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On generalization and refinements of weighted Majorization theorem, Favard and Berwald inequalities

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Abstract

In this article, we give generalizations and refinements of weighted majorization theorem, weighted Faward and weighted Berwald inequalities for functions with nondecreasing increments of convex type.

Keywords: Functions with non decreasing increments, Jensen's inequality, Majorization theorem, Favard inequality, Berwald inequality
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1 Introduction

1.1 Functions with nondecreasing increments

In the year 1964, H. D. Brunk introduced a new class of functions called it functions with non-decreasing increment see [3], which is defined below

Definition 1.1. A function $\psi : \mathbf{I} \to \mathbb{R}$, $\mathbf{I} \subset \mathbb{R}^m$, which is said to have nondecreasing increments if

$$\psi(\mathbf{u}+\mathbf{s}) - \psi(\mathbf{u}) \leq \psi(\mathbf{v}+\mathbf{s}) - \psi(\mathbf{v}) \tag{1.1}$$

whenever $\mathbf{u} \in I, \mathbf{v}+\mathbf{s} \in I$, $\mathbf{0} \leq \mathbf{s} \in \mathbb{R}^m$.

Generally it is not a continuous function even if $m = 1$. Important fact about ψ is, it becomes convex along the lines whose direction cosines are nonnegative, such that the line can be expressed in the form of $\mathbf{z} = \mathbf{u}t + \mathbf{v}$ where $\mathbf{0} \leq \mathbf{u} \in \mathbb{R}^m, \mathbf{v} \in \mathbb{R}^m$. Throughout the paper we let $\mathbb{R}^m$ denotes the m^{th} dimensional euclidean vector lattice of points $\mathbf{u} = (u_1, \ldots, u_m)$, u_j is real for $j = 1, 2, \ldots, m$, with the partial ordering $\mathbf{u} = (u_1, \ldots, u_m) \leq \mathbf{v} = (v_l, \ldots, v_m)$ iff $u_j \leq v_j$ for $j = 1, 2, \ldots, m$. Note that in the manuscript, we use **FWNDI** as an abbreviation for functions with nondecreasing increments.

For other details and properties of the functions with non-decreasing increment see [3, 12]. Khan et al. [5] also uses higher order functions having non-decreasing increments to generalize other inequalities. While in [6], authors proved some generalizations of other significant inequalities for the functions having non-decreasing increments.

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Convex Type: A continuous mapping $\psi(\mathbf{z})$ is said to be a function having non-decreasing increments of convex type if the condition $\mathbf{v} \leq \mathbf{z} \leq \mathbf{u}+\mathbf{v}$ holds, such that a function $\vartheta : [0,1] \to \mathbb{R}$ is defined as $\vartheta(p) = \psi(\mathbf{u}p + \mathbf{v})\ for\ 0 \leq p \leq 1$, is a convex function. Hence a **FWNDI** along positively oriented lines with positive direction cosines is convex over the domain and we will call this function as **FWNDI** convex type.

1.1.1 Majorization, Favard and Berwald inequalities

In [10], Marshall and Olkin give details of majorization theory with profoundness. Pečarić et al. [14, pp. 319], described the notion of majorization, as a measure of the diversity of the components of an n-dimensional vector (an "n"tuple) and introduced some related inequalities such as weighted majorzation theorem. Furthermore, in [1, 7], authors reproduced generalizations and refinements of weighted majorization theorem. In 1933, J. Favard in his article [4] introduced an integral inequality based on convex function which is known as Favard inequality, while in [2] L. Berwald given the generalized form of Favard's inequality, one of the classical result re known as Berwald inequality. In [8, 7], authors reproduced the weighted generalized version of Favard's and Berwald's inequalities for convex function and in [1] Adil Khan et al. produced the refinements of Favard's and Berwald's inequalities on the basis of convex functions.
Here is consider the following generalizations of weighted majorization theorem, Favard and Berwald inequalities in [7].

Theorem 1.2. Suppose that $\theta, \psi : [0,\infty) \to \mathbb{R}$, such that θ is strictly increasing and ψ is a convex function with $\psi \circ \theta^{-1}$ is convex. Since for a given real positive "n"tuples of $\mathbf{a}, \mathbf{b}\, and\, \mathbf{w}$, that satisfy the below conditions,

$$\sum_1^k{}_i\, w_i\theta(b^{(i)}) \leq \sum_1^k{}_i\, w_i\theta(a^{(i)}), \quad for \quad k = 1, 2, \cdots, n-1. \tag{1.2}$$

and

$$\sum_1^n{}_i\, w_i\theta(b^{(i)}) = \sum_1^n{}_i\, w_i\theta(a^{(i)}). \tag{1.3}$$

If $\mathbf{b}$ is decreasing "n"tuple, then

$$\sum_1^n{}_i\, w_i\psi(b^{(i)}) \leq \sum_1^n{}_i\, w_i\psi(a^{(i)}). \tag{1.4}$$

If $\mathbf{a}$ is increasing "n"tuple, then

$$\sum_1^n{}_i\, w_i\psi(a^{(i)}) \leq \sum_1^n{}_i\, w_i\psi(b^{(i)}). \tag{1.5}$$

See [7] for the generalization of weighted Favard and weighted Berwald inequalities.
Consider an another important proposition from [14, pp. 12-13], which would be used throughout the paper for the functions having nondecreasing increments of convex type as given below,

Proposition 1.3. Let the convex function ψ on an open set U of a real norm space $\mathbb{R}^n$. Then $\forall\ u, u_o \in U$, the inequality

$$\psi(u) - \psi(u_o) \geq \langle \psi'_+(u_o), u - u_o \rangle. \tag{1.6}$$

holds, where $\langle *, * \rangle$ represents an inner product on the space $\mathbb{R}^n$, and $\psi'_+(u_o)$ appear for right directional derivatives of ψ at u_o.

we will also need the following important lemma throughout the article which was proved in [9], see below

Proposition 1.4. Let $\psi : \mathbf{U} \to \mathbb{R}$ be the functions having non-decreasing increments of convex type then for $\mathbf{u}, \mathbf{u_o} \in \mathbf{U}$, we get

$$\psi(\mathbf{u}) - \psi(\mathbf{u_o}) - \langle \psi'_+(\mathbf{u_o}), \mathbf{u} - \mathbf{u_o} \rangle \geq \Big| |\psi(u) - \psi(\mathbf{u_o})| - \|\psi'_+(\mathbf{u_o})\| \|\mathbf{u} - \mathbf{u_o}\| \Big|, \tag{1.7}$$

holds, where $< *, * >$ represents a usual inner product on $\mathbb{R}^m$ and $\psi'_+(\mathbf{u}) = (\psi'_{1+}(\mathbf{u}), \psi'_{2+}(\mathbf{u}), \dots, \psi'_{m+}(\mathbf{u}))$ and $(\psi'_{1+}, \psi'_{2+}, \dots, \psi'_{m+})$ are right directional partial derivatives of ψ,

Convexity plays a significant role for the development of inequalities, one may focus on to generalize those inequalities in several variables which have already defind for one variable convex function, unfortunately, we may not generalize all those inequalities in terms of several variable convex function, alternavely we have an opportunity to give this generalization for **FWNDI** of convex type. We notice that weighted Majorization, weighted Faward and Berwald inequalities could follow the conditions of **FWNDI**, and so we obtain the required targets.
The manuscript comprises on three different sections. After introduction, we make first section for weighted majorization theorems, second for weighted Faward inequalities and third section for weighted Berwald inequalities.

2 Main Result

2.1 Generalization and Refinement of Weighted Majorization Theorem

In the present section we give generalizations and refinements for the result of [7] and [1], which was produced for convex function. We will reproduce these weighted majorization theorem results on the basis of functions having non-decreasing increments of convex type. Consider the following vectors lattice on summation, since for any $\mathbf{a}^{(i)}, \mathbf{b}^{(i)} \in \mathbf{U}, \mathbf{U} \subset \mathbb{R}^m$ *and* $w_i \geq 0$, we obtain

$$\sum_1^n {}_i\, w_i \mathbf{a}^{(i)} = \Big(\sum_1^n {}_i\, w_i a_1^{(i)}, \cdots, \sum_1^n {}_i\, w_i a_m^{(i)} \Big) \in \mathbf{U},$$

with the partial ordering

$$\begin{aligned} &\textstyle\sum_1^n {}_i\, w_i \mathbf{b}^{(i)} = \Big(\sum_1^n {}_i\, w_i b_1^{(i)}, \cdots, \sum_1^n {}_i\, w_i b_m^{(i)} \Big) \\ &\leq \textstyle\sum_1^n {}_i\, w_i \mathbf{a}^{(i)} = \Big(\sum_1^n {}_i\, w_i a_1^{(i)}, \cdots, \sum_1^n {}_i\, w_i a_m^{(i)} \Big), \end{aligned}$$

iff

$$\sum_1^n {}_i\, w_i b_j^{(i)} \leq \sum_1^n {}_i\, w_i a_j^{(i)}, \quad for \quad j \in 1, 2, \cdots, m.$$

Here we state our first main result.

Theorem 2.1. Let $\psi : \mathbf{U} \to \mathbb{R}$ $(\mathbf{U} \subset \mathbb{R}^m)$, a continuous function having non-decreasing increments of convex type. Since for a given positive "n"tuples of
$\mathbf{a}^{(i)}, \mathbf{b}^{(i)} \in \mathbf{U}$, and $w_i \geq 0$ for $i \in \{1, 2, \dots, n\}$ satisfies that

$$\sum_1^k {}_i\, w_i \mathbf{b}^{(i)} \leq \sum_1^k {}_i\, w_i \mathbf{a}^{(i)}, \quad \text{with} \quad k = 1, 2, \cdots, n-1, \tag{2.1}$$

and

$$\sum_1^n {}_i\, w_i \mathbf{b}^{(i)} = \sum_1^n {}_i\, w_i \mathbf{a}^{(i)}, \tag{2.2}$$

If $\mathbf{b}^{(i)}$ is decreasing "n"tuples, then

$$\sum_1^n {}_i\, w_i \psi(\mathbf{b}^{(i)}) \leq \sum_1^n {}_i\, w_i \psi(\mathbf{a}^{(i)}). \tag{2.3}$$

If $\mathbf{a}^{(i)}$ is increasing "n"tuples, then

$$\sum_{1}^{n}{}_{i} w_i\psi(\mathbf{a}^{(i)}) \leq \sum_{1}^{n}{}_{i} w_i\psi(\mathbf{b}^{(i)}). \tag{2.4}$$

Proof . As ψ is continuous function having non-decreasing increments of convex type, therefore $\mathbf{a}, \mathbf{b}, \in U$ using proposition 1.3 we get the following result,

$$\psi(\mathbf{a}) \geq \psi(\mathbf{b}) + \Big\langle \psi'_{+}(\mathbf{b}), \mathbf{a} - \mathbf{b} \Big\rangle, \tag{2.5}$$

Take $\mathbf{a} = \mathbf{a}^{(i)} \in \mathbf{U}$ and $\mathbf{b} = \mathbf{b}^{(i)} \in \mathbf{U}$, in (2.5), we have

$$\psi(\mathbf{a}^{(i)}) - \psi(\mathbf{b}^{(i)}) - \Big\langle \psi'_{+}(\mathbf{b}^{(i)}), \mathbf{a}^{(i)} - \mathbf{b}^{(i)} \Big\rangle \geq 0,$$

where

$$\psi'_{+}(\mathbf{b}^{(i)}) = (\psi'_{1+}(\mathbf{b}^{(i)}), \cdots, \psi'_{m+}(\mathbf{b}^{(i)})),$$

Multiplying $w_i \geq 0$ and then applying summation, indexing with i runs from 1 to n, we obtain

$$\sum_{1}^{n}{}_{i} w_i\psi(\mathbf{a}^{(i)}) - \sum_{1}^{n}{}_{i} w_i\psi(\mathbf{b}^{(i)}) \geq \sum_{1}^{n}{}_{i} w_i \Big\langle \psi'_{+}(\mathbf{b}^{(i)}), \mathbf{a}^{(i)} - \mathbf{b}^{(i)} \Big\rangle, \tag{2.6}$$

By expending the inner product on right hand side, we have

$$\begin{aligned}
\sum_{1}^{n}{}_{i} w_i\psi(\mathbf{a}^{(i)}) - \sum_{1}^{n}{}_{i} w_i\psi(\mathbf{b}^{(i)}) \quad &\geq \quad \sum_{1}^{n}{}_{i} w_i \Big[\sum_{1}^{m}{}_{j} \psi'_{j+}(\mathbf{b}^{(i)})(a_j^{(i)} - b_j^{(i)}) \Big] \\
&= \quad \sum_{1}^{n}{}_{i} w_i\psi'_{1+}(\mathbf{b}^{(i)})(a_1^{(i)} - b_1^{(i)}) \\
&+ \quad \sum_{1}^{n}{}_{i} w_i\psi'_{2+}(\mathbf{b}^{(i)})(a_2^{(i)} - b_2^{(i)}) \\
&\vdots \\
&+ \quad \sum_{1}^{n}{}_{i} w_i\psi'_{m+}(\mathbf{b}^{(i)})(a_m^{(i)} - b_m^{(i)}).
\end{aligned} \tag{2.7}$$

When $\mathbf{b}^{(i)}$ is decreasing "n"tuple, using (2.1) and the fact that ψ is a **FWNDI**, we obtain

$$\sum_{1}^{n}{}_{i} w_i\psi'_{j+}(\mathbf{b}^{(i)})(a_j^{(i)} - b_j^{(i)}) = \sum_{1}^{n-1}{}_{k} \big(A_j^{(k)} - B_j^{(k)}\big)\big(\psi'_{j+}(\mathbf{b}^{(k)}) - \psi'_{j+}(\mathbf{b}^{(k+1)}\big) \geq 0, \tag{2.8}$$

for each fixed $j = 1, 2, \cdots, m$.
where,

$$A_j^{(k)} = \sum_{1}^{k}{}_{i} w_i a_j^{(i)}; \quad B_j^{(k)} = \sum_{1}^{k}{}_{i} w_i b_j^{(i)}.$$

We obtain the R.H.S of (2.7) as a nonnegative quantity, therefore we have

$$\sum_{1}^{n}{}_{i} w_i\psi(\mathbf{a}^{(i)}) - \sum_{1}^{n}{}_{i} w_i\psi(\mathbf{b}^{(i)}) \geq 0,$$

which implies that

$$\sum_{1}^{n}{}_{i} w_i\psi(\mathbf{b}^{(i)}) \leq \sum_{1}^{n}{}_{i} w_i\psi(\mathbf{a}^{(i)}).$$

Similarly when $\mathbf{a}^{(i)}$ is increasing "n"tuple, we obtain (2.4). $\square$

Remark 2.2. For the convenience of the reader we refer [11, pp. 46] for understanding the fact used in (2.8).

In the paper $\boldsymbol{\theta}(\mathbf{a}^{(i)}) = (\theta(a_1^{(i)}), \theta(a_2^{(i)}), \cdots, \theta(a_m^{(i)}))$, for each fixed $i = 1, \cdots, n$ denotes a map from $\mathbf{U}$ to $\mathbf{U}$ such that components $\theta(a_j^{(i)})$ of $\boldsymbol{\theta}(\mathbf{a}^{(i)})$ are continuous real valued and strictly increasing.

Theorem 2.3. Suppose that $\boldsymbol{\theta} : \mathbf{U} \to \mathbf{U}$, $\mathbf{U} \subset \mathbb{R}^m$, such that $\boldsymbol{\theta}$ is strictly increasing on $\mathbf{U}$ and $\psi : \mathbf{U} \to \mathbb{R}$ with $\psi(\mathbf{a}^{(i)}) = \psi \circ \boldsymbol{\theta}^{-1}(\mathbf{u}^{(i)})$ be a continuous **FWNDI** of convex type. Since for a given positive "n"tuples of $\mathbf{a}^{(i)}, \mathbf{b}^{(i)}, \mathbf{u}^{(i)} \in \mathbf{U}$, $w_i \geq 0\, for\ i \in \{1, 2, \ldots, n\}$, satisfy the below conditions,

$$\sum_1^k {}_i\, w_i \boldsymbol{\theta}(\mathbf{b}^{(i)}) \leq \sum_1^k {}_i\, w_i \boldsymbol{\theta}(\mathbf{a}^{(i)}), \quad \text{with} \quad k = 1, 2, \cdots, n-1. \tag{2.9}$$

and

$$\sum_1^n {}_i\, w_i \boldsymbol{\theta}(\mathbf{b}^{(i)}) = \sum_1^n {}_i\, w_i \boldsymbol{\theta}(\mathbf{a}^{(i)}). \tag{2.10}$$

If $\mathbf{b}^{(\mathbf{i})}$ is decreasing "n"tuples, then

$$\sum_1^n {}_i\, w_i \psi(\mathbf{b}^{(i)}) \leq \sum_1^n {}_i\, w_i \psi(\mathbf{a}^{(i)}). \tag{2.11}$$

If $\mathbf{a}^{(\mathbf{i})}$ is increasing "n"tuples, then

$$\sum_1^n {}_i\, w_i \psi(\mathbf{a}^{(i)}) \leq \sum_1^n {}_i\, w_i \psi(\mathbf{b}^{(i)}). \tag{2.12}$$

Proof . Using Theorem 2.1 for **FWNDI** $\psi(\mathbf{a}^{(i)}) = \psi \circ \boldsymbol{\theta}^{-1}(\mathbf{u}^{(i)}) = \bar{\psi}(\mathbf{u}^{(i)})$ such that

$$\mathbf{u}^{(i)} = \boldsymbol{\theta}(\mathbf{a}^{(i)}) \quad \text{and} \quad \mathbf{v}^{(i)} = \boldsymbol{\theta}(\mathbf{b}^{(i)}),$$

$\forall \mathbf{u}^{(i)}, \mathbf{v}^{(i)}$ in $\mathbf{U}$ we get the required inequalities. $\square$ The following important result is the refinement of Theorem 2.1.

Theorem 2.4. Let $\psi : \mathbf{U} \to \mathbb{R}$ ($\mathbf{U} \subset \mathbb{R}^m$), a continuous function having non-decreasing increments of convex type. Since for a given positive "n"tuples of
$\mathbf{a}^{(i)}, \mathbf{b}^{(i)} \in \mathbf{U}$, $w_i \geq 0\, for\ i \in 1, 2, \ldots, n$ satifies that

$$\sum_1^k {}_i\, w_i \mathbf{b}^{(i)} \leq \sum_1^k {}_i\, w_i \mathbf{a}^{(i)}, \quad \text{with} \quad k = 1, 2, \cdots, n-1, \tag{2.13}$$

and

$$\sum_1^n {}_i\, w_i \mathbf{b}^{(i)} = \sum_1^n {}_i\, w_i \mathbf{a}^{(i)}, \tag{2.14}$$

If $\mathbf{b}^{(\mathbf{i})}$ is decreasing "n"tuples, then

$$\begin{aligned} \sum_1^n {}_i\, w_i \psi(\mathbf{a}^{(i)}) - \sum_1^n {}_i\, w_i \psi(\mathbf{b}^{(i)}) \geq \Big| \sum_1^n {}_i\, w_i |\psi(\mathbf{a}^{(i)}) - \psi(\mathbf{b}^{(i)})| \\ - \sum_1^n {}_i\, w_i \|\psi'_+(\mathbf{b}^{(i)})\| \, \|\mathbf{a}^{(i)} - \mathbf{b}^{(i)}\| \Big|. \end{aligned} \tag{2.15}$$

If $\mathbf{a^{(i)}}$ is increasing "n" tuples, then

$$\sum_1^n {}_i w_i \psi(\mathbf{b}^{(i)}) - \sum_1^n {}_i w_i \psi(\mathbf{a}^{(i)}) \geq \Big| \sum_1^n {}_i w_i |\psi(\mathbf{b}^{(i)}) - \psi(\mathbf{a}^{(i)})| - \sum_1^n {}_i w_i \|\psi'_+(\mathbf{a}^{(i)})\| \, \|\mathbf{b}^{(i)} - \mathbf{a}^{(i)}\| \Big|. \tag{2.16}$$

Proof . As ψ is a continuous function having non-decreasing increments of convex type, therefore $\mathbf{a^{(i)}b^{(i)}}, \in U$ using proposition 1.3 we get the following result,

$$\psi(\mathbf{a}^{(i)}) - \psi(\mathbf{b}^{(i)}) - \Big\langle \psi'_+(\mathbf{b}^{(i)}), \mathbf{a}^{(i)} - \mathbf{b}^{(i)} \Big\rangle \geq 0,$$

which implies that

$$\psi(\mathbf{a}^{(i)}) - \psi(\mathbf{b}^{(i)}) - \Big\langle \psi'_+(\mathbf{b}^{(i)}), \mathbf{a}^{(i)} - \mathbf{b}^{(i)} \Big\rangle = \Big| \psi(\mathbf{a}^{(i)}) - \psi(\mathbf{b}^{(i)}) - \Big\langle \psi'_+(\mathbf{b}^{(i)}), \mathbf{a}^{(i)} - \mathbf{b}^{(i)} \Big\rangle \Big|,$$

Using triangle inequality, multiplying by $w_i \geq 0$, taking sum for i runs from 1 to n, then applying Cauchy-Schwarz inequality on the R.H.S, we have

$$\sum_1^n {}_i w_i \psi(\mathbf{a}^{(i)}) - \sum_1^n {}_i w_i \psi(\mathbf{b}^{(i)}) - \sum_1^n {}_i w_i \Big\langle \psi'_+(\mathbf{b}^{(i)}), \mathbf{a}^{(i)} - \mathbf{b}^{(i)} \Big\rangle$$
$$\geq \Big| \sum_1^n {}_i w_i |\psi(\mathbf{a}^{(i)}) - \psi(\mathbf{b}^{(i)})| - \sum_1^n {}_i w_i \|\psi'_+(\mathbf{b}^{(i)})\| \, \|\mathbf{a}^{(i)} - \mathbf{b}^{(i)}\| \Big|.$$

If $\mathbf{b^{(i)}}$ is decreasing so using the fact that,

$$\sum_1^n {}_i w_i \Big\langle \psi'_+(\mathbf{b}^{(i)}), \mathbf{a}^{(i)} - \mathbf{b}^{(i)} \Big\rangle \geq 0,$$

hence, we get the required result

$$\sum_1^n {}_i w_i \psi(\mathbf{a}^{(i)}) - \sum_1^n {}_i w_i \psi(\mathbf{b}^{(i)})$$
$$\geq \Big| \sum_1^n {}_i w_i |\psi(\mathbf{a}^{(i)}) - \psi(\mathbf{b}^{(i)})| - \sum_1^n {}_i w_i \|\psi'_+(\mathbf{b}^{(i)})\| \, \|\mathbf{a}^{(i)} - \mathbf{b}^{(i)}\| \Big|.$$

Analogously when $\mathbf{a^{(i)}}$ is increasing we get (2.16). □ In the next theorem we will give refinement of Theorem 2.3,

Theorem 2.5. Under the assumptions of Theorem 2.3, with

$$\sum_1^k {}_i w_i \boldsymbol{\theta}(\mathbf{b}^{(i)}) \leq \sum_1^k {}_i w_i \boldsymbol{\theta}(\mathbf{a}^{(i)}), \quad \text{with} \quad k = 1, 2, \cdots, n-1, \tag{2.17}$$

and

$$\sum_1^n {}_i w_i \boldsymbol{\theta}(\mathbf{b}^{(i)}) = \sum_1^n {}_i w_i \boldsymbol{\theta}(\mathbf{a}^{(i)}). \tag{2.18}$$

If $\mathbf{b^{(i)}}$ is decreasing "n"tuples, then

$$\begin{aligned}
&\sum_1^n{}_i\, w_i\psi(\mathbf{a}^{(i)}) - \sum_1^n{}_i\, w_i\psi(\mathbf{b}^{(i)}) \\
&\geq \Big|\sum_1^n{}_i\, w_i|\psi(\mathbf{a}^{(i)}) - \psi(\mathbf{b}^{(i)})| \\
&-\sum_1^n{}_i\, w_i\|(\phi\circ\boldsymbol{\theta}^{-1})'_+(\boldsymbol{\theta}(\mathbf{b}^{(i)}))\|\,\|\boldsymbol{\theta}(\mathbf{a}^{(i)}) - \boldsymbol{\theta}(\mathbf{b}^{(i)})\|\Big|. \qquad (2.19)
\end{aligned}$$

If $\mathbf{a^{(i)}}$ is increasing "n"tuples, then

$$\begin{aligned}
&\sum_1^n{}_i\, w_i\psi(\mathbf{b}^{(i)}) - \sum_1^n{}_i\, w_i\psi(\mathbf{a}^{(i)}) \\
&\geq \Big|\sum_1^n{}_i\, w_i|\psi(\mathbf{b}^{(i)}) - \psi(\mathbf{a}^{(i)})| \\
&-\sum_1^n{}_i\, w_i\|(\phi\circ\boldsymbol{\theta}^{-1})'_+(\boldsymbol{\theta}(\mathbf{a}^{(i)}))\|\,\|\boldsymbol{\theta}(\mathbf{b}^{(i)}) - \boldsymbol{\theta}(\mathbf{a}^{(i)})\|\Big|. \qquad (2.20)
\end{aligned}$$

Proof . Using Theorem 2.3 for **FWNDI** $\psi(\mathbf{a}^{(i)}) = \psi\circ\boldsymbol{\theta}^{-1}(\mathbf{u}^{(i)}) = \bar{\psi}(\mathbf{u}^{(i)})$ such that

$$\mathbf{u}^{(i)} = \boldsymbol{\theta}(\mathbf{a}^{(i)}) \quad \text{and} \quad \mathbf{v}^{(i)} = \boldsymbol{\theta}(\mathbf{b}^{(i)}),$$

$\forall\mathbf{u}^{(i)}, \mathbf{v}^{(i)}$ in $\mathbf{U}$ we get the required inequalities. □

2.2 Generalization and Refinement of Weighted Favard Inequality

Here we would give generalizations and refinements of the result of [7] and [1], which was produced for convex function. We would produce generalizaion and refinement of discrete weighted Favard inequality for **FWNDI** of convex type.
The following weighted version of majorization lemma will be needed to prove upcoming results.

Lemma 2.6. Let $\mathbf{s}$ be a positive "n"tuple. If $\mathbf{x}$ is an increasing "n"tuple such that $\mathbf{s} = \mathbf{s}^{(i)}, \mathbf{x} = \mathbf{x}^{(i)} \in U \subset R^m$, then

$$\sum_1^k{}_i\,\mathbf{s}^{(i)}\mathbf{x}^{(i)}\sum_1^n{}_i\,\mathbf{s}^{(i)} \quad\leq\quad \sum_1^n{}_i\,\mathbf{s}^{(i)}\mathbf{x}^{(i)}\sum_1^k{}_i\,\mathbf{s}^{(i)}. \qquad (2.21)$$

Proof .

$$\begin{aligned}
\sum_1^k{}_i\,\mathbf{s}^{(i)}\mathbf{x}^{(i)}\sum_1^n{}_i\,\mathbf{s}^{(i)} &= \Big(\sum_1^n{}_i\,\mathbf{s}^{(i)}\mathbf{x}^{(i)} - \sum_{k+1}^n{}_i\,\mathbf{s}^{(i)}\mathbf{x}^{(i)}\Big)\sum_1^k{}_i\,\mathbf{s}^{(i)} + \sum_1^k{}_i\,\mathbf{s}^{(i)}\mathbf{x}^{(i)}\sum_{k+1}^n{}_i\,\mathbf{s}^{(i)} \\
&= \sum_1^n{}_i\,\mathbf{s}^{(i)}\mathbf{x}^{(i)}\sum_1^k{}_i\,\mathbf{s}^{(i)} - \sum_{k+1}^n{}_i\,\mathbf{s}^{(i)}\mathbf{x}^{(i)}\sum_1^k{}_i\,\mathbf{s}^{(i)} + \sum_1^k{}_i\,\mathbf{s}^{(i)}\mathbf{x}^{(i)}\sum_{k+1}^n{}_i\,\mathbf{s}^{(i)} \\
&\leq \sum_1^n{}_i\,\mathbf{s}^{(i)}\mathbf{x}^{(i)}\sum_1^k{}_i\,\mathbf{s}^{(i)} - \mathbf{x}^{(j)}\sum_{k+1}^n{}_i\,\mathbf{s}^{(i)}\sum_1^k{}_i\,\mathbf{s}^{(i)} + \mathbf{x}^{(j)}\sum_1^k{}_i\,\mathbf{s}^{(i)}\sum_{k+1}^n{}_i\,\mathbf{s}^{(i)} \\
&\leq \sum_1^n{}_i\,\mathbf{s}^{(i)}\mathbf{x}^{(i)}\sum_1^k{}_i\,\mathbf{s}^{(i)}.
\end{aligned}$$

If $\mathbf{x}$ is decreasing "n"tuple, reverse inequality holds (2.21).
□

$$\text{If}\quad \mathbf{a}^{(i)} = (a_1^{(i)}, \cdots, a_m^{(i)}) \in \mathbf{U} \quad\text{and}\quad \mathbf{b}^{(i)} = (b_1^{(i)}, \cdots, b_m^{(i)}) \in \mathbf{U}; \quad \mathbf{b^{(i)}} \neq \mathbf{0},$$

such that we define,
$$\frac{\mathbf{a}^{(i)}}{\mathbf{b}^{(i)}} = \Big(\frac{a_1^{(i)}}{b_1^{(i)}}, \cdots, \frac{a_m^{(i)}}{b_m^{(i)}}\Big), \quad \text{for fixed} \quad i = 1, 2, \cdots, n,$$

and

$$\sum_1^n{}_i\, w_i \mathbf{a}^{(i)} = \Big(\sum_1^n{}_i\, w_i a_1^{(i)}, \cdots, \sum_1^n{}_i\, w_i a_m^{(i)}\Big) \in \mathbf{U},$$

which implies

$$\frac{\mathbf{a}^{(i)}}{\sum_1^n{}_i\, w_i \mathbf{a}^{(i)}} = \left(\frac{a_1^{(i)}}{\sum_1^n{}_i\, w_i a_1^{(i)}}, \cdots, \frac{a_m^{(i)}}{\sum_1^n{}_i\, w_i a_m^{(i)}}\right) \in \mathbf{U}.$$

In the following result we would generalize the weighted Favard inequality for the function having non-decreasing increment of convex type.

Theorem 2.7. Let $\psi : \mathbf{I} \to \mathbb{R}$ ($\mathbf{I} = (\mathbf{0}, \mathbf{1})^{\mathbf{m}} \subset \mathbb{R}^m$), a continuous function having non-decreasing increments of convex type. Since for a given positive "n"tuples of $\mathbf{a}^{(i)}, \mathbf{b}^{(i)} \in \mathbf{I}$, $w_i \geq 0\, for\ i \in 1, 2, \ldots, n$,
Let ${}^{\mathbf{a}^{(\mathbf{i})}}/_{\mathbf{b}^{(\mathbf{i})}}$ be an increasing "n"tuple. If $\mathbf{a}^{(\mathbf{i})}$ is a decreasing "n"tuples, then

$$\sum_1^n{}_i\, w_i \psi\left(\frac{\mathbf{a}^{(i)}}{\sum_1^n{}_i\, w_i \mathbf{a}^{(i)}}\right) \leq \sum_1^n{}_i\, w_i \psi\left(\frac{\mathbf{b}^{(i)}}{\sum_1^n{}_i\, w_i \mathbf{b}^{(i)}}\right). \tag{2.22}$$

If $\mathbf{b}^{(\mathbf{i})}$ is an increasing "n"tuples, then reverse inequality holds in (2.22).

Let ${}^{\mathbf{a}^{(\mathbf{i})}}/_{\mathbf{b}^{(\mathbf{i})}}$ be a decreasing "n"tuple. If $\mathbf{b}^{(\mathbf{i})}$ is an decreasing "n"tuples, then

$$\sum_1^n{}_i\, w_i \psi\left(\frac{\mathbf{b}^{(i)}}{\sum_1^n{}_i\, w_i \mathbf{b}^{(i)}}\right) \leq \sum_1^n{}_i\, w_i \psi\left(\frac{\mathbf{a}^{(i)}}{\sum_1^n{}_i\, w_i \mathbf{a}^{(i)}}\right). \tag{2.23}$$

If $\mathbf{a}^{(\mathbf{i})}$ is an increasing "n"tuples, then reverse inequality holds in (2.23).

Proof . Using lemma 2.6 with $\mathbf{s}^{(\mathbf{i})} = \mathbf{b}^{(\mathbf{i})} w_i$, such that $\mathbf{x}^{(i)} = \frac{\mathbf{a}^{(i)}}{\mathbf{b}^{(i)}}$ be increasing "n"tuple, we obtain

$$\sum_1^k{}_i\, w_i \mathbf{a}^{(i)} \sum_1^n{}_i\, w_i \mathbf{b}^{(i)} \quad \leq \quad \sum_1^n{}_i\, w_i \mathbf{a}^{(i)} \sum_1^k{}_i\, w_i \mathbf{b}^{(i)},$$

which implies that

$$\sum_1^k{}_i\, w_i \left(\frac{\mathbf{a}^{(i)}}{\sum_1^n{}_i\, w_i \mathbf{a}^{(i)}}\right) \leq \sum_1^k{}_i\, w_i \left(\frac{\mathbf{b}^{(i)}}{\sum_1^n{}_i\, w_i \mathbf{b}^{(i)}}\right), \tag{2.24}$$

now using Theorem 2.1 with $\mathbf{a}^{(\mathbf{i})}$ is a decreasing "n"tuple, we obtain

$$\sum_1^n{}_i\, w_i \psi\left(\frac{\mathbf{a}^{(i)}}{\sum_1^n{}_i\, w_i \mathbf{a}^{(i)}}\right) \leq \sum_1^n{}_i\, w_i \psi\left(\frac{\mathbf{b}^{(i)}}{\sum_1^n{}_i\, w_i \mathbf{b}^{(i)}}\right).$$

If $\mathbf{b}^{(\mathbf{i})}$ is an increasing "n"tuple, from Theorem 2.1, we get

$$\sum_1^n{}_i\, w_i \psi\left(\frac{\mathbf{b}^{(i)}}{\sum_1^n{}_i\, w_i \mathbf{b}^{(i)}}\right) \leq \sum_1^n{}_i\, w_i \psi\left(\frac{\mathbf{a}^{(i)}}{\sum_1^n{}_i\, w_i \mathbf{a}^{(i)}}\right).$$

The remaining cases can be obtained analogous to the above cases by exchanging the role of $\mathbf{a}^{(\mathbf{i})}$ and $\mathbf{b}^{(\mathbf{i})}$. $\square$

Now we would give refinement of above inequality for the function having non-decreasing increment of convex type,

Theorem 2.8. Let the assumptions of Theorem 2.7 hold. Suppose that

$$\mathbf{c}^{(i)} = \frac{\mathbf{a}^{(i)}}{\sum_1^n {}_i\, w_i \mathbf{a}^{(i)}} \quad \text{and} \quad \mathbf{d}^{(i)} = \frac{\mathbf{b}^{(i)}}{\sum_1^n {}_i\, w_i \mathbf{b}^{(i)}},$$

let $^{\mathbf{a}^{(\mathbf{i})}}/_{\mathbf{b}^{(\mathbf{i})}}$ be an increasing "n"tuple. If $\mathbf{a}^{(\mathbf{i})}$ is a decreasing "n"tuples, then

$$\sum_1^n {}_i\, w_i \psi(\mathbf{d}^{(i)}) - \sum_1^n {}_i\, w_i \psi(\mathbf{c}^{(i)}) \quad \geq \quad \left| \sum_1^n {}_i\, w_i |\psi(\mathbf{d}^{(i)}) - \psi(\mathbf{c}^{(i)})| - \sum_1^n {}_i\, w_i \|\psi'_+(\mathbf{c}^{(i)})\| \, \|\mathbf{d}^{(i)} - \mathbf{c}^{(i)}\| \right|. \tag{2.25}$$

If $\mathbf{b}^{(\mathbf{i})}$ is an increasing "n"tuple, then

$$\sum_1^n {}_i\, w_i \psi(\mathbf{c}^{(i)}) - \sum_1^n {}_i\, w_i \psi(\mathbf{d}^{(i)}) \quad \geq \quad \left| \sum_1^n {}_i\, w_i |\psi(\mathbf{c}^{(i)}) - \psi(\mathbf{d}^{(i)})| - \sum_1^n {}_i\, w_i \|\psi'_+(\mathbf{d}^{(i)})\| \, \|\mathbf{c}^{(i)} - \mathbf{d}^{(i)}\| \right|. \tag{2.26}$$

Let $^{\mathbf{a}^{(\mathbf{i})}}/_{\mathbf{b}^{(\mathbf{i})}}$ be decreasing "n"tuple. If $\mathbf{b}^{(\mathbf{i})}$ is decreasing "n"tuples, then (2.25) holds.
If $\mathbf{a}^{(\mathbf{i})}$ is increasing "n"tuple, then (2.26) holds.

Proof . Using lemma 2.6 with $\mathbf{s}^{(\mathbf{i})} = \mathbf{b}^{(\mathbf{i})} w_i$, such that $\mathbf{x}^{(i)} = \frac{\mathbf{a}^{(i)}}{\mathbf{b}^{(i)}}$ be increasing "n"tuple, we obtain

$$\sum_1^k {}_i\, w_i \mathbf{c}^{(i)} \leq \sum_1^k {}_i\, w_i \mathbf{d}^{(i)},$$

for $\mathbf{a}^{(i)}$ is a decreasing "n"tuple implies that $\mathbf{c}^{(i)}$ is also a decreasing, now using Theorem 2.4, we get

$$\sum_1^n {}_i\, w_i \psi(\mathbf{d}^{(i)}) - \sum_1^n {}_i\, w_i \psi(\mathbf{c}^{(i)}) \quad \geq \quad \left| \sum_1^n {}_i\, w_i |\psi(\mathbf{d}^{(i)}) - \psi(\mathbf{c}^{(i)})| - \sum_1^n {}_i\, w_i \|\psi'_+(\mathbf{c}^{(i)})\| \, \|\mathbf{d}^{(i)} - \mathbf{c}^{(i)}\| \right|. \tag{2.27}$$

For $\mathbf{b}^{(i)}$ is an increasing "n"tuple implies that $\mathbf{d}^{(i)}$ is also an increasing, now using Theorem 2.4, we get

$$\sum_1^n {}_i\, w_i \psi(\mathbf{c}^{(i)}) - \sum_1^n {}_i\, w_i \psi(\mathbf{d}^{(i)}) \quad \geq \quad \left| \sum_1^n {}_i\, w_i |\psi(\mathbf{c}^{(i)}) - \psi(\mathbf{d}^{(i)})| - \sum_1^n {}_i\, w_i \|\psi'_+(\mathbf{d}^{(i)})\| \, \|\mathbf{c}^{(i)} - \mathbf{d}^{(i)}\| \right|. \tag{2.28}$$

Similarly remaining cases can be proved analogously by replacing the role of $\mathbf{a}^{(i)}$ and $\mathbf{b}^{(i)}$. □

2.3 Generalization and Refinement of Weighted Berwald Inequality

In this section we would produce generalizaion and refinement of discrete weighted Berwald inequality for **FWNDI** of convex type. Since,

$$\frac{\mathbf{a}^{(i)}}{\mathbf{b}^{(i)}} = \Big(\frac{a_1^{(i)}}{b_1^{(i)}}, \cdots, \frac{a_m^{(i)}}{b_m^{(i)}} \Big), \quad \text{for fixed} \quad i = 1, 2, \cdots, n.$$

is bounded above if $\frac{a_j^{(i)}}{b_j^{(i)}} < z \quad \forall j = 1, \cdots, m$, such that $z \in \mathbb{R}_+$ (set of nonnegetive reals) and $b_j^{(i)} > 0$ which implies that

$$a_j^{(i)} < z b_j^{(i)}. \tag{2.29}$$

Theorem 2.9. Suppose that $\boldsymbol{\theta} : \mathbf{I} \to \mathbf{I}$, $(\mathbf{I} = (\mathbf{0}, \mathbf{1})^{\mathbf{m}} \subset \mathbb{R}^m)$, such that $\boldsymbol{\theta}$ is strictly increasing on $\mathbf{I}$ and $\psi : \mathbf{I} \to \mathbb{R}$ with $\psi(\mathbf{a}^{(i)}) = \psi \circ \boldsymbol{\theta}^{-1}(\mathbf{u}^{(i)})$ be a continuous **FWNDI** of convex type. Since for a given positive "n"tuples of $\mathbf{a}^{(i)}, \mathbf{b}^{(i)}, \mathbf{u}^{(i)} \in \mathbf{I}$, $w_i \geq 0 \, for \; i \in \{1, 2, \ldots, n\}$. Let z be such that

$$\sum_1^n {}_i\, w_i \boldsymbol{\theta}(z \mathbf{b}^{(i)}) = \sum_1^n {}_i\, w_i (\boldsymbol{\theta}(\mathbf{a}^{(i)})), \tag{2.30}$$

let ${}^{\mathbf{a}^{(\mathbf{i})}}/_{\mathbf{b}^{(\mathbf{i})}}$ be a decreasing "n"tuple. If $\mathbf{a^{(i)}}$ is an increasing "n"tuple then

$$\sum_{1}^{n}{}_{i}\, w_i\psi(\mathbf{a}^{(i)}) \leq \sum_{1}^{n}{}_{i}\, w_i\psi(z\mathbf{b}^{(i)}). \tag{2.31}$$

If $\mathbf{b^{(i)}}$ is a decreasing "n"tuple then reverse inequality holds in (2.31).

Let ${}^{\mathbf{a}^{(\mathbf{i})}}/_{\mathbf{b}^{(\mathbf{i})}}$ be an increasing "n"tuple. If $\mathbf{b^{(i)}}$ is an increasing "n"tuple then

$$\sum_{1}^{n}{}_{i}\, w_i\psi(z\mathbf{b}^{(i)}) \leq \sum_{1}^{n}{}_{i}\, w_i\psi(\mathbf{a}^{(i)}). \tag{2.32}$$

If $\mathbf{a^{(i)}}$ is a decreasing "n"tuple then reverse inequality holds in (2.32).

Proof . $\boldsymbol{\theta}$ is continuous with respect to its components, therefore

$$\mathbf{F}(z) = \sum_{1}^{n}{}_{i}\, w_i\boldsymbol{\theta}(z\mathbf{b}^{(i)}),$$

for $z \geq 0$ a real number, is also continuous. Using $\mathbf{a}^{(i)} > \mathbf{0}$ and $\boldsymbol{\theta}$ is strictly increasing, we have $\mathbf{F}(\mathbf{0}) = \sum_1^n{}_i\, w_i\boldsymbol{\theta}(\mathbf{0}) < \sum_1^n{}_i\, w_i\boldsymbol{\theta}(\mathbf{a}^{(i)})$. Using equation (2.29), we have for all m tuples $\mathbf{a}^{(i)} < z\mathbf{b}^{(i)}$ which implies that

$$\sum_{1}^{n}{}_{i}\, w_i\boldsymbol{\theta}(\mathbf{a}^{(i)}) < \sum_{1}^{n}{}_{i}\, w_i\boldsymbol{\theta}(z\mathbf{b}^{(i)}) = \mathbf{F}(z).$$

This shows the existence of z.

As ${}^{\mathbf{a}^{(\mathbf{i})}}/_{\mathbf{b}^{(\mathbf{i})}}$ is a decreasing, $\boldsymbol{\theta}$ is strictly increasing with respect to its components and from equation (2.30), defined as

$$\sum_{1}^{n}{}_{i}\, w_i\boldsymbol{\theta}(z\mathbf{b}^{(i)}) = \sum_{1}^{n}{}_{i}\, w_i\boldsymbol{\theta}(\mathbf{a}^{(i)}),$$

which implies that,

$$\sum_{1}^{n}{}_{i}\, w_i\theta(zb_j^{(i)}) = \sum_{1}^{n}{}_{i}\, w_i\theta(a_j^{(i)}), \quad \forall j = 1, \cdots, m, \tag{2.33}$$

then $\exists$ a k such that

$$\begin{aligned} a_j^{(i)} &\geq zb_j^{(i)} \quad for \quad i = 1, 2, \cdots, k, \\ a_j^{(i)} &\leq zb_j^{(i)} \quad for \quad i = k+1, \cdots, n. \end{aligned}$$

(see [7]). Therefore, for m-tuples, we have

$$\mathbf{a}^{(i)} \geq z\mathbf{b}^{(i)} \quad for \quad i = 1, 2, \cdots, k, \tag{2.34}$$

$$\mathbf{a}^{(i)} \leq z\mathbf{b}^{(i)} \quad for \quad i = k+1, \cdots, n, \tag{2.35}$$

so we get

$$\sum_{1}^{k}{}_{i}\, w_i\boldsymbol{\theta}(z\mathbf{b}^{(i)}) \leq \sum_{1}^{k}{}_{i}\, w_i\boldsymbol{\theta}(\mathbf{a}^{(i)}). \tag{2.36}$$

For reader, we would like to give proof of inequality (2.36), If $i = 1, \cdots, k$ then it can be achieved by using (2.34) immediately. If $i = k+1, \cdots, n$ then (2.36) follows (2.35), such that

$$\begin{aligned}
\sum_{1}^{k}{}_{i} w_i \boldsymbol{\theta}(z\mathbf{b}^{(i)}) &= \sum_{1}^{n}{}_{i} w_i \boldsymbol{\theta}(z\mathbf{b}^{(i)}) - \sum_{k+1}^{n}{}_{i} w_i \boldsymbol{\theta}(z\mathbf{b}^{(i)}) \\
&\leq \sum_{1}^{n}{}_{i} w_i \boldsymbol{\theta}(\mathbf{a}^{(i)}) - \sum_{k+1}^{n}{}_{i} w_i \boldsymbol{\theta}(\mathbf{a}^{(i)}) \\
&= \sum_{1}^{k}{}_{i} w_i \boldsymbol{\theta}(\mathbf{a}^{(i)}).
\end{aligned}$$

If $\mathbf{a}^{(i)}$ is an increasing an "n"tuple using (2.30), (2.36) and using Theorem 2.3, we have

$$\sum_{1}^{n}{}_{i} w_i \psi(\mathbf{a}^{(i)}) \leq \sum_{1}^{n}{}_{i} w_i \psi(z\mathbf{b}^{(i)}).$$

If $\mathbf{b}^{(i)}$ is a decreasing a "n"tuple, then

$$\sum_{1}^{n}{}_{i} w_i \psi(z\mathbf{b}^{(i)}) \leq \sum_{1}^{n}{}_{i} w_i \psi(\mathbf{a}^{(i)}).$$

The remaining cases can be proved analogously. ☐ The following result is the refinement of above inequality.

Theorem 2.10. Let the assumption of Theorem 2.9 holds. Let z be such that

$$\sum_{1}^{n}{}_{i} w_i \boldsymbol{\theta}(z\mathbf{b}^{(i)}) = \sum_{1}^{n}{}_{i} w_i (\boldsymbol{\theta}(\mathbf{a}^{(i)})), \tag{2.37}$$

let $\mathbf{a}^{(\mathbf{i})}/\mathbf{b}^{(\mathbf{i})}$ be a decreasing "n"tuple. If $\mathbf{a}^{(\mathbf{i})}$ is an increasing "n"tuple then

$$\sum_{1}^{n}{}_{i} w_i \psi(z\mathbf{b}^{(i)}) - \sum_{1}^{n}{}_{i} w_i \psi(\mathbf{a}^{(i)} \geq \Big| \sum_{1}^{n}{}_{i} w_i |\psi(z\mathbf{b}^{(i)}) - \psi(\mathbf{a}^{(i)})| - \sum_{1}^{n}{}_{i} w_i \|\psi \circ \boldsymbol{\theta}^{-1}(\mathbf{u}^{(i)})\| \, \|\boldsymbol{\theta}^{-1}(\mathbf{v}^{(i)}) - \boldsymbol{\theta}^{-1}(\mathbf{u}^{(i)})\| \Big|. \tag{2.38}$$

If $\mathbf{b}^{(\mathbf{i})}$ is a decreasing "n"tuples, then

$$\sum_{1}^{n}{}_{i} w_i \psi(\mathbf{a}^{(i)}) - \sum_{1}^{n}{}_{i} w_i \psi(z\mathbf{b}^{(i)}) \geq \Big| \sum_{1}^{n}{}_{i} w_i |\psi(\mathbf{a}^{(i)}) - \psi(z\mathbf{b}^{(i)})| \sum_{1}^{n}{}_{i} w_i \|(\psi \circ \boldsymbol{\theta}^{-1}(\mathbf{u}^{(i)})\| \, \|\boldsymbol{\theta}^{-1}(\mathbf{u}^{(i)}) - \boldsymbol{\theta}^{-1}(\mathbf{v}^{(i)})\| \Big|. \tag{2.39}$$

Let $\mathbf{a}^{(\mathbf{i})}/\mathbf{b}^{(\mathbf{i})}$ be an increasing "n"tuple. If $\mathbf{b}^{(\mathbf{i})}$ is an increasing "n"tuple then (2.38) holds. If $\mathbf{a}^{(\mathbf{i})}$ is a decreasing "n"tuple (2.39) holds.

Proof . The existence of z and the equation

$$\sum_{1}^{k}{}_{i} w_i \boldsymbol{\theta}(z\mathbf{b}^{(i)}) \leq \sum_{1}^{k}{}_{i} w_i \boldsymbol{\theta}(\mathbf{a}^{(i)}), \tag{2.40}$$

has already proved in Theorem 2.9. Therefore using (2.37), (2.40) and Theorem 2.5, we obtain the required refinements (2.38) and (2.39). ☐

Conclusion In the paper we retrieved weighted majorization theorems and its refinements for the functions having non-decreasing increments of convex type. Also we have achieved weighted Favard's inequality and its refinements for the functions having non-decreasing increments of convex type, while at last we derived weighted Berwald inequality and its refinement for the functions having non-decreasing increments of convex type.

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