - S_{q-1}\Big)| : t \in \mathcal{A}\Big\} + \\ &\quad \mathcal{X}^*\mathcal{K}_2^* \max\Big\{|L^{-2}\Big(S_{p-1} - S_{q-1}\Big)| : t \in \mathcal{A}\Big\} \\ &\leq \Big(lT + \frac{1}{2}\mathcal{X}^*\mathcal{K}_2^*T^2\Big)\Big\|S_{p-1} - S_{q-1}\Big\|. \end{aligned}$$

Therefore we have the following inequality

$$\max\Big\{|L^{-1}\Big(N(S_{p-1}) - N(S_{q-1})\Big)| : t \in \mathcal{A}\Big\} \leq \omega^*\Big\|S_{p-1} - S_{q-1}\Big\| \tag{3.7}$$

where $\omega^* = lT + \frac{1}{2}\mathcal{X}^*\mathcal{K}_2^*T^2$. Combining inequality (3.5) and (3.7) we have

$$||S_p - S_q|| \leq \omega^*||S_{p-1} - S_{q-1}||.$$

For $p = q + 1$,

$$||S_{q+1} - S_q|| \leq \omega^*||S_q - S_{q-1}|| \leq {\omega^*}^2||S_{q-1} - S_{q-2}|| \leq ... \leq {\omega^*}^q||S_1 - S_0||.$$

Implementing triangle inequality,

$$||S_p - S_q|| \leq ||S_{q+1} - S_q|| + ||S_{q+2} - S_{q+1}|| + ... + ||S_p - S_{p-1}||$$

$$\leq \Big(\omega^{*q} + \omega^{*q+1} + \omega^{*q+2} + ... + \omega^{*p-1}\Big)||S_1 - S_0||$$
$$\leq \omega^{*q}\Big(\tfrac{1-\omega^{*p-q}}{1-\omega^*}\Big)||y_1(t)||.$$

As $0 < \omega^* < 1$, so $1 - \omega^{*p-q} < 1$.
Therefore

$$||S_p - S_q|| \leq \tfrac{\omega^{*q}}{1-\omega^*} \max\Big\{|y_1(t)| : t \in \mathcal{A}\Big\}.$$

Since $|y_1(t)| < \infty$ for $t \in \mathcal{A}$, so $||S_p - S_q|| \longrightarrow 0$ as $q \longrightarrow \infty$.

Hence, $\{S_p\}$ is a Cauchy sequence in the considered Banach space. This completes the proof.

Theorem 3.2.2 The maximum truncation error for the approximate solution to the problem (1.1) – (1.2) is estimated to be

$$\max\Big\{|y(t) - \sum_{i=0}^{q} y_i(t)| : t \in \mathcal{A}\Big\} \leq \frac{\omega^{*q}}{1-\omega^*} \max\Big\{|y_1(t)| : t \in \mathcal{A}\Big\}.$$

Proof. From Theorem 3.2.1 we obtain

$$||S_p - S_q|| \leq \tfrac{\omega^{*q}}{1-\omega^*} \max\Big\{|y_1(t)| : t \in \mathcal{A}\Big\}.$$

As $p \longrightarrow \infty$ then $S_p \longrightarrow y(t)$, therefore we have

$$||y(t) - S_q|| \leq \tfrac{\omega^{*q}}{1-\omega^*}\max\Big\{|y_1(t)| : t \in \mathcal{A}\Big\}$$

and the maximum truncation error is to be estimated as

$$\max\Big\{|y(t) - \textstyle\sum_{i=0}^{q} y_i(t)| : t \in \mathcal{A}\Big\} \leq \tfrac{\omega^{*q}}{1-\omega^*} \max\Big\{|y_1(t)| : t \in \mathcal{A}\Big\}$$

This concludes the proof.

3.3. Boundedness of solution

In this section we impose additional condition on α and proceed with the following assumptions

(A3) α is considered to be constant function such that $\Big(\epsilon + \alpha\tau\Big)$ is non vanishing for all choices of ϵ and τ.

(A4) $y : \mathcal{A} \to \mathbb{R}$ is smooth function such that there exists some positive constant $\mathcal{M}$ such that $|\mathcal{O}(y^{'})| < \mathcal{M}$, where $\mathcal{O}(y^{'})$ stands for higher order derivatives of y involved in Taylor series expansion of $y(t-\tau)$ so that

$$|y(t-z)| \leq |y(t)| + \mathcal{M}\Big(exp(T) - 1\Big), \text{ where } t, z \in \mathcal{A}.$$

Now we present a theorem in support of the boundedness of solution.

Theorem 3.3.1 Under the assumptions (A1)-(A4) equation (1.1) posses bounded solution provided the following condition holds

$$\frac{\Lambda_{t,\mu,\epsilon,\alpha,\tau} exp(\omega\alpha t) + \mathcal{M}\Big(exp(T)-1\Big)\mathcal{X}^*\mathcal{K}_2^*\Big\{\frac{(\omega\alpha t-1)exp(\omega\alpha t)+1}{\alpha^2\omega}\Big\}}{1 - \omega\mathcal{X}^*\mathcal{K}_2^* \int_0^t z exp(\omega\alpha z)dz} \leq \mathcal{B}^*$$

where $\Lambda_{t,\mu,\epsilon,\alpha,\tau} = |\mu| + \Big\{\omega\alpha\mathcal{M}(e^{-\tau} - 1) + \omega\mathcal{G}\Big\} + \Big\{\omega|\mu|\mathcal{X}^*\mathcal{K}_2^* + \omega\mathcal{X}^*|h(\mu)|\Big\}$
$\int_0^t \sigma exp(-\omega\alpha\sigma)d\sigma$

and $\mathcal{B}^*$ is some constant with $t - \sigma = z$.

Proof. Applying truncated Taylor series in equation (1.1)

$$\Big(\epsilon + \alpha\tau\Big)\frac{dy(t)}{dt} = \alpha y(t) + \int_0^t \mathcal{X}(t,s)h(y(s))ds + g(t) + \alpha R_n(\tau, \mathcal{O}(y^{'}(t))).$$

where $R_n(\tau, \mathcal{O}(y^{'}(t)))$ is remainder term after truncation, we have

$$\frac{dy(t)}{dt} = \omega\alpha y(t) + \omega \int_0^t \mathcal{X}(t,s)h(y(s))ds + \omega g(t) + \omega\alpha R_n(\tau, \mathcal{O}(y^{'}(t))), \text{ with } \omega = \Big(\epsilon + \alpha\tau\Big)^{-1}.$$

This implies

$$\frac{dy(t)}{dt} - \omega\alpha y(t) = \omega H(t),$$

where $H(t) = \int_0^t \mathcal{X}(t,s)h(y(s))ds + g(t) + \alpha R_n(\tau, \mathcal{O}(y^{'}(t)))$. We introduce integrating factor and proceed as follows

$$exp(-\omega\alpha t)y(t) = y(0) + \int_0^t \omega exp(-\omega\alpha\sigma)\Big\{ \int_0^\sigma \mathcal{X}(\sigma,s)h(y(s))ds + g(\sigma) + \alpha R_n(\tau, \mathcal{O}(y^{'}(\sigma))\Big\}d\sigma.$$

Thus we obtain the following estimates

$$\begin{aligned} exp(-\omega\alpha t)|y(t)| \leq & |\mu| + \omega\alpha \int_0^t exp(-\omega\alpha\sigma)|R_n(\tau, \mathcal{O}(y^{'}(\sigma)))|d\sigma + \omega \int_0^t exp(-\omega\alpha\sigma)|g(\sigma)|d\sigma \\ & + \omega \int_0^t \int_0^\sigma exp(-\omega\alpha\sigma)|\mathcal{X}(\sigma,s)||h(y(s))|dsd\sigma. \end{aligned}$$

Assumptions (A1)-(A4) provide

$$\begin{aligned} exp(-\omega\alpha t)|y(t)| \leq & |\mu| + \Big\{\omega\alpha\mathcal{M}(e^{-\tau}-1) + \omega\mathcal{G}\Big\} \int_0^t exp(-\omega\alpha\sigma)d\sigma + \omega\mathcal{X}^* \int_0^t \int_0^\sigma exp(-\omega\alpha\sigma)|h(y(s)) - h(y(0))|dsd\sigma \\ & + \omega\mathcal{X}^* \int_0^t \int_0^\sigma exp(-\omega\alpha\sigma)|h(y(0))|dsd\sigma. \end{aligned}$$

Consequently we obtain

$$\begin{aligned} exp(-\omega\alpha t)|y(t)| \leq & \mu + \Big\{\omega\alpha\mathcal{M}(e^{-\tau}-1) + \omega\mathcal{G}\Big\} \int_0^t exp(-\omega\alpha\sigma)d\sigma + \omega\mathcal{X}^*\mathcal{K}_2^* \int_0^t \int_0^\sigma exp(-\omega\alpha\sigma)|y(s) - \mu|dsd\sigma \\ & + \omega\mathcal{X}^*|h(\mu)| \int_0^t \sigma exp(-\omega\alpha\sigma)d\sigma. \end{aligned}$$

This leads to

$$exp(-\omega\alpha t)|y(t)| \leq \Lambda_{t,\mu,\epsilon,\alpha,\tau} + \omega\mathcal{X}^*\mathcal{K}_2^* \int_0^t (t-\sigma)|y(\sigma)|exp(-\omega\alpha\sigma)d\sigma,$$

where

$$\Lambda_{t,\mu,\epsilon,\alpha,\tau} = |\mu| + \Big\{\omega\alpha\mathcal{M}(e^{-\tau}-1) + \omega\mathcal{G}\Big\} + \Big\{\omega|\mu|\mathcal{X}^*\mathcal{K}_2^* + \omega\mathcal{X}^*|h(\mu)|\Big\} \int_0^t \sigma exp(-\omega\alpha\sigma)d\sigma.$$

This implies

$$|y(t)| \leq \Lambda_{t,\mu,\epsilon,\alpha,\tau} exp(\omega\alpha t) + \omega\mathcal{X}^*\mathcal{K}_2^* \int_0^t (t-\sigma)|y(\sigma)|exp\Big\{\omega\alpha(t-\sigma)\Big\}d\sigma.$$

We perform the substitution $t - \sigma = z$ in the integral part and considering the assumption (A4) we obtain

$$|y(t)| \leq \Lambda_{t,\mu,\epsilon,\alpha,\tau} exp(\omega\alpha t) + \Big\{\omega\mathcal{X}^*\mathcal{K}_2^* \int_0^t z exp(\omega\alpha z)dz\Big\}|y(t)| + \mathcal{M}\Big(exp(T) - 1\Big)\omega\mathcal{X}^*\mathcal{K}_2^* \int_0^t z exp(\omega\alpha z)dz.$$

Finally, we obtain

$$|y(t)| \le \frac{\Lambda_{t,\mu,\epsilon,\alpha,\tau} exp(\omega\alpha t)+\mathcal{M}\Big(exp(T)-1\Big)\mathcal{X}^*\mathcal{K}_2^*\Big\{\frac{(\omega\alpha t-1)exp(\omega\alpha t)+1}{\alpha^2\omega}\Big\}}{1-\omega\mathcal{X}^*\mathcal{K}_2^*\int_0^t zexp(\omega\alpha z)dz}.$$

Since $t \in (0,T]$ with $0 < T < T^*$, the quantity on the right side of the above inequality is bounded by some constant $\mathcal{B}^*$. This completes the proof.

3.4. Stability analysis

In this section, we have discussed stability criteria for equation (1.1) under some relevant assumptions. Motivated from the paper [25] we propose a theorem to prove Hyer-Ulam-Rassias stability. For this purpose, we consider the complete generalized metric space $(\tilde{Y},\tilde{d})$, where $\tilde{Y}$ be the set of all real valued continuous functions on $[0,T]$ and $\tilde{d}(\tilde{u},\tilde{w}) = \text{Inf}\ \{\tilde{k} \in [0,\infty) : |\tilde{u}(t)-\tilde{w}(t)| \le \tilde{k}\tilde{\sigma}(t) \text{ for } t \in [0,T]\}$. Before entering into the main result related to stability, first, we introduce the following assumptions

(A5) The kernel $\mathcal{X}(t,s)$ and $\alpha(t) : [0,T] \to \mathbb{R}$ are bounded by some non-negative constant $\mathcal{K}_2$ and $\mathcal{K}_1^*$ respectively.

(A6) There exists some positive constant $\mathcal{K}$ such that $|u_1^{\xi}(t)-u_2^{\xi}(t)| \le \mathcal{K}_1|u_1(t)-u_2(t)|$ where ξ denotes ξ-th derivative.

(A7) $u : [0,T] \to \mathbb{R}$ is a continuously differentiable function satisfies

$\Big|\frac{du}{dt} - \frac{1}{\epsilon}\Big(\alpha(t)u(t-\tau) + \int_0^t \mathcal{X}(t,s)h(u(s))ds + g(t)\Big)\Big| \le \tilde{\sigma}(t)$, for all $t \in [0,T]$, where $\tilde{\sigma} : [0,T] \to \mathbb{R}$ is a continuous function.

(A8) There exists $\mathcal{N} > 0$ such that $\int_0^t \tilde{\sigma}(t^*)dt^* \le \mathcal{N}\tilde{\sigma}(t)$ for all $t \in [0,T]$.

(A9) There exists a unique continuous function $w_0 : [0,T] \to \mathbb{R}$ such that

$$w_0 = \frac{1}{\epsilon}\Big(\int_0^t \alpha(s)w_0(s-\tau)ds + \int_0^t\int_0^s \mathcal{X}(s,s^{'})h(w_0(s))ds^{'}ds + g(t)\Big).$$

Theorem 3.4.1 Under the assumptions (A1) and (A5)-(A9) the integro differential equation

$$\epsilon\frac{dy(t)}{dt} = \alpha(t)y(t-\tau) + \int_0^t \mathcal{X}(t,s)h(y(s))ds + g(t)$$

with a prescribed initial value for $y(t)$, possess Hyers-Ulam-Rassias stability in a complete generalized metric space .

Proof. On $(\tilde{Y},\tilde{d})$ we define the operator $\tilde{T} : \tilde{Y} \to \tilde{Y}$ as

$$\tilde{T}\tilde{y}(t) = \tilde{y}(0) + \frac{1}{\epsilon}\Big(\int_0^t \alpha(s)\tilde{y}(s-\tau)ds + \int_0^t\int_0^s \mathcal{X}(s,s^{'})h(\tilde{y}(s))ds^{'}ds + g(t)\Big) \tag{3.8}$$

Initially, we establish that $\tilde{T}$ is strictly contraction. For that, we consider $\tilde{u}(t)$, $\tilde{w}(t) \in \tilde{Y}$ and proceed as follows

$$\begin{aligned}
|(\tilde{T}\tilde{u})(t) - (\tilde{T}\tilde{w})(t)| =& \Big|\frac{1}{\epsilon}\int_0^t \alpha(s)\Big(\tilde{u}(s-\tau) - \tilde{w}(s-\tau)\Big)ds + \frac{1}{\epsilon}\int_0^t\int_0^s \mathcal{X}(s,s^{'})\Big(h(\tilde{u}(s^{'})) - h(\tilde{w}(s^{'}))\Big)ds^{'}ds\Big| \\
\le& \Big|\frac{1}{\epsilon}\int_0^t \alpha(s)\sum_{n=0}^{\infty}\frac{\tau^n}{n!}\Big(\tilde{u}^n(s) - \tilde{w}^n(s)\Big)ds\Big| + \Big|\frac{1}{\epsilon}\int_0^t\int_0^s \mathcal{X}(s,s^{'})\Big(h(\tilde{u}(s^{'})) - h(\tilde{v}(s^{'}))\Big)ds^{'}ds\Big| \\
\le& \frac{\mathcal{K}_1}{|\epsilon|}\int_0^t \Big|\alpha(s)\Big|\sum_{n=0}^{\infty}\frac{\tau^n}{n!}\Big|\tilde{u}(s) - \tilde{w}(s)\Big|ds + \frac{\mathcal{K}_2}{|\epsilon|}\int_0^t\int_0^s \Big|h(\tilde{u}(s^{'})) - h(\tilde{w}(s^{'}))\Big|ds^{'}ds \\
\le& \frac{\mathcal{K}_1\mathcal{K}_1^*e^{\tau}}{|\epsilon|}\int_0^t \Big|\tilde{u}(s) - \tilde{w}(s)|ds + \frac{\mathcal{K}_2\mathcal{K}_2^*}{|\epsilon|}\int_0^t\int_0^s \Big|\tilde{u}(s^{'}) - \tilde{w}(s^{'})\Big|ds^{'}ds^{'} \\
\le& \frac{\mathcal{K}_1\mathcal{K}_1^*e^{\tau}}{|\epsilon|}\mathcal{K}_{\tilde{u}\tilde{w}}\int_0^t \tilde{\sigma}(s)ds + \frac{\mathcal{K}_2\mathcal{K}_2^*}{|\epsilon|}\mathcal{K}_{\tilde{u}\tilde{w}}\int_0^t\int_0^s \tilde{\sigma}(s^{'})ds^{'}ds \\
\le& \mathcal{K}_{\tilde{u}\tilde{w}}\tilde{\sigma}(t)\Big(\frac{\mathcal{K}_1\mathcal{K}_1^*e^{\tau}}{|\epsilon|}\mathcal{N} + \frac{\mathcal{K}_2\mathcal{K}_2^*}{|\epsilon|}\mathcal{N}^2\Big),
\end{aligned}$$

thus, $\tilde{d}(\tilde{T}\tilde{u},\tilde{T}\tilde{w}) \le \mathcal{K}_{\tilde{u}\tilde{w}}\tilde{\sigma}(t)\Big(\frac{\mathcal{K}_1\mathcal{K}_1^*e^{\tau}}{|\epsilon|}\mathcal{N} + \frac{\mathcal{K}_2\mathcal{K}_2^*}{|\epsilon|}\mathcal{N}^2\Big)$. Hence, we obtain $\tilde{d}(\tilde{T}\tilde{u},\tilde{T}\tilde{w}) \le \Big(\frac{\mathcal{K}_1\mathcal{K}_1^*e^{\tau}}{|\epsilon|}\mathcal{N} + \frac{\mathcal{K}_2\mathcal{K}_2^*}{|\epsilon|}\mathcal{N}^2\Big)\tilde{d}(\tilde{u},\tilde{w})$ for all $\tilde{u},\tilde{w} \in \tilde{Y}$, with $0 < \Big(\frac{\mathcal{K}_1\mathcal{K}_1^*e^{\tau}}{|\epsilon|}\mathcal{N} + \frac{\mathcal{K}_2\mathcal{K}_2^*}{|\epsilon|}\mathcal{N}^2\Big) < 1$.

For any $\tilde{u}_0 \in \tilde{Y}$, there exists some non negative constant $\mathcal{K}$ such that

$$\begin{aligned}
|\tilde{T}\tilde{u}_0(t) - \tilde{u}_0(t)| =& \Big|\tilde{u}(0) + \frac{1}{\epsilon}\Big(\int_0^t \alpha(s)\tilde{u}_0(s-\tau)ds + \int_0^t\int_0^s \mathcal{X}(s,s^{'})h(\tilde{u}_0(s))ds^{'}ds + g(t)\Big) - \tilde{u}_0\Big| \\
\le& \mathcal{K}\tilde{\sigma}(t),
\end{aligned}$$

for all $t \in [0,T]$. As $\min\Big\{\tilde{\sigma}(t) : t \in [0,T]\Big\} > 0$ and $\tilde{u}_0$, $\alpha(s)\tilde{u}_0(s-\tau)$, $\mathcal{X}(s,s^{'})h(\tilde{u}_0(s))$ are bounded, we get $\tilde{d}(\tilde{T}\tilde{u}_0,\tilde{u}_0) < \infty$. By Theorem 2.3. we have some continuous function $w_0 : [0,T] \to \mathbb{R}$ such that $\tilde{T}^n\tilde{u}_0 \to w_0$ in $(\tilde{Y},\tilde{d})$ and $\tilde{T}w_0 = w_0$. Since $\tilde{u},\tilde{u}_0$ are bounded in $[0,T]$ and $\min\Big\{\tilde{\sigma}(t) : t \in [0,T]\Big\} > 0$, there exists a constant $\mathcal{K}_{\tilde{u}} \in [0,\infty)$ such that, $\tilde{d}(\tilde{u}_0(t),\tilde{u}(t)) \le \mathcal{K}_{\tilde{u}}\tilde{\sigma}(t)$. Thus we have $|\tilde{u}_0(t)-\tilde{u}(t)| < \infty$, for all $\tilde{u} \in \tilde{Y}$, which yields $\{\tilde{u} \in \tilde{Y}|\tilde{d}(\tilde{u}_0,\tilde{u}) < \infty\} = \tilde{Y}$. The conclusion of the above mentioned theorem suggests that w_0 is the unique continuous function.

Now we consider assumption (A9) and integrating each term on $[0,t]$ we get,

$$-\int_0^t \tilde{\sigma}(t^*)dt^* < \tilde{u}(t)-\tilde{u}(0)-\frac{1}{\epsilon}\Big(\int_0^t \alpha(s)\tilde{u}(s-\tau)ds+\int_0^t\int_0^s \mathcal{X}(s,s^{'})h(\tilde{u}(s))ds^{'}ds+g(t)\Big) < \int_0^t \tilde{\sigma}(t^*)dt^*.$$

This implies that

$$\Big|\tilde{u}(t)-\tilde{u}(0)-\frac{1}{\epsilon}\Big(\int_0^t \alpha(s)\tilde{u}(s-\tau)ds+\int_0^t\int_0^s \mathcal{X}(s,s^{'})h(\tilde{u}(s))ds^{'}ds+g(t)\Big)\Big| < \int_0^t \tilde{\sigma}(t^*)dt^* \int_0^t \tilde{\sigma}(t^*)dt^*.$$

Thus,

$$\Big|\tilde{u}(t)-(\tilde{T}\tilde{u})(t)\Big| \le \int_0^t \tilde{\sigma}(t^*)dt^* \le \mathcal{N}\tilde{\sigma}(t).$$

Then we have

$$\tilde{d}(\tilde{u},\tilde{T}\tilde{u}) \le \mathcal{N}\tilde{\sigma}(t).$$

By implementing Theorem 2.3. and the above inequality we conclude that

$$\tilde{d}(\tilde{u},w_0) \le \frac{1}{1-\Big(\frac{\mathcal{K}_1\mathcal{K}_1^*e^{\tau}}{|\epsilon|}\mathcal{N}+\frac{\mathcal{K}_2\mathcal{K}_2^*}{|\epsilon|}\mathcal{N}^2\Big)}\tilde{d}(\tilde{T}\tilde{u},\tilde{u})$$

and so

$$\tilde{d}(\tilde{u},w_0) \le \frac{\mathcal{N}\tilde{\sigma}(t)}{1-\Big(\frac{\mathcal{K}_1\mathcal{K}_1^*e^{\tau}}{|\epsilon|}\mathcal{N}+\frac{\mathcal{K}_2\mathcal{K}_2^*}{|\epsilon|}\mathcal{N}^2\Big)}.$$

Thus Hyer-Ulam-Rassias stability of considered integro-differential equation is established. In this context, Hyer-Ulam stability occurs as a particular case when $\tilde{\sigma}(t)$ turns out to be a constant function.

4 Numerical results and discussion

In support of the proposed testimony, we consider the following examples,
Example 1.

$$\epsilon v'(t) = 1-e^{-1}-v(t-\tau)+2\int_0^t v^2(p)dp \tag{4.1}$$

with $v(0)=1$ as initial condition, $t \in (0,T] = (0,\frac{1}{2}]$. The exact solution is $v(t)=e^{-t}$ for the case of $\epsilon=1$ and $\tau=0$ (no delay). The following representative tables demonstrate the numerical results obtained by the proposed method. Also, a comparative study between the present method and the spline function method used in the literature [26] is illustrated in table 6 for some test values of ϵ and τ. For that purpose error estimation is done using $\mathcal{E}v(t) = v(t)-v_{\epsilon,\tau}(t)$ where $v_{\epsilon,\tau}(t)$ is approximation for particular choice of ϵ and τ.

Table 1: Fixing $\epsilon=1$ and pure delay term as $\tau = 0, 0.25, 0.75, 0.5$

t	0	0.1	0.2	0.3	0.4	0.5
$v_{1,0}$	0.98393	0.82877	0.75587	0.70933	0.68432	0.67690
$v_{1,0.25}$	0.99621	0.85219	0.7514 1	0.68487	0.64543	0.62746
$v_{1,0.5}$	0.95087	0.76881	0.65231	0.58425	0.55198	0.54613
$v_{1,0.75}$	0.97149	0.65349	0.48719	0.41273	0.39383	0.40854

Table 2: Fixing $\epsilon = 4$ and pure delay term as $\tau = 0, 0.25, 0.75, 0.5$

t	0	0.1	0.2	0.3	0.4	0.5
$v_{4,0}$	1.12087	1.05151	0.99602	0.95280	0.92040	0.89755
$v_{4,0.25}$	0.91709	0.87185	0.84070	0.82195	0.81413	0.82244
$v_{4,0.5}$	0.91746	0.87060	0.83823	0.81860	0.81017	0.81157
$v_{4,0.75}$	0.91816	0.86947	0.83572	0.81508	0.80593	0.80687

Table 3: Fixing $\epsilon = 6$ and pure delay term as $\tau = 0, 0.25, 0.75, 0.5$

t	0	0.1	0.2	0.3	0.4	0.5
$v_{6,0}$	0.91445	0.87749	0.85263	0.83855	0.83405	0.83808
$v_{6,0.25}$	0.91463	0.87709	0.85179	0.83738	0.83264	0.83650
$v_{6,0.5}$	0.91485	0.87665	0.85084	0.83605	0.83103	0.83471
$v_{6,0.75}$	0.91816	0.86947	0.83572	0.81508	0.80593	0.80687

Table 4: Fixing $\epsilon = 10$ and pure delay term as $\tau = 0, 0.25, 0.75, 0.5$

t	0	0.1	0.2	0.3	0.4	0.5
$v_{10,0}$	0.97329	0.93527	0.90810	0.89065	0.88190	0.88095
$v_{10,0.25}$	0.91269	0.88113	0.86045	0.84953	0.84734	0.85235
$v_{10,0.5}$	0.91277	0.88098	0.86013	0.84907	0.84678	0.85235
$v_{10,0.75}$	0.91284	0.88081	0.85978	0.84858	0.84619	0.85322

Table 5: Fixing $\epsilon = 20$ and pure delay term as $\tau = 0, 0.25, 0.75, 0.5$

t	0	0.1	0.2	0.3	0.4	0.5
$v_{20,0}$	0.91122	0.88407	0.86686	0.85860	0.85838	0.86541
$v_{20,0.25}$	0.91124	0.88409	0.86689	0.85864	0.85845	0.86551
$v_{20,0.5}$	0.91125	0.88400	0.85838	0.85838	0.86011	0.86511
$v_{20,0.75}$	0.91128	0.88397	0.86664	0.85827	0.85827	0.85790

The above representative tables demonstrate the numerical verification for the proposed method. For fixed perturbation parameter ϵ and delay term τ it is clear from the tables that the solution for different inputs does not fluctuate on a rapid scale, which has already been established in Theorem 3.4.1 from a theoretical perspective. Now we present the graphical illustration for the numerical results.

The graphs in figure 1 corresponding to table 1-5 depict the behaviour of the solution obtained by the proposed method. As the value of the perturbation parameter increases, the fluctuation of the solution curve reduces. This fact is also supported by the discussion of Hyer-Ulam-Rassias stability. Also we present the following comparative study in table 6 between the proposed method and the spline method. The graphical overview is considered in figure 2.

Table 6: Comparison of absolute error between the developed method and spline method
(Example 1)

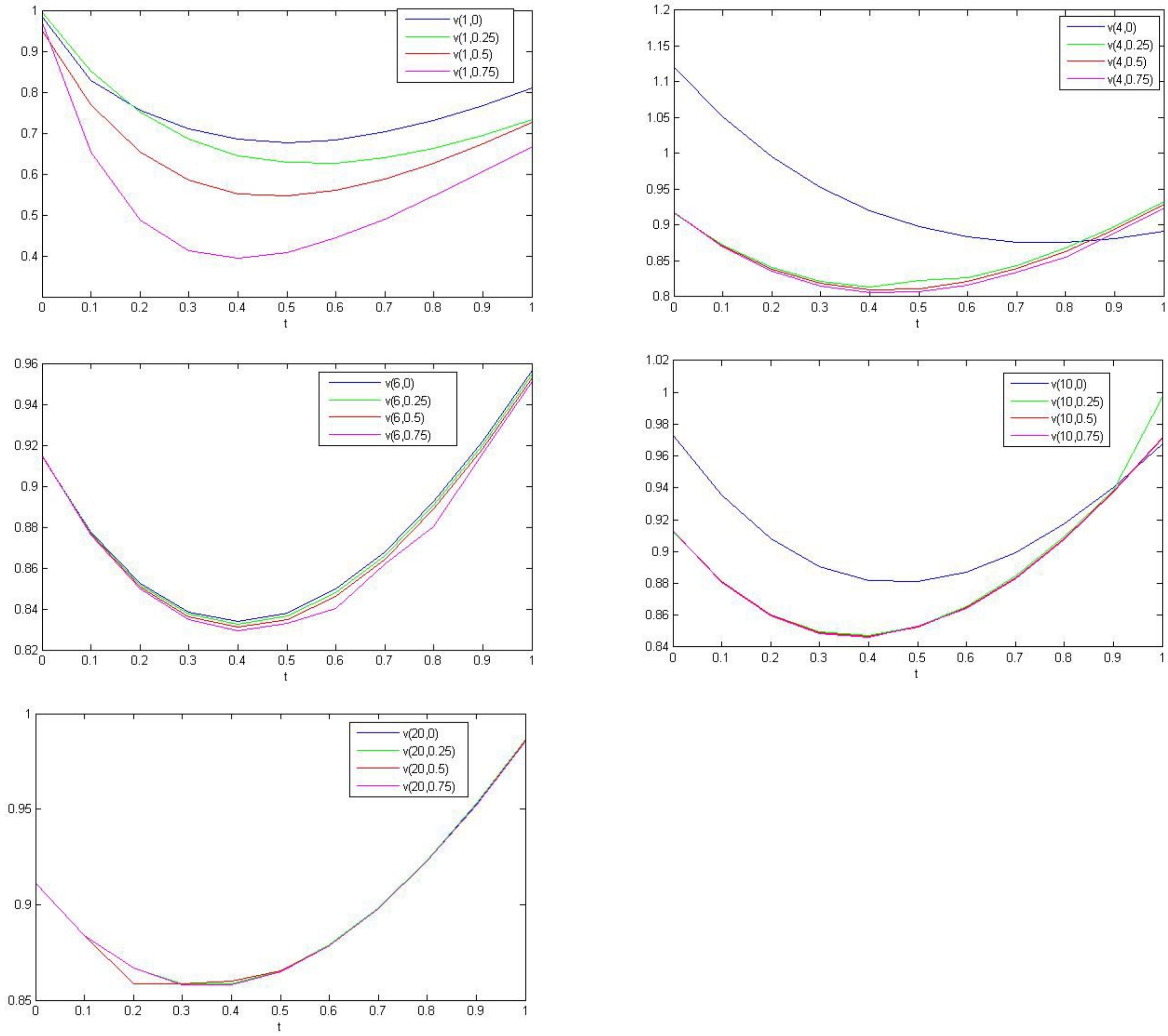

Figure 1: Graphical representation of exact and approximate solutions for Example 1

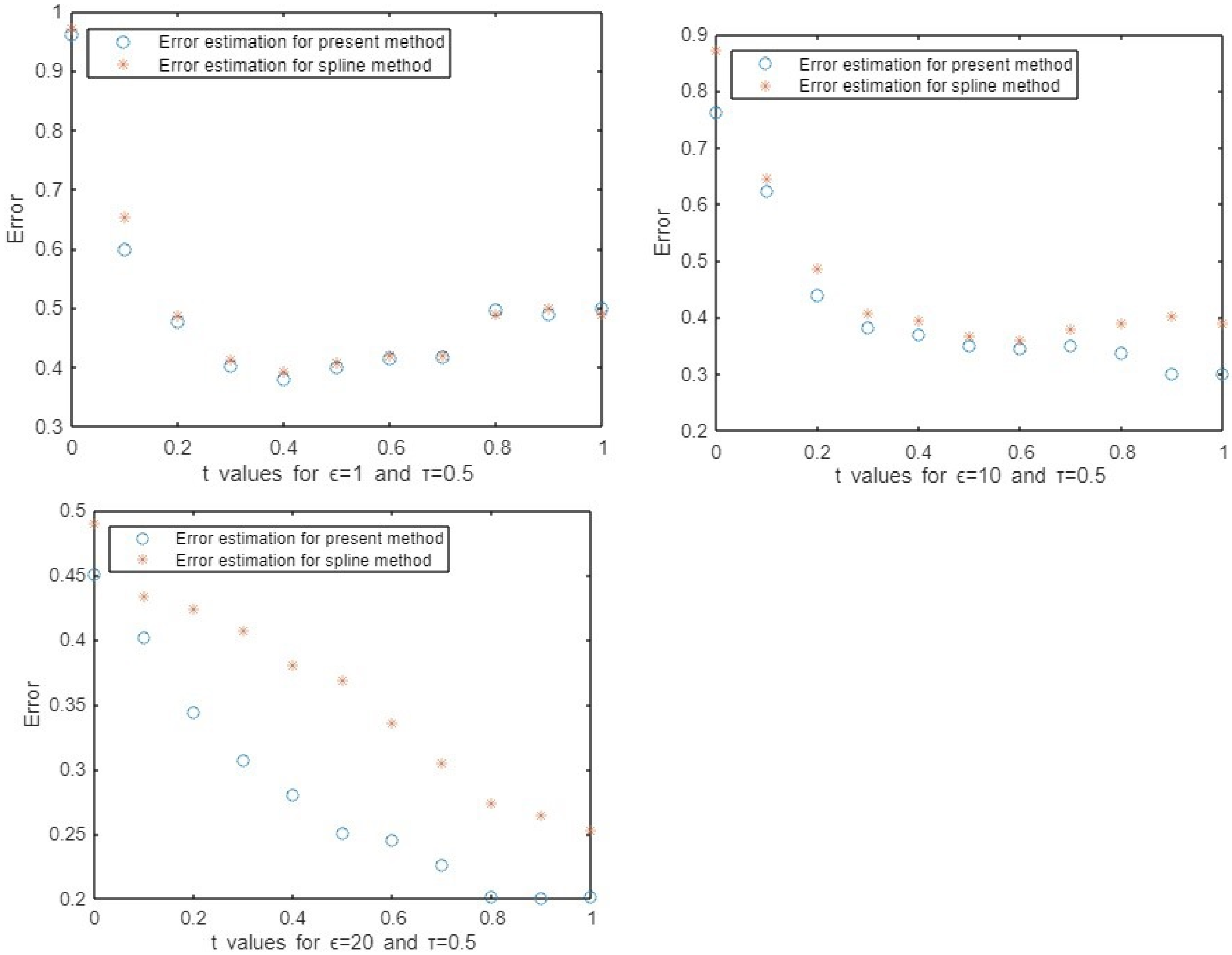

Figure 2: Abs. error comparison of present method and spline method for different values of ϵ and τ (Example 1)

t	(ϵ,τ)	$\mathcal{E}_{present}$	$\mathcal{E}_{spline}$	(ϵ,τ)	$\mathcal{E}_{present}$	$\mathcal{E}_{spline}$
	$\epsilon=1,\ \tau=0.5$			$\epsilon=10,\ \tau=0.5$		
0		0.96149	0.97149		0.76149	0.87149
0.1		0.60009	0.65349		0.62303	0.64549
0.2		0.47719	0.48719		0.44019	0.48719
0.3		0.38003	0.41273		0.38273	0.40673
0.4		0.40114	0.39393		0.37003	0.39443
0.5		0.38003	0.40854		0.35114	0.36854
	$\epsilon=20,\ \tau=0.5$			$\epsilon=50,\ \tau=0.5$		
0		0.45001	0.49006		0.26001	0.41170
0.1		0.40209	0.43400		0.24860	0.39001
0.2		0.34399	0.42390		0.24311	0.36500
0.3		0.30663	0.40673		0.22109	0.33340
0.4		0.27993	0.37993		0.21011	0.32017
0.5		0.25014	0.36548		0.20016	0.31004

Example 2. We consider the following problem,

$$\epsilon v^{'}(t) + v^3(t) + 3v(t-\tau) + \int_0^t v^2(p)dp = e^{-\frac{3t}{\epsilon}} - \frac{1}{2}\epsilon e^{-\frac{2t}{\epsilon}} + 2e^{-\frac{t}{\epsilon}} + \frac{\epsilon}{2}, t \in (0,1]$$

with the initial condition $y(0) = 1$. For no delay the exact solution is given by $y(t) = e^{-\frac{t}{\epsilon}}$. We applied the proposed technique for different values of ϵ and τ, that yield the following representative tables (Table 7 –Table 11). Table 12 demonstrates the comparative study between proposed method and spline method.

Table 7: Fixing $\epsilon = 1$ and pure delay term as $\tau = 0, 0.25, 0.75, 0.5$

t	0	0.1	0.2	0.3	0.4	0.5
$v_{1,0}$	1.00000	0.90501	0.82000	0.74509	0.68001	0.62505
$v_{1,0.25}$	1.00000	0.90499	0.81033	0.73021	0.68490	0.62499
$v_{1,0.5}$	1.00000	0.90450	0.81105	0.73112	0.68520	0.62381
$v_{1,0.75}$	1.00000	0.90433	0.81005	0.73160	0.68411	0.62250

Table 8: Fixing $\epsilon = 4$ and pure delay term as $\tau = 0, 0.25, 0.75, 0.5$

t	0	0.1	0.2	0.3	0.4	0.5
$v_{1,0}$	1.00000	0.97533	0.95125	0.92789	0.90522	0.88279
$v_{1,0.25}$	1.00000	0.97580	0.95000	0.92604	0.90478	0.88277
$v_{1,0.5}$	1.00000	0.97609	0.95240	0.92533	0.90667	0.88340
$v_{1,0.75}$	1.00000	0.97611	0.95339	0.92602	0.90468	0.88199

Table 9: Fixing $\epsilon = 6$ and pure delay term as $\tau = 0, 0.25, 0.75, 0.5$

t	0	0.1	0.2	0.3	0.4	0.5
$v_{1,0}$	1.00000	0.98410	0.96864	0.95321	0.93822	0.92345
$v_{1,0.25}$	1.00000	0.98421	0.96820	0.95105	0.92900	0.92100
$v_{1,0.5}$	1.00000	0.98001	0.95330	0.95015	0.92009	0.92020
$v_{1,0.75}$	1.00000	0.97998	0.95299	0.96002	0.93900	0.92119

Table 10: Fixing $\epsilon = 10$ and pure delay term as $\tau = 0, 0.25, 0.75, 0.5$

t	0	0.1	0.2	0.3	0.4	0.5
$v_{1,0}$	1.00000	0.99020	0.98089	0.97180	0.96323	0.95501
$v_{1,0.25}$	1.00000	0.98006	0.98129	0.97544	0.96601	0.97002
$v_{1,0.5}$	1.00000	0.98312	0.98060	0.97666	0.95000	0.96055
$v_{1,0.75}$	1.00000	0.98007	0.98690	0.96800	0.95522	0.96801

Table 11: Fixing $\epsilon = 20$ and pure delay term as $\tau = 0, 0.25, 0.75, 0.5$

t	0	0.1	0.2	0.3	0.4	0.5
$v_{1,0}$	1.00000	0.99710	0.99835	1.00361	1.01311	1.02670
$v_{1,0.25}$	1.00000	0.99350	0.98077	1.00348	1.01590	1.02669
$v_{1,0.5}$	1.00000	0.99669	0.98604	1.07091	1.66096	1.02556
$v_{1,0.75}$	1.00000	0.98900	0.98778	1.07000	1.66560	1.02670

Table 12: Comparison of absolute error between the developed method and spline method (Example 2)

t	(ϵ, τ)	$\mathcal{E}_{present}$	$\mathcal{E}_{spline}$	(ϵ, τ)	$\mathcal{E}_{present}$	$\mathcal{E}_{spline}$
	$\epsilon = 1$, $\tau = 0.5$			$\epsilon = 10$, $\tau = 0.5$		
0		0.90085	0.90088		0.90080	0.90112
0.1		0.90501	0.90489		0.99055	0.99009
0.2		0.82001	0.81877		0.98201	0.98022
0.3		0.74501	0.74008		0.97451	0.97044
0.4		0.68000	0.67031		0.96802	0.96081
0.5		0.62505	0.60655		0.96250	0.95120
	$\epsilon = 20$, $\tau = 0.5$			$\epsilon = 50$, $\tau = 0.5$		
0		0.90009	0.90011		0.90090	0.90099
0.1		0.99500	0.99508		0.99805	0.99800
0.2		0.99001	0.99000		0.99605	0.99606
0.3		0.99811	0.99004		0.99502	0.99001
0.4		0.98060	0.98100		0.98011	0.98100
0.5		0.98122	0.98000		0.98055	0.98800

Table 13: Absolute error estimation for different choices of t, ϵ and τ (Example 1)

t	ϵ	τ	Absolute error (by proposed method)	Absolute error (by method [26])
0.000	0.10000	0.00001	0.00899	0.00989
0.125	0.10000	0.00001	0.00898	0.00992
0.250	0.10000	0.00001	0.00888	0.00990
0.375	0.10000	0.00001	0.00851	0.00917
0.500	0.10000	0.00001	0.00849	0.00911
0.625	0.10000	0.00001	0.00848	0.00910
0.750	0.10000	0.00001	0.00840	0.00903
0.875	0.10000	0.00001	0.00839	0.00901
0.100	0.10000	0.00001	0.00838	0.00900
0.000	0.50000	0.00001	0.00800	0.01655
0.125	0.50000	0.00001	0.00799	0.01980
0.250	0.50000	0.00001	0.00799	0.01660
0.375	0.50000	0.00001	0.00789	0.01749
0.500	0.50000	0.00001	0.00784	0.01452
0.625	0.50000	0.00001	0.00781	0.01449
0.750	0.50000	0.00001	0.00772	0.01529
0.875	0.50000	0.00001	0.00770	0.01400
0.100	0.50000	0.00001	0.00765	0.01409

Table 14: Absolute error estimation for different choices of t, ϵ and τ (Example 2)

t	ϵ	τ	Absolute error (by proposed method)	Absolute error (by method [26])
0.000	0.10000	0.00001	0.00032	0.00645
0.125	0.10000	0.00001	0.00029	0.00699
0.250	0.10000	0.00001	0.00020	0.00675
0.375	0.10000	0.00001	0.00019	0.00677
0.500	0.10000	0.00001	0.00015	0.00670
0.625	0.10000	0.00001	0.00014	0.00665
0.750	0.10000	0.00001	0.00011	0.00640
0.875	0.10000	0.00001	0.00011	0.00634
0.100	0.10000	0.00001	0.00010	0.00629
0.000	0.50000	0.00001	0.00018	0.00039
0.125	0.50000	0.00001	0.00014	0.00038
0.250	0.50000	0.00001	0.00012	0.00033
0.375	0.50000	0.00001	0.00010	0.00032
0.500	0.50000	0.00001	0.00010	0.00032
0.625	0.50000	0.00001	0.00011	0.00031
0.750	0.50000	0.00001	0.00010	0.00030
0.875	0.50000	0.00001	0.00010	0.00027
0.100	0.50000	0.00001	0.00009	0.00026

5 Conclusion

The proposed method is applied to solve a singularly perturbed Volterra integro-differential equation. The numerical outcomes from this investigation suggest that the developed method is preferable and converges rapidly. This observation is also supported by the comparative study with the spline method. The most appealing advantage of the present method is that it provides an adequate result for higher values of perturbation parameter. This particular fact makes this scheme attractive for numerical treatment for a large scale of ϵ. The numerical scheme is compatible with the pure delay term and its variation, although the same method with variable delay might be an interesting future scope of the present article. From the boundedness condition, stability analysis, numerical discussion, and comparative study, it is evident that the method can be a useful tool to find an approximate solution of the considered class of equation.

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