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$H(\cdot,\cdot)$-φ-η-accretive operator with an application to a system of generalized variational inclusion problems in q-uniformly smooth Banach spaces

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Abstract

In this paper, we study a new system of generalized variational-like inclusion problems involving generalized $H(\cdot,\cdot)$-φ-η-accretive operators in real q-uniformly smooth Banach spaces. We define the resolvent operator associated with $H(\cdot,\cdot)$-φ-η-accretive operator and prove it is single-valued and Lipschitz continuous. Moreover, we suggest a perturbed Mann-type iterative algorithm with errors for approximating the solution of a system of generalized variational-like inclusion problems. Furthermore, we discuss the convergence and stability analysis of the iterative sequence generated by the algorithm.

Keywords: $H(\cdot,\cdot)$-φ-η-accretive operator, q-uniformly smooth Banach spaces, Resolvent operator technique, Perturbed Mann-type iterative algorithm, Convergence analysis, Stability analysis
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1 Introduction

The mathematical study of variational inequality was initially started independently by Stampacchia [24] and Fichera [7] in the early 1960's to study the problems in potential theory and elasticity, respectively. Since then the ideas and techniques of variational inequalities are being used to interpret the basic principles of pure and applied sciences in elegant and simple form. An important aspect in the theory of variational inequalities is the approximation solvability of its solution. In the recent past several researchers studied the approximation solvability of some important classes of variational inequalities and their generalizations.

Motivated and inspired by the research work going on in the approximation solvability of variational inequalities and their generalizations (see for example [1]-[6],[8]-[22],[25]-[30]), in this paper, we introduce and study a new system of generalized variational-like inclusion problem involving generalized $H(\cdot,\cdot)$-φ-η-accretive operator in real q-uniformly smooth Banach spaces. Using resolvent operator associated with $H(\cdot,\cdot)$-φ-η-accretive operator, we prove it is single-valued and Lipschitz continuous. Moreover, we prove the existence of solution for the system of generalized variational-like inclusion problem. Further, we suggest a perturbed Mann-type iterative algorithm with errors for approximating

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the solution of the system of generalized variational-like inclusion problem. Also, we discuss the convergence and stability analysis of the iterative sequences generated by the iterative algorithm.

2 Preliminaries and Formulation of Problem

Let X be a real Banach space equipped with norm $\|\cdot\|$ and $X^\star$ be the topological dual space of X. Let $\langle\cdot,\cdot\rangle$ be the dual pair between X and $X^\star$ and 2^X be the power set of X.

Definition 2.1. [25] For $q>1$, $J_q : X \to 2^{X^\star}$ is said to be a generalized duality mapping, if it is defined by

$$J_q(u) = \{f \in X^\star : \langle u, f\rangle = \|u\|^q, \|u\|^{q-1} = \|f\|\}, \quad \forall u \in X.$$

In particular, J_2 is the usual normalized duality mapping on X. It is well known (see, e.g., [25]) that

$$J_q(u) = \|u\|^{q-1} J_2(u), \quad \forall u (\neq 0) \in X.$$

Note that if $X \equiv H$, a real Hilbert space, then J_2 becomes the identity mapping on X.

Definition 2.2. [25] A Banach space X is said to be *smooth* if, for every $u \in X$ with $\|u\| = 1$, there exists a unique $f \in X^\star$ such that $\|f\| = f(u) = 1$.

The modulus of smoothness of X is the function $\rho_X : [0,\infty) \to [0,\infty)$, defined by

$$\rho_X(t) = \sup\left\{\frac{\|u+v\| + \|u-v\|}{2} - 1 : u, v \in X, \|u\| = 1, \|v\| = t\right\}.$$

Definition 2.3. [25] A Banach space X is said to be

(i) uniformly smooth if $\lim\limits_{t\to 0} \dfrac{\rho_X(t)}{t} = 0$,

(ii) q-uniformly smooth, for $q>1$, if there exists a constant $c>0$ such that $\rho_X(t) \le ct^q$, $t \in [0,\infty)$.

It is well known (see, e.g., [28]) that

$$L_q \text{ (or } l_q\text{) is } \begin{cases} q\text{-uniformly smooth}, & \text{if } 1 < q \le 2, \\ 2\text{-uniformly smooth}, & \text{if } q \ge 2. \end{cases}$$

Note that if X is uniformly smooth, J_q becomes single-valued. In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [25] established the following lemma.

Lemma 2.4. Let $q>1$ be a real number and let X be a smooth Banach space. Then the following statements are equivalent:

(i) X is q-uniformly smooth.

(ii) There is a constant $c_q > 0$ such that for every $u, v \in X$, the following inequality holds

$$\|u+v\|^q \le \|u\|^q + q\langle v, J_q(u)\rangle + c_q\|v\|^q.$$

Definition 2.5. [1] Let X be a *q-uniformly smooth* Banach space. Let $A, B : X \to X$, $\eta, H : X \times X \to X$ be single-valued mappings and $M : X \times X \to 2^X$ be a multi-valued mapping. Then

(i) A is said to be *η-accretive*, if

$$\big\langle Au - Av, J_q\big(\eta(u,v)\big)\big\rangle \ge 0, \quad \forall u, v \in X.$$

(ii) A is said to be *strictly η-accretive*, if

$$\big\langle Au - Av, J_q\big(\eta(u,v)\big)\big\rangle > 0, \quad \forall u, v \in X$$

and equality holds if and only if $u = v$.

(iii) A is said to be *δ-strongly η-accretive*, if there exists a constant $\delta > 0$ such that

$$\big\langle Au - Av, J_q\big(\eta(u,v)\big)\big\rangle \geq \delta\|u-v\|^q, \quad \forall u,v \in X.$$

(iv) A is said to be *λ-Lipschitz continuous*, if there exists a constant $\lambda > 0$ such that

$$\big\|Au - Av\big\| \leq \lambda\|u-v\|, \quad \forall u,v \in X.$$

(v) $H(A,\cdot)$ is said to be *α-strongly η-accretive* with respect to A, if there exists a constant $\alpha > 0$ such that

$$\big\langle H(Au,z) - H(Av,z), J_q\big(\eta(u,v)\big)\big\rangle \geq \alpha\|u-v\|^q, \quad \forall u,v,z \in X.$$

(vi) $H(\cdot,B)$ is said to be *β-relaxed η-accretive* with respect to B, if there exists a constant $\beta > 0$ such that

$$\big\langle H(z,Bu) - H(z,Bv), J_q\big(\eta(u,v)\big)\big\rangle \geq -\beta\|u-v\|^q, \quad \forall u,v,z \in X.$$

(vii) $H(\cdot,\cdot)$ is said to be *(γ,δ)-mixed Lipschitz continuous*, if there exist constants $\gamma > 0, \delta > 0$ such that

$$\big\|H(u,z) - H(v,t)\big\| \leq \gamma\|u-v\| + \delta\|z-t\|, \quad \forall u,v,z,t \in X.$$

(viii) $H(\cdot,\cdot)$ is said to be *$\alpha\beta$-symmetric η-accretive* with respect to A and B, if $H(A,\cdot)$ is *α-strongly η-accretive* with respect to A and $H(\cdot,B)$ is *β-relaxed η-accretive* with respect to B with $\alpha \geq \beta$, and $\alpha = \beta$ if and only if $u = v$, for all $u,v \in X$.

(ix) M is said to be *η-accretive*, if

$$\big\langle x - y, J_q\big(\eta(u,v)\big)\big\rangle \geq 0, \quad \forall u,v \in X, x \in M(u,z), y \in M(v,z) \text{ for each fixed } z \in X.$$

(x) M is said to be *strictly η-accretive*, if M is *η-accretive* and equality holds if and only if $u = v$.

Throughout the rest of the paper unless otherwise stated, we assume X_i to be a q_i-uniformly smooth Banach space.

Definition 2.6. For each $i = 1,2, j \in \{1,2\}\setminus i$, X_i is a q_i-uniformly smooth Banach space. Let $\varphi_i, A_i, B_i : X_i \to X_i$, $H_i, \eta_i : X_i \times X_i \to X_i$ be single-valued mappings, $M_i : X_i \times X_i \to 2^{X_i}$ be a multi-valued mapping. Then $M_i(\cdot, z_i)$ is said to be *$H_i(A_i,B_i)$-φ-η-accretive mapping with respect to A_i and B_i*, if for each fixed $z_i \in X_i, \varphi_i \circ M_i(\cdot,z_i)$ is *η-accretive* and $\big(H_i(A_i,B_i) + \rho_i\varphi_i \circ M_i(\cdot,z_i)\big)X_i = X_i$, for all $\rho_i > 0$.

Remark 2.7. If $\varphi_i(u) = u, \ \forall u \in X, \eta(u,v) = u - v$ and $M_i(\cdot,\cdot) = M_i(\cdot)$, then $H_i(A_i,B_i)$-φ_i-η-accretive operator reduces to a class of $H(\cdot,\cdot)$-accretive operator considered by Zou and Huang [29].

Example 2.8. Let $X = \mathbb{R}, Az = 0, Bz = \sin z, H(Az,Bz) = Az + Bz$ and $M(\omega,z) = \omega^2 + z^2$, for all $z \in X$ and for each fixed $\omega \in X$. Let $\varphi \circ M(\omega,z) = \dfrac{\partial}{\partial z}[M(\omega,z)] = 2z$ and $\eta(z_1,z_2) = \dfrac{z_1 - z_2}{2}$, for all $z_1, z_2 \in X$. Then

$$\begin{aligned}\big\langle \varphi \circ M(\omega,z_1) - \varphi \circ M(\omega,z_2), \eta(z_1,z_2)\big\rangle &= \left\langle 2z_1 - 2z_2, \frac{z_1 - z_2}{2}\right\rangle \\ &= (z_1 - z_2)^2 \\ &\geq 0,\end{aligned}$$

which means that $\varphi \circ M(\omega,\cdot)$ is η-accretive in the second argument. Also, for any $z \in X$, it follows from above that

$$\begin{aligned}\big(H(A,B) + \lambda\varphi \circ M(\omega,\cdot))(z) &= H(Az,Bz) + \lambda\varphi \circ M(\omega,z) \\ &= Az + Bz + \lambda\varphi \circ M(\omega,z) \\ &= 0 + \sin z + 2\lambda z \\ &= 2\lambda z + \sin z,\end{aligned}$$

which means that $\big(H(A,B) + \lambda\varphi \circ M(\omega,\cdot)\big)$ is surjective. Thus M is $H(\cdot,\cdot)$-φ-η-accretive operator with respect to the mappings A and B.

Example 2.9. Let X, A, B, H, η and M be same as in Example 2.8. Let for each fixed $\omega \in X, \varphi \circ M(\omega, z) = e^{\omega^2+z^2}$, for all $z \in X$. Then

$$\begin{aligned}\big(H(A,B) + \lambda\varphi \circ M(\omega,\cdot)\big)(z) &= H(Az, Bz) + \lambda\varphi \circ M(\omega, z)\\ &= Az + Bz + \lambda\varphi \circ M(\omega, z)\\ &= 0 + \sin z + \lambda e^{\omega^2+z^2},\end{aligned}$$

which shows $0 \notin \big(H(A,B) + \lambda\varphi \circ M(\omega,\cdot)\big)(X)$, that is $\big(H(A,B) + \lambda\varphi \circ M(\omega,\cdot)\big)$ is not surjective, hence M is not $H(\cdot,\cdot)$-φ-η-accretive operator with respect to the mappings A and B.

Theorem 2.10. For each $i = 1, 2, j \in \{1, 2\} \setminus i$, let $\varphi_i, A_i, B_i, H_i, \eta_i$ be same as in Definition 2.6, and let H_i be an $\alpha_i\beta_i$-symmetric η-accretive mapping with respect to A_i and B_i $(\alpha_i > \beta_i)$, $M_i : X_i \times X_i \to 2^{X_i}$ be a $H_i(A_i, B_i)$-φ_i-η_i-accretive mapping with respect to A_i and B_i. Then for each fixed $z_i \in X_i$,

(i) if $\langle x - y, J_q(u - v)\rangle \geq 0$ holds for all $(v, y) \in Graph\big(\varphi_i \circ M_i(\cdot, z_i)\big)$ implies $(u, x) \in Graph\big(\varphi_i \circ M_i(\cdot, z_i)\big)$, where $Graph\big(\varphi_i \circ M_i(\cdot, z_i)\big) = \{(u, x) \in X_i \times X_i : x \in \varphi_i \circ M_i(u, z_i)\}$.

(ii) the mapping $\big(H_i(A_i, B_i) + \rho_i\varphi_i \circ M_i(\cdot, z_i)\big)^{-1}$ is single-valued for all $\rho_i > 0$.

Definition 2.11. For each $i = 1, 2, j \in \{1, 2\} \setminus i$, let $\varphi_i, A_i, B_i, H_i, \eta_i$ and M_i be same as defined in Theorem 2.10. Then for each fixed $z_i \in X_i$, the *resolvent operator* $R^{H_i(A_i,B_i),\varphi_i}_{M_i(\cdot,z_i),\rho_i} : X_i \to X_i$ is defined by

$$R^{H_i(A_i,B_i),\varphi_i}_{M_i(\cdot,z_i),\rho_i}(u_i) = (H_i(A_i, B_i) + \rho_i\varphi_i \circ M_i(\cdot, z_i))^{-1}(u_i), \quad \forall u_i \in X_i.$$

Now, we give the following result which guarantees the Lipschitz continuity of the resolvent operator $R^{H_i(A_i,B_i),\varphi_i}_{M_i(\cdot,z_i),\rho_i}$, the proof of which can directly follows from the definition of $R^{H_i(A_i,B_i),\varphi_i}_{M_i(\cdot,z_i),\rho_i}$ and hence is omitted.

Theorem 2.12. For each $i = 1, 2, j \in \{1, 2\} \setminus i$, let $\varphi_i, A_i, B_i, H_i, \eta_i, M_i$ be same as defined in Theorem 2.10, and let η_i be τ_i-Lipschitz continuous. Then for each fixed $z_i \in X_i$, the resolvent operator $R^{H_i(A_i,B_i),\varphi_i}_{M_i(\cdot,z_i),\rho_i} : X_i \to X_i$ is Lipschitz continuous with constant L_i, that is,

$$\|R^{H_i(A_i,B_i),\varphi_i}_{M_i(\cdot,z_i),\rho_i}(x) - R^{H_i(A_i,B_i),\varphi_i}_{M_i(\cdot,z_i),\rho_i}(y)\| \leq L_i\|x - y\|, \quad \forall x, y \in X_i, \text{ where } L_i = \frac{\tau_i^{q-1}}{(\alpha_i - \beta_i)}.$$

Definition 2.13. For each $i = 1, 2, j \in \{1, 2\} \setminus i$, let $\varphi_i, A_i, B_i, H_i, \eta_i, M_i$ be same as defined in Theorem 2.10. Let $\{M_i^n\}, M_i^n : X_i \times X_i \to 2^{X_i}$ be a sequence of $H_i(A_i, B_i)$-φ_i-η_i-accretive mappings with respect to A_i and B_i, respectively, for $n = 0, 1, 2, \ldots$. Then the sequence $\{\varphi_i \circ M_i^n\}$ is graph convergent to $\varphi_i \circ M_i$, denoted by $\varphi_i \circ M_i^n \xrightarrow{H_iG} \varphi_i \circ M_i$, if for every $(u, v) \in Graph(\varphi_i \circ M_i)$, there exists a sequence $\{(u_n, v_n)\} \subset Graph(\varphi_i \circ M_i^n)$ such that $u_n \to u$ and $v_n \to v$ as $n \to \infty$.

Lemma 2.14. [18] Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be non-negative sequences satisfying

$$a_{n+1} \leq (1 - t_n)a_n + b_nt_n + c_n, \quad \forall n \geq 0,$$

where $\{t_n\}_{n=0}^{\infty} \in [0, 1]$, $\sum\limits_{n=0}^{\infty} t_n = +\infty$, $\lim\limits_{n\to\infty} b_n = 0$ and $\sum\limits_{n=0}^{\infty} c_n < \infty$. Then $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 2.15. [23] For $n \geq 0$, let $A : X \to X$ be a single-valued mapping and $u_0 \in X, u_{n+1} = T(A, u_n)$ be an iteration procedure which yields a sequence of points $\{u_n\}_{n\geq 0} \subset X$, where T is a continuous mapping. Suppose that $\{u \in X : Au = u\} \neq \emptyset$ and $\{u_n\}_{n\geq 0}$ converges to a fixed point $u^\star$ of A. Let $\{v_n\}_{n\geq 0} \subset X,\ h_n = \|v_{n+1} - T(A, v_n)\|$. If $\lim\limits_{n\to\infty} h_n = 0$ implies that $\lim\limits_{n\to\infty} v_n = u^\star$, then the iteration procedure defined by $u_{n+1} = T(A, u_n)$ is said to be A-stable or stable with respect to A.

Let for each $i = 1, 2, j \in \{1, 2\}\backslash i$, X_i be a q_i-uniformly smooth Banach space with norm $\|\cdot\|_i$. Let $\varphi_i, A_i, B_i, P_i, N_i, g_i, p_i : X_i \to X_i$, $Q_i : X_j \to X_i, F_i, \eta_i : X_i \times X_i \to X_i$ be single-valued mappings, and let $H_i : X_i \times X_i \to X_i$ be an $\alpha_i\beta_i$-symmetric η_i-accretive mapping with respect to A_i and B_i $(\alpha_i > \beta_i)$, $M_i : X_i \times X_i \to 2^{X_i}$ be a $H_i(A_i, B_i)$-φ_i-η_i-accretive mapping with respect to A_i and B_i. We consider the following system of generalized variational-like inclusion problem (SGVLIP): Find $(u, v) \in X_1 \times X_2$ where $u \in X_1, v \in X_2$ and for fixed $z_1 \in X_1, z_2 \in X_2$ such that

$$\left.\begin{aligned} &\theta_1 \in N_1\big(u - p_1(u)\big) + F_1\big(P_1(u), Q_1(v)\big) + M_1\big((g_1 - p_1)(u), z_1\big), \\ &\theta_2 \in N_2\big(v - p_2(v)\big) + F_2\big(Q_2(u), P_2(v)\big) + M_2\big((g_2 - p_2)(v), z_2\big), \end{aligned}\right\} \tag{2.1}$$

where θ_1 and θ_2 are zero vectors of X_1 and X_2, respectively.

We remark that for appropriate and suitable choices of the above defined mappings, SGVLIP (2.1) includes a number of variational inequalities, variational inclusions and variational-like inclusions as special cases, see for example [1]-[4],[6],[8]-[14],[16, 17, 29, 30] and the related references cited therein.

3 Existence of Solution

First, we give the following technical lemma:

Lemma 3.1. For $i = 1, 2, j \in \{1, 2\} \setminus i$, let $A_i, B_i, P_i, N_i, g_i, p_i : X_i \to X_i$, $Q_i : X_j \to X_i, F_i, \eta_i : X_i \times X_i \to X_i$ be single-valued mappings, $H_i : X_i \times X_i \to X_i$ be an $\alpha_i\beta_i$-symmetric η_i-accretive mapping with respect to A_i and B_i $(\alpha_i > \beta_i)$, and let $\varphi_i : X_i \to X_i$ be mappings such that $\varphi_i(w_i + w_i') = \varphi_i(w_i) + \varphi_i(w_i')$, for all $w_i, w_i' \in X_i$ and $\mathrm{Ker}(\varphi_i) = \{\theta_i\}$, where $\mathrm{Ker}(\varphi_i) = \{w_i \in X_i : \varphi_i(w_i) = \theta_i\}$ and $M_i : X_i \times X_i \to 2^{X_i}$ be a $H_i(A_i, B_i)$-φ_i-η_i-accretive mapping with respect to A_i and B_i. Then $(u, v) \in X_1 \times X_2$ is a solution of SGVLIP (2.1), where $u \in X_1, v \in X_2$ if and only if it satisfies:

$$\left.\begin{aligned} (g_1 - p_1)(u) &= R^{H_1(A_1,B_1),\varphi_1}_{M_1(\cdot,z_1),\rho_1}\big[H_1\big(A_1(g_1 - p_1)(u), B_1(g_1 - p_1)(u)\big) \\ &\quad -\rho_1\varphi_1 \circ N_1(u - p_1(u)) - \rho_1\varphi_1 \circ F_1\big(P_1(u), Q_1(v)\big)\big] \\ (g_2 - p_2)(v) &= R^{H_2(A_2,B_2),\varphi_2}_{M_2(\cdot,z_2),\rho_2}\big[H_2\big(A_2(g_2 - p_2)(v), B_2(g_2 - p_2)(v)\big) \\ &\quad -\rho_2\varphi_2 \circ N_2(v - p_2(v)) - \rho_2\varphi_2 \circ F_2\big(Q_2(u), P_2(v)\big)\big], \end{aligned}\right\} \tag{3.1}$$

where $\rho_1, \rho_2 > 0$ are constants and for fixed z_i, $R^{H_i(A_i,B_i),\varphi_i}_{M_i(\cdot,z_i),\rho_i}(u_i) = \big(H_i(A_i, B_i) + \rho_i\varphi_i \circ M_i(\cdot, z_i)\big)^{-1}(u_i),\ \forall u_i \in X_i$.

Proof . From the definition of $R^{H_1(A_1,B_1),\varphi_1}_{M_1(\cdot,z_1),\rho_1}$, we have

$$\begin{aligned} &\big[H_1\big(A_1(g_1 - p_1)(u), B_1(g_1 - p_1)(u)\big) - \rho_1\varphi_1 \circ N_1(u - p_1(u)) - \rho_1\varphi_1 \circ F_1\big(P_1(u), Q_1(v)\big)\big] \\ &\qquad \in H_1\big(A_1(g_1 - p_1)(u), B_1(g_1 - p_1)(u)\big) + \rho_1\varphi_1 \circ M_1\big((g_1 - p_1)(u), z_1\big)\big) \\ \Longrightarrow \quad &\theta_1 \in \varphi_1 \circ N_1(u - p_1(u)) + \varphi_1 \circ F_1\big(P_1(u), Q_1(v)\big) + \varphi_1 \circ M_1\big((g_1 - p_1)(u), z_1\big)\big). \end{aligned}$$

Thus $\theta_1 \in N_1(u - p_1(u)) + F_1\big(P_1(u), Q_1(v)\big) + M_1\big((g_1 - p_1)(u), z_1\big)\big)$.

Similarly, we have $\theta_2 \in N_2(v - p_2(v)) + F_2\big(Q_2(u), P_2(v)\big) + M_2\big((g_2 - p_2)(v), z_2\big)$. □

Now, we give the following result which guarantees the existence of solution for SGVLIP (2.1).

Theorem 3.2. For $i = 1, 2, j \in \{1, 2\} \setminus i$, let $A_i, B_i, P_i, N_i, g_i, p_i : X_i \to X_i$, $Q_i : X_j \to X_i, F_i, \eta_i : X_i \times X_i \to X_i$ be single-valued mappings, $\varphi_i : X_i \to X_i$ be a mapping satisfying $\varphi_i(w_i + w_i') = \varphi_i(w_i) + \varphi_i(w_i')$, for all $w_i, w_i' \in X_i$ and $\mathrm{Ker}(\varphi_i) = \{\theta_i\}$, where $\mathrm{Ker}(\varphi_i) = \{w_i \in X_i : \varphi_i(w_i) = \theta_i\}$, and let $H_i : X_i \times X_i \to X_i$ be an $\alpha_i\beta_i$-symmetric η_i-accretive mapping with respect to A_i and B_i and (γ_i, δ_i)-mixed Lipschitz continuous, respectively, P_i be L_{P_i}-Lipschitz continuous, Q_i be L_{Q_i}-Lipschitz continuous, $\varphi_i \circ N_i$ be L_{N_i}-Lipschitz continuous, $(g_i - p_i)$ be σ_i-strongly η_i-accretive and μ_i-Lipschitz continuous, and $A_i(g_i - p_i)$ be L_{A_i}-Lipschitz continuous, $B_i(g_i - p_i)$ be a L_{B_i}-Lipschitz continuous mapping, respectively. Let $M_i : X_i \times X_i \to 2^{X_i}$ be a $H_i(A_i, B_i)$-φ_i-η_i-accretive mapping with respect to A_i and B_i, p_i be r_i-strongly-η_i-accretive and s_i-Lipschitz continuous, $\varphi_1 \circ F_1$ be ζ_1-strongly accretive in the first argument and (L_{F_1}, l_{F_1})-mixed Lipschitz continuous, $\varphi_2 \circ F_2$ be ζ_2-strongly accretive in the second argument and (l_{F_2}, L_{F_2})-mixed Lipschitz continuous, respectively. In addition, suppose that the following conditions are satisfied:

$$k_i = m_i + L_j \rho_j l_{F_j} < 1, \tag{3.2}$$

where

$$\begin{aligned} m_i &= a_i + L_i(b_i + c_i + \rho_i d_i), \quad a_i = \left[1 - q_i\sigma_i + q_i\mu_i(1 + \tau_i^{q_i-1}) + c_{q_i}\mu_i^{q_i}\right]^{1/q_i}, \\ b_i &= \left[1 - q_i(\alpha_i - \beta_i)\mu_i^{q_i} + q_i(\gamma_i + \delta_i)\left(1 + \tau_i^{q_i-1}\right) + c_{q_i}\left((\gamma_i L_{A_i})^{q_i} + (\delta_i L_{B_i})^{q_i}\right)\right]^{1/q_i}, \\ c_i &= \left[1 - \rho_i q_i \zeta_i + \rho_i q_i L_{P_i}\left(1 + \tau_i^{q_i-1}\right) + \rho_i^{q_i} c_{q_i} L_{F_i}^{q_i} L_{P_i}^{q_i}\right]^{1/q_i}, \\ d_i &= L_{N_i}\left[1 - q_i r_i + q_i s_i\left(1 + \tau_i^{q_i-1}\right) + c_{q_i} s_i^{q_i}\right]^{1/q_i}, \quad L_i = \frac{\tau_i^{q_i-1}}{(\alpha_i - \beta_i)}. \end{aligned}$$

Then SGVLIP (2.1) has a solution.

Proof . For each $(u, v) \in X_1 \times X_2$, define a mapping $G : X_1 \times X_2 \to X_1 \times X_2$ by

$$G(u, v) = (S_1(u, v), S_2(u, v)), \quad \forall (u, v) \in X_1 \times X_2, \tag{3.3}$$

where $S_1 : X_1 \times X_2 \to X_1$ and $S_2 : X_1 \times X_2 \to X_2$ are defined by

$$\begin{aligned} S_1(u, v) = u - (g_1 - p_1)(u) + R^{H_1(A_1,B_1),\varphi_1}_{M_1(\cdot,z_1),\rho_1}\Big[&H_1(A_1(g_1 - p_1)(u), B_1(g_1 - p_1)(u)) \\ &- \rho_1\varphi_1 \circ N_1(u - p_1(u)) - \rho_1\varphi_1 \circ F_1(P_1(u), Q_1(v))\Big], \quad \rho_1 > 0 \qquad (3.4) \\ S_2(u, v) = v - (g_2 - p_2)(v) + R^{H_2(A_2,B_2),\varphi_2}_{M_2(\cdot,z_2),\rho_2}\Big[&H_2(A_2(g_2 - p_2)(v), B_2(g_2 - p_2)(v)) \\ &- \rho_2\varphi_2 \circ N_2(v - p_2(v)) - \rho_2\varphi_2 \circ F_2(Q_2(u), P_2(v))\Big], \quad \rho_2 > 0. \qquad (3.5) \end{aligned}$$

For any $(u_1, v_1), (u_2, v_2) \in X_1 \times X_2$, it follows from (3.4), (3.5) and Lipschitz continuity of $R^{H_1(A_1,B_1),\varphi_1}_{M_1(\cdot,z_1),\rho_1}$ and $R^{H_2(A_2,B_2),\varphi_2}_{M_2(\cdot,z_2),\rho_2}$ that

$$\begin{aligned} \left\|S_1(u_1, v_1) - S_1(u_2, v_2)\right\|_1 \le &\left\|(u_1 - u_2) - \Big((g_1 - p_1)(u_1) - (g_1 - p_1)(u_2)\Big)\right\|_1 \\ &+ \Big\|R^{H_1(A_1,B_1),\varphi_1}_{M_1(\cdot,z_1),\rho_1}\left[H_1(A_1(g_1 - p_1)(u_1), B_1(g_1 - p_1)(u_1))\right. \\ &\left.- \rho_1\varphi_1 \circ N_1(u_1 - p_1(u_1)) - \rho_1\varphi_1 \circ F_1(P_1(u_1), Q_1(v_1))\right] \\ &- R^{H_1(A_1,B_1),\varphi_1}_{M_1(\cdot,z_1),\rho_1}\left[H_1(A_1(g_1 - p_1)(u_2), B_1(g_1 - p_1)(u_2))\right. \\ &\left.- \rho_1\varphi_1 \circ N_1(u_2 - p_1(u_2)) - \rho_1\varphi_1 \circ F_1(P_1(u_2), Q_1(v_2))\right]\Big\|_1 \\ \le &\left\|(u_1 - u_2) - \Big((g_1 - p_1)(u_1) - (g_1 - p_1)(u_2)\Big)\right\|_1 \\ &+ L_1\Big\|H_1(A_1(g_1 - p_1)(u_1), B_1(g_1 - p_1)(u_1)) \\ &- H_1(A_1(g_1 - p_1)(u_2), B_1(g_1 - p_1)(u_2)) - (u_1 - u_2)\Big\|_1 \\ &+ L_1\Big\|(u_1 - u_2) - \rho_1[\varphi_1 \circ F_1(P_1(u_1), Q_1(v_1)) - \varphi_1 \circ F_1(P_1(u_2), Q_1(v_1))]\Big\|_1 \\ &+ L_1\rho_1\Big\|\varphi_1 \circ F_1(P_1(u_2), Q_1(v_1)) - \varphi_1 \circ F_1(P_1(u_2), Q_1(v_2))\Big\|_1 \\ &+ L_1\rho_1\Big\|\varphi_1 \circ N_1(u_1 - p_1(u_1)) - \varphi_1 \circ N_1(u_2 - p_1(u_2))\Big\|_1. \end{aligned} \tag{3.6}$$

Since $(g_i - p_i)$ is σ_i-strongly η_i-accretive, μ_i-Lipschitz continuous and using Lemma 2.4, we have

$$\left\|(u_1 - u_2) - \big((g_1 - p_1)(u_1) - (g_1 - p_1)(u_2)\big)\right\|_1^{q_1}$$

$$
\begin{aligned}
&\le \|u_1-u_2\|_1^{q_1} - q_1\big\langle (g_1-p_1)(u_1)-(g_1-p_1)(u_2), J_{q_1}\big(\eta_1(u_1,u_2)\big)\big\rangle_1\\
&\quad - q_1\big\langle (g_1-p_1)(u_1)-(g_1-p_1)(u_2), J_{q_1}(u_1-u_2)-J_{q_1}\big(\eta_1(u_1,u_2)\big)\big\rangle_1\\
&\quad + c_{q_1}\big\|(g_1-p_1)(u_1)-(g_1-p_1)(u_2)\big\|_1^{q_1}\\
&\le \|u_1-u_2\|_1^{q_1} - q_1\big\langle (g_1-p_1)(u_1)-(g_1-p_1)(u_2), J_{q_1}\big(\eta_1(u_1,u_2)\big)\big\rangle_1\\
&\quad + q_1\big\|(g_1-p_1)(u_1)-(g_1-p_1)(u_2)\big\|_1 \times \big[\|u_1-u_2\|_1^{q_1-1} + \big\|\eta_1(u_1,u_2)\big\|_1^{q_1-1}\big]\\
&\quad + c_{q_1}\big\|(g_1-p_1)(u_1)-(g_1-p_1)(u_2)\big\|_1^{q_1}\\
&\le \big(1-q_1\sigma_1+q_1\mu_1(1+\tau_1^{q_1-1})+c_{q_1}\mu_1^{q_1}\big)\big\|u_1-u_2\big\|_1^{q_1}.
\end{aligned}
$$

This implies

$$
\big\|(u_1-u_2)-\big((g_1-p_1)(u_1)-(g_1-p_1)(u_2)\big)\big\|_1 \le a_1\big\|u_1-u_2\big\|_1, \tag{3.7}
$$

where $a_1=\big(1-q_1\sigma_1+q_1\mu_1(1+\tau_1^{q_1-1})+c_{q_1}\mu_1^{q_1}\big)^{1/q_1}$.

Since $H_i(A_i,B_i)$ is an $\alpha_i\beta_i$-symmetric η_i-accretive mapping with respect to A_i and B_i and (γ_i,δ_i)-mixed Lipschitz continuous, $A_i(g_i-p_i)$ is L_{A_i}-Lipschitz continuous, $B_i(g_i-p_i)$ is L_{B_i}-Lipschitz continuous, respectively, by using Lemma 2.4, we have

$$
\begin{aligned}
&\big\|H_1\big(A_1(g_1-p_1)(u_1),B_1(g_1-p_1)(u_1)\big)-H_1\big(A_1(g_1-p_1)(u_2),B_1(g_1-p_1)(u_2)\big)-(u_1-u_2)\big\|_1^{q_1}\\
&\le \big\|u_1-u_2\big\|_1^{q_1}\\
&\quad - q_1\big\langle H_1\big(A_1(g_1-p_1)(u_1),B_1(g_1-p_1)(u_1)\big)-H_1\big(A_1(g_1-p_1)(u_2),B_1(g_1-p_1)(u_2)\big), J_{q_1}\big(\eta_1(u_1,u_2)\big)\big\rangle_1\\
&\quad - q_1\big\langle H_1\big(A_1(g_1-p_1)(u_1),B_1(g_1-p_1)(u_1)\big)-H_1\big(A_1(g_1-p_1)(u_2),B_1(g_1-p_1)(u_2)\big),\\
&\qquad\qquad J_{q_1}(u_1-u_2)-J_{q_1}\big(\eta_1(u_1,u_2)\big)\big\rangle_1\\
&\quad + c_{q_1}\big\|H_1\big(A_1(g_1-p_1)(u_1),B_1(g_1-p_1)(u_1)\big)-H_1\big(A_1(g_1-p_1)(u_2),B_1(g_1-p_1)(u_2)\big)\big\|_1^{q_1}\\
&\le \big\|u_1-u_2\big\|_1^{q_1}\\
&\quad - q_1\big\langle H_1\big(A_1(g_1-p_1)(u_1),B_1(g_1-p_1)(u_1)\big)-H_1\big(A_1(g_1-p_1)(u_2),B_1(g_1-p_1)(u_2)\big), J_{q_1}\big(\eta_1(u_1,u_2)\big)\big\rangle_1\\
&\quad - q_1\Big\|H_1\big(A_1(g_1-p_1)(u_1),B_1(g_1-p_1)(u_1)\big)-H_1\big(A_1(g_1-p_1)(u_2),B_1(g_1-p_1)(u_2)\big)\big\|_1\\
&\quad \times \big[\big\|u_1-u_2\big\|_1^{q_1-1}+\|\eta_1(u_1,u_2)\|_1^{q_1-1}\big]\\
&\quad + c_{q_1}\big\|H_1\big(A_1(g_1-p_1)(u_1),B_1(g_1-p_1)(u_1)\big)-H_1\big(A_1(g_1-p_1)(u_2),B_1(g_1-p_1)(u_2)\big)\big\|_1^{q_1}\\
&\le \big\|u_1-u_2\big\|_1^{q_1}\\
&\quad - q_1\big\langle H_1\big(A_1(g_1-p_1)(u_1),B_1(g_1-p_1)(u_1)\big)-H_1\big(A_1(g_1-p_1)(u_2),B_1(g_1-p_1)(u_1)\big), J_{q_1}\big(\eta_1(u_1,u_2)\big)\big\rangle_1\\
&\quad - q_1\big\langle H_1\big(A_1(g_1-p_1)(u_2),B_1(g_1-p_1)(u_1)\big)-H_1\big(A_1(g_1-p_1)(u_2),B_1(g_1-p_1)(u_2)\big), J_{q_1}\big(\eta_1(u_1,u_2)\big)\big\rangle_1\\
&\quad - q_1\big\|H_1\big(A_1(g_1-p_1)(u_1),B_1(g_1-p_1)(u_1)\big)-H_1\big(A_1(g_1-p_1)(u_2),B_1(g_1-p_1)(u_2)\big)\big\|_1\\
&\quad \times \big[\big\|u_1-u_2\big\|_1^{q_1-1}+\|\eta_1(u_1,u_2)\|_1^{q_1-1}\big]\\
&\quad + c_{q_1}\big\|H_1\big(A_1(g_1-p_1)(u_1),B_1(g_1-p_1)(u_1)\big)-H_1\big(A_1(g_1-p_1)(u_2),B_1(g_1-p_1)(u_2)\big)\big\|_1^{q_1}\\
&\le \big\|u_1-u_2\big\|_1^{q_1} - q_1(\alpha_1-\beta_1)\big\|(g_1-p_1)(u_1)-(g_1-p_1)(u_2)\big\|_1^{q_1}\\
&\quad + q_1(\gamma_1+\delta_1)\big\|u_1-u_2\big\|_1 \times \big[\big\|u_1-u_2\big\|_1^{q_1-1}+\|\tau_1^{q_1-1}\big\|u_1-u_2\big\|_1^{q_1-1}\big]\\
&\quad + c_{q_1}\big[\gamma_1^{q_1}\big\|A_1(g_1-p_1)(u_1)-A_1(g_1-p_1)(u_2)\big\|_1^{q_1}+\delta_1^{q_1}\big\|B_1(g_1-p_1)(u_1)-B_1(g_1-p_1)(u_2)\big\|_1^{q_1}\big]\\
&\le \big\{1-q_1(\alpha_1-\beta_1)\mu_1^{q_1}+q_1(\gamma_1+\delta_1)\big[1+\tau_1^{q_1-1}\big]\big\}\big\|u_1-u_2\big\|_1^{q_1}+c_{q_1}\big[(\gamma_1L_{A_1})^{q_1}+(\delta_1L_{B_1})^{q_1}\big]\big\|u_1-u_2\big\|_1^{q_1}\\
&\le \big\{1-q_1(\alpha_1-\beta_1)\mu_1^{q_1}+q_1(\gamma_1+\delta_1)\big[1+\tau_1^{q_1-1}\big]+c_{q_1}\big[(\gamma_1L_{A_1})^{q_1}+(\delta_1L_{B_1})^{q_1}\big]\big\}\|u_1-u_2\|_1^{q_1}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&\big\|H_1\big(A_1(g_1-p_1)(u_1),B_1(g_1-p_1)(u_1)\big)-H_1\big(A_1(g_1-p_1)(u_2),B_1(g_1-p_1)(u_2)\big)-(u_1-u_2)\big\|_1\\
&\qquad\qquad \le b_1\big\|u_1-u_2\big\|_1,
\end{aligned} \tag{3.8}
$$

where

$$
b_1=\big\{1-q_1(\alpha_1-\beta_1)\mu_1^{q_1}+q_1(\gamma_1+\delta_1)\big[1+\tau_1^{q_1-1}\big]+c_{q_1}\big[(\gamma_1L_{A_1})^{q_1}+(\delta_1L_{B_1})^{q_1}\big]\big\}^{1/q_1}.
$$

Now, since $\varphi_1 \circ F_1$ is ζ_1-strongly η_1-accretive mapping in the first argument and is L_{F_1}-Lipschitz continuous in the first argument and l_{F_1}-Lipschitz continuous in the second argument, respectively by using Lemma 2.4, we have

$$\begin{aligned}
&\big\|(u_1-u_2)-\rho_1\big[\varphi_1\circ F_1\big(P_1(u_1),Q_1(v_1)\big)-\varphi_1\circ F_1\big(P_1(u_2),Q_1(v_1)\big)\big]\big\|_1^{q_1}\\
&\quad\le \big\|u_1-u_2\big\|_1^{q_1}-\rho_1q_1\big\langle\varphi_1\circ F_1(P_1(u_1),Q_1(v_1))-\varphi_1\circ F_1(P_1(u_2),Q_1(v_1)),J_{q_1}\big(\eta_1(u_1,u_2)\big)\big\rangle_1\\
&\qquad-\rho_1q_1\big\langle\varphi_1\circ F_1(P_1(u_1),Q_1(v_1))-\varphi_1\circ F_1(P_1(u_2),Q_1(v_1)),J_{q_1}(u_1-u_2)-J_{q_1}\big(\eta_1(u_1,u_2)\big)\big\rangle_1\\
&\qquad+\rho_1^{q_1}c_{q_1}\big\|\varphi_1\circ F_1(P_1(u_1),Q_1(v_1))-\varphi_1\circ F_1(P_1(u_2),Q_1(v_1))\big\|_1^{q_1}\\
&\quad\le \big\|u_1-u_2\big\|_1^{q_1}-\rho_1q_1\zeta_1\big\|u_1-u_2\big\|_1^{q_1}+\rho_1q_1\big\|\varphi_1\circ F_1(P_1(u_1),Q_1(v_1))-\varphi_1\circ F_1(P_1(u_2),Q_1(v_1))\big\|_1\\
&\qquad\times\Big[\big\|u_1-u_2\big\|_1^{q_1-1}+\big\|\eta_1(u_1,u_2)\big\|_1^{q_1-1}\Big]+\rho_1^{q_1}c_{q_1}L_{F_1}^{q_1}\big\|P_1(u_1)-P_1(u_2)\big\|_1^{q_1}\\
&\quad\le \big\|u_1-u_2\big\|_1^{q_1}-\rho_1q_1\zeta_1\big\|u_1-u_2\big\|_1^{q_1}+\rho_1q_1L_{F_1}\big\|P_1(u_1)-P_1(u_2)\big\|_1\\
&\qquad\times\Big[\big\|u_1-u_2\big\|_1^{q_1-1}+\tau_1^{q_1-1}\big\|u_1-u_2\big\|_1^{q_1-1}\Big]+\rho_1^{q_1}c_{q_1}L_{F_1}^{q_1}\Big\|P_1(u_1)-P_1(u_2)\Big\|_1^{q_1}\\
&\quad\le\Big[1-\rho_1q_1\zeta_1+\rho_1q_1L_{F_1}L_{P_1}\big(1+\tau_1^{q_1-1}\big)+\rho_1^{q_1}c_{q_1}L_{F_1}^{q_1}L_{P_1}^{q_1}\Big]\big\|u_1-u_2\big\|_1^{q_1}.
\end{aligned}$$

This implies

$$\big\|(u_1-u_2)-\rho_1\big[\varphi_1\circ F_1\big(P_1(u_1),Q_1(v_1)\big)-\varphi_1\circ F_1\big(P_1(u_2),Q_1(v_1)\big)\big]\big\|_1\le c_1\big\|u_1-u_2\big\|_1, \tag{3.9}$$

where

$$c_1=\Big[1-\rho_1q_1\zeta_1+\rho_1q_1L_{F_1}L_{P_1}\big(1+\tau_1^{q_1-1}\big)+\rho_1^{q_1}c_{q_1}L_{F_1}^{q_1}L_{P_1}^{q_1}\Big]^{1/q_1}$$

and

$$\begin{aligned}
\big\|\varphi_1\circ F_1(P_1(u_2),Q_1(v_1))-\varphi_1\circ F_1(P_1(u_2),Q_1(v_2))\big\|_1&\le l_{F_1}\big\|Q_1(v_1)-Q_1(v_2)\big\|_1\\
&\le l_{F_1}L_{Q_1}\big\|v_1-v_2\big\|_2.
\end{aligned}\tag{3.10}$$

As p_1 is r_1-strongly η_1-accretive and s_1-Lipschitz continuous, by using Lemma 2.4, we have

$$\begin{aligned}
&\big\|u_1-u_2-\big(p_1(u_1)-p_1(u_2)\big)\big\|_1^{q_1}\\
&\qquad\le\big\|u_1-u_2\big\|_1^{q_1}-q_1\big\langle p_1(u_1)-p_1(u_2),J_{q_1}\big(\eta_1(u_1,u_2)\big)\big\rangle\\
&\qquad\quad-q_1\big\langle p_1(u_1)-p_1(u_2),J_{q_1}(u_1-u_2)-J_{q_1}\big(\eta_1(u_1,u_2)\big)\big\rangle+c_{q_1}\Big\|p_1(u_1)-p_1(u_2)\Big\|_1^{q_1}\\
&\qquad\le\big\|u_1-u_2\big\|_1^{q_1}-q_1\big\langle p_1(u_1)-p_1(u_2),J_{q_1}\big(\eta_1(u_1,u_2)\big)\big\rangle\\
&\qquad\quad+q_1\big\|p_1(u_1)-p_1(u_2)\big\|\times\Big[\big\|u_1-u_2\big\|_1^{q_1-1}+\big\|\eta_1(u_1,u_2)\big\|_1^{q_1-1}\Big]+c_{q_1}\big\|p_1(u_1)-p_1(u_2)\big\|_1^{q_1}\\
&\qquad\le\big[1-q_1r_1+q_1s_1\big(1+\tau_1^{q_1-1}\big)+c_{q_1}s_1^{q_1}\big]\big\|u_1-u_2\big\|_1^{q_1}.
\end{aligned}\tag{3.11}$$

Again, since $\varphi_i\circ N_i$ is L_{N_i}-Lipschitz continuous, by using (3.11) and Lemma 2.4, we have

$$\begin{aligned}
&\big\|\varphi_1\circ N_1(u_1-p_1(u_1))-\varphi_1\circ N_1(u_2-p_1(u_2))\big\|_1\\
&\qquad\le L_{N_1}\big\|u_1-u_2-\big(p_1(u_1)-p_1(u_2)\big)\big\|_1\\
&\qquad\le L_{N_1}\big[1-q_1r_1+q_1s_1\big(1+\tau_1^{q_1-1}\big)+c_{q_1}s_1^{q_1}\big]^{1/q_1}\big\|u_1-u_2\big\|_1\\
&\qquad\le d_1\big\|u_1-u_2\big\|_1,
\end{aligned}\tag{3.12}$$

where $d_1=L_{N_1}\big[1-q_1r_1+q_1s_1\big(1+\tau_1^{q_1-1}\big)+c_{q_1}s_1^{q_1}\big]^{1/q_1}$.

From (3.6)-(3.12), we have

$$\begin{aligned}
\big\|S_1(u_1,v_1)-S_1(u_2,v_2)\big\|_1&\le\big[a_1+L_1\big(b_1+c_1+\rho_1d_1\big)\big]\big\|u_1-u_2\big\|_1+L_1\rho_1l_{F_1}L_{Q_1}\big\|v_1-v_2\big\|_2\\
&\le m_1\big\|u_1-u_2\big\|_1+L_1\rho_1l_{F_1}L_{Q_1}\big\|v_1-v_2\big\|_2.
\end{aligned}\tag{3.13}$$

Similarly, we have

$$\big\|S_2(u_1,v_1)-S_2(u_2,v_2)\big\|_2\le\big[a_2+L_2\big(b_2+c_2+\rho_2d_2\big)\big]\big\|v_1-v_2\big\|_2+L_2\rho_2l_{F_2}L_{Q_2}\big\|u_1-u_2\big\|_1$$

$$\leq m_2\big\|v_1 - v_2\big\|_2 + L_2\rho_2 l_{F_2} L_{Q_2}\big\|u_1 - u_2\big\|_1. \tag{3.14}$$

From (3.13) and (3.14), we have

$$\begin{aligned}
\big\|S_1(u_1,v_1) - S_1(u_2,v_2)\big\|_1 &+ \big\|S_2(u_1,v_1) - S_2(u_2,v_2)\big\|_2 \\
&\leq \big(m_1 + L_2\rho_2 l_{F_2} L_{Q_2}\big)\big\|u_1 - u_2\big\|_1 + \big(m_2 + L_1\rho_1 l_{F_1} L_{Q_1}\big)\big\|v_1 - v_2\big\|_2 \\
&\leq k_1\big\|u_1 - u_2\big\|_1 + k_2\big\|v_1 - v_2\big\|_2 \\
&\leq \max\{k_1, k_2\}\big(\big\|u_1 - u_2\big\|_1 + \big\|v_1 - v_2\big\|_2\big) \\
&\leq k\big\{\big\|u_1 - u_2\big\|_1 + \big\|v_1 - v_2\big\|_2\big\},
\end{aligned} \tag{3.15}$$

where $k = \max\{k_1, k_2\}$ and

$$\begin{aligned}
k_i &= m_i + L_j\rho_j l_{F_j} L_{Q_j} < 1, \quad m_i = a_i + L_i(b_i + c_i + \rho_i d_i), \\
a_i &= \big[1 - q_i\sigma_i + q_i\mu_i\big(1 + \tau_i^{q_i-1}\big) + c_{q_i}\mu_i^{q_i}\big]^{1/q_i}, \\
b_i &= \big[1 - q_i(\alpha_i - \beta_i)\mu_i^{q_i} + q_i(\gamma_i + \delta_i)\big(1 + \tau_i^{q_i-1}\big) + c_{q_i}\big((\gamma_i L_{A_i})^{q_i} + (\delta_i L_{B_i})^{q_i}\big)\big]^{1/q_i} \\
c_i &= \big[1 - \rho_i q_i\zeta_i + \rho_i q_i L_{P_i} L_{F_i}\big(1 + \tau_i^{q_i-1}\big) + \rho_i^{q_i} c_{q_i} L_{F_i}^{q_i} L_{P_i}^{q_i}\big]^{1/q_i}, \\
d_i &= L_{N_i}\big[1 - q_i r_i + q_i s_i(1 + \tau_i^{q_i-1}) + c_{q_i} s_i^{q_i}\big]^{1/q_i}, \quad L_i = \frac{\tau_i^{q_i-1}}{(\alpha_i - \beta_i)}.
\end{aligned}$$

Now, define the norm $\|\cdot\|_\star$ on $X_1 \times X_2$ by

$$\big\|(u,v)\big\|_\star = \|u\|_1 + \|v\|_2, \ \ \forall\ (u,v) \in X_1 \times X_2. \tag{3.16}$$

We observe that $\big(X_1 \times X_2, \|\cdot\|_\star\big)$ is a Banach space. Hence, it follows from (3.3), (3.15) and (3.16) that

$$\begin{aligned}
\big\|G(u_1,v_1) - G(u_2,v_2)\big\|_\star &\leq \big\|(S_1(u_1,v_1), S_2(u_1,v_1)) - (S_1(u_2,v_2), S_2(u_2,v_2))\big\|_\star \\
&\leq \big\|S_1(u_1,v_1) - S_1(u_2,v_2), S_2(u_1,v_1) - S_2(u_2,v_2)\big\|_\star \\
&\leq \big\|S_1(u_1,v_1) - S_1(u_2,v_2)\big\|_1 + \big\|S_2(u_1,v_1) - S_2(u_2,v_2)\big\|_2 \\
&\leq k\big\{\big\|(u_1 - u_2\big\|_1 + \big\|v_1 - v_2\big\|_2\big\}.
\end{aligned} \tag{3.17}$$

Since $k = \max\{k_1, k_2\} < 1$ by (3.2), it follows from (3.17) that G is a contraction mapping. Hence, by Banach contraction principle, it admits a unique fixed point $(u,v) \in X_1 \times X_2$ that is

$$G(u,v) = (u,v).$$

Which implies that

$$\left.\begin{aligned}
(g_1 - p_1)(u) \ &= R^{H_1(A_1,B_1),\varphi_1}_{M_1(\cdot,z_1),\rho_1}\big[H_1\big(A_1(g_1 - p_1)(u), B_1(g_1 - p_1)(u)\big) \\
&\qquad -\rho_1\varphi_1 \circ N_1(u - p_1(u)) - \rho_1\varphi_1 \circ F_1(P_1(u), Q_1(v))\big] \\
(g_2 - p_2)(v) \ &= R^{H_2(A_2,B_2),\varphi_2}_{M_2(\cdot,z_2),\rho_2}\big[H_2\big(A_2(g_2 - p_2)(v), B_2(g_2 - p_2)(v)\big) \\
&\qquad -\rho_2\varphi_2 \circ N_2(v - p_2(v)) - \rho_2\varphi_2 \circ F_2(Q_2(u), P_2(v))\big],
\end{aligned}\right\}$$

It follows from Lemma 3.1, that (u,v) is a solution of SGVLIP (2.1). This completes the proof. □

4 Mann-Type Perturbed Iterative Algorithm, Convergence and Stability Analysis

Lemma 3.1 is very important from the numerical point of view as it allows us to suggest the following Mann-type perturbed iterative algorithm for finding the approximate solution of SGVLIP (2.1).

Iterative Algorithm 4.1. For each $i = 1, 2, j \in \{1, 2\}\setminus i$, given $(u_0, v_0) \in X_1 \times X_2$, where $u_0 \in X_1, v_0 \in X_2$, compute the sequences $\{u_n\}, \{v_n\}$, by the iterative schemes:

$$\begin{aligned} u_{n+1} &= (1-\alpha_n)u_n + \alpha_n\Big\{u_n - (g_1 - p_1)(u_n) + R^{H_1(A_1,B_1),\varphi_1}_{M_1^n(\cdot,z_1^n),\rho_1}\big[H_1\big(A_1(g_1-p_1)(u_n), B_1(g_1-p_1)(u_n)\big) \\ &\quad - \rho_1\varphi_1 \circ N_1(u_n - p_1(u_n)) - \rho_1\varphi_1 \circ F_1(P_1(u_n), Q_1(v_n))\big]\Big\} + \alpha_n e_n, \\ v_{n+1} &= (1-\alpha_n)v_n + \alpha_n\Big\{v_n - (g_2 - p_2)(v_n) + R^{H_2(A_2,B_2),\varphi_2}_{M_2^n(\cdot,z_2^n),\rho_2}\big[H_2\big(A_2(g_2-p_2)(v_n), B_2(g_2-p_2)(v_n)\big) \\ &\quad - \rho_2\varphi_2 \circ N_2(v_n - p_2(v_n)) - \rho_2\varphi_2 \circ F_2(Q_2(u_n), P_2(v_n))\big]\Big\} + \alpha_n e'_n, \end{aligned}$$

where $n = 0, 1, 2, \cdots, \rho_i > 0$ are constants, M_i^n is a $H_i(A_i, B_i)$-φ_i-η_i-accretive mapping and $\{e_n, e'_n\}_{n\geq 0}$ is sequence in $X_1 \times X_2$ introduced to take into account possible inexact computation which satisfies $\lim_{n\to\infty} \|e_n\| = \lim_{n\to\infty} \|e'_n\| = 0$ and $\{\alpha_n\}$ is a sequence of real numbers such that $\alpha_n \in [0, 1]$ and $\sum_{n=0}^{\infty} \alpha_n = +\infty$.

Theorem 4.2. Let all the conditions of Theorem 3.2 hold. For $i \in \{1, 2\}, j \in \{1, 2\} \setminus i$, let $M_i^n : X_i \times X_i \to 2^{X_i}$ be a $H_i(A_i, B_i)$-φ_i-η_i-accretive mapping with respect to A_i and B_i, respectively such that $\varphi_i \circ M_i^n(\cdot, z_i^n) \xrightarrow{HG} \varphi_i \circ M_i(\cdot, z_i)$ as $n \to \infty$ for each $z_i \in X_i$, respectively. Further, suppose $\{(\bar{u}_n, \bar{v}_n)\}_{n\geq 0}$ is a sequence in $X_1 \times X_2$ and define $\epsilon_n = \omega_n + \omega'_n$ for $n \geq 0$ by

$$\epsilon_n = \big\|(\bar{u}_{n+1}, \bar{v}_{n+1}) - (\omega_n, \omega'_n)\big\|_\star,$$

where

$$\begin{aligned} \omega_n &= \Big\|\bar{u}_{n+1} - \Big[(1-\alpha_n)\bar{u}_n + \alpha_n\Big\{\bar{u}_n - (g_1 - p_1)(\bar{u}_n) + R^{H_1(A_1,B_1),\varphi_1}_{M_1^n(\cdot,\bar{z}_1^n),\rho_1}[H_1(A_1(g_1-p_1)(\bar{u}_n), B_1(g_1-p_1)(\bar{u}_n)) \\ &\quad -\rho_1\varphi_1 \circ N_1(\bar{u}_n - p_1(\bar{u}_n)) - \rho_1\varphi_1 \circ F_1(P_1(\bar{u}_n), Q_1(\bar{v}_n))]\Big\} + \alpha_n e_n\Big]\Big\|_1, \\ \omega'_n &= \Big\|\bar{v}_{n+1} - \Big[(1-\alpha_n)\bar{v}_n + \alpha_n\Big\{\bar{v}_n - (g_2 - p_2)(\bar{v}_n) + R^{H_2(A_2,B_2),\varphi_2}_{M_2^n(\cdot,\bar{z}_2^n),\rho_2}[H_2(A_2(g_2-p_2)(\bar{v}_n), B_2(g_2-p_2)(\bar{v}_n)) \\ &\quad -\rho_2\varphi_2 \circ N_2(\bar{v}_n - p_2(\bar{v}_n)) - \rho_2\varphi_2 \circ F_2(Q_2(\bar{u}_n), P_2(\bar{v}_n))]\Big\} + \alpha_n e'_n\Big]\Big\|_2. \end{aligned} \tag{4.1}$$

If there exist positive constants ρ_1, ρ_2 such that (3.2) holds then:

(a) the iterative sequence $\{(u_n, v_n)\}_{n\geq 0}$ generated by Iterative Algorithm 4.1 converges to the solution $\{(u, v)\}$ of SGVLIP (2.1).

(b) For any sequences $\{\bar{u}_n, \bar{v}_n\}_{n\geq 0}$, $\lim_{n\to\infty}(\bar{u}_n, \bar{v}_n) = (u, v)$ if and only if $\lim_{n\to\infty} \epsilon_n = 0$, where $\epsilon_n = \omega_n + \omega'_n$, for all $n \geq 0$.

Proof . By Theorem 3.2, there exists a solution (u, v) of SGVLIP (2.1). From Lemma 3.1, we have

$$\left.\begin{aligned} u &= (1-\alpha_n)u + \alpha_n\Big\{u - (g_1 - p_1)(u) + R^{H_1(A_1,B_1),\varphi_1}_{M_1(\cdot,z_1),\rho_1}\big[H_1(A_1(g_1-p_1)(u), B_1(g_1-p_1(u)) \\ &\quad -\rho_1\varphi_1 \circ N_1(u - p_1(u)) - \rho_1\varphi_1 \circ F_1(P_1(u), Q_1(v))\big]\Big\}, \\ v &= (1-\alpha_n)v + \alpha_n\Big\{v - (g_1 - p_1)(v) + R^{H_2(A_2,B_2),\varphi_2}_{M_2(\cdot,z_2),\rho_2}\big[H_2(A_2(g_2-p_2)(v), B_2(g_2-p_2(v)) \\ &\quad -\rho_2\varphi_2 \circ N_2(v - p_2(v)) - \rho_2\varphi_2 \circ F_2(Q_2(u), P_2(v))\big]\Big\}. \end{aligned}\right\} \tag{4.2}$$

Now, from Algorithm 4.1, (4.2) and using the same arguments used in estimating (3.6)-(3.14), we have

$$\begin{aligned} \big\|u_{n+1} - u\big\|_1 &\leq (1-\alpha_n)\big\|u_n - u\big\|_1 + \alpha_n\big\|(u_n - u) - \big((g_1-p_1)(u_n) - (g_1-p_1)(u)\big)\big\|_1 \\ &\quad + \alpha_n\Big\|R^{H_1(A_1,B_1),\varphi_1}_{M_1^n(\cdot,z_1^n),\rho_1}\big[H_1(A_1(g_1-p_1)(u_n), B_1(g_1-p_1)(u_n)) - \rho_1\varphi_1 \circ N_1(u_n - p_1(u_n)) \\ &\quad - \rho_1\varphi_1 \circ F_1(P_1(u_n), Q_1(v_n))\big] \\ &\quad - R^{H_1(A_1,B_1),\varphi_1}_{M_1(\cdot,z_1),\rho_1}\big[H_1(A_1(g_1-p_1)(u), B_1(g_1-p_1)(u)) - \rho_1\varphi_1 \circ N_1(u - p_1(u)) \\ &\quad - \rho_1\varphi_1 \circ F_1(P_1(u), Q_1(v))\big]\Big\|_1 + \alpha_n\big\|e_n\big\|_1 \end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_n)\big\|u_n-u\big\|_1+\alpha_n\big\|(u_n-u)-\big((g_1-p_1)(u_n)-(g_1-p_1)(u)\big)\big\|_1\\
&\quad+\alpha_n\Big\|R^{H_1(A_1,B_1),\varphi_1}_{M_1^n(\cdot,z_1^n),\rho_1}\big[H_1(A_1(g_1-p_1)(u_n),B_1(g_1-p_1)(u_n))-\rho_1\varphi_1\circ N_1(u_n-p_1(u_n))\\
&\quad-\rho_1\varphi_1\circ F_1(P_1(u_n),Q_1(v_n))\big]\\
&\quad-R^{H_1(A_1,B_1),\varphi_1}_{M_1^n(\cdot,z_1^n),\rho_1}\big[H_1(A_1(g_1-p_1)(u),B_1(g_1-p_1)(u))-\rho_1\varphi_1\circ N_1(u-p_1(u))\\
&\quad-\rho_1\varphi_1\circ F_1(P_1(u),Q_1(v))\big]\Big\|_1\\
&\quad+\alpha_n\Big\|R^{H_1(A_1,B_1),\varphi_1}_{M_1^n(\cdot,z_1^n),\rho_1}\big[H_1(A_1(g_1-p_1)(u),B_1(g_1-p_1)(u))-\rho_1\varphi_1\circ N_1(u-p_1(u))\\
&\quad-\rho_1\varphi_1\circ F_1(P_1(u),Q_1(v))\big]-R^{H_1(A_1,B_1),\varphi_1}_{M_1(\cdot,z_1),\rho_1}\big[H_1(A_1(g_1-p_1)(u),B_1(g_1-p_1)(u))\\
&\quad-\rho_1\varphi_1\circ N_1(u-p_1(u))-\rho_1\varphi_1\circ F_1(P_1(u),Q_1(v))\big]\Big\|_1+\alpha_n\big\|e_n\big\|_1\\
&\leq (1-\alpha_n)\big\|u_n-u\big\|_1+\alpha_n\big[a_1+L_1(b_1+c_1+\rho_1d_1)\big]\big\|u_n-u\big\|_1\\
&\quad+\alpha_nL_1\rho_1l_{F_1}L_{Q_1}\big\|v_n-v\big\|_2+\alpha_nf_n+\alpha_n\big\|e_n\big\|_1\\
&\leq \big[(1-\alpha_n)+\alpha_nm_1\big]\big\|u_n-u\big\|_1+\alpha_nL_1\rho_1l_{F_1}L_{Q_1}\big\|v_n-v\big\|_2+\alpha_nf_n+\alpha_n\big\|e_n\big\|_1,
\end{aligned}
\tag{4.3}
$$

where

$$
\begin{aligned}
f_n=\Big\|&R^{H_1(A_1,B_1),\varphi_1}_{M_1^n(\cdot,z_1^n),\rho_1}\big[H_1(A_1(g_1-p_1)(u),B_1(g_1-p_1)(u))-\rho_1\varphi_1\circ N_1(u-p_1(u))-\rho_1\varphi_1\circ F_1(P_1(u),Q_1(v))\big]\\
&-R^{H_1(A_1,B_1),\varphi_1}_{M_1(\cdot,z_1),\rho_1}\big[H_1(A_1(g_1-p_1)(u),B_1(g_1-p_1)(u))-\rho_1\varphi_1\circ N_1(u-p_1(u))-\rho_1\varphi_1\circ F_1(P_1(u),Q_1(v))\big]\Big\|_1\\
&\longrightarrow 0,\quad \text{as}\ \ n\to\infty.
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
\big\|v_{n+1}-v\big\|_2&\leq (1-\alpha_n)\big\|v_n-v\big\|_2+\alpha_n\big[a_2+L_2(b_2+c_2+\rho_2d_2)\big]\big\|v_n-v\big\|_2\\
&\quad+\alpha_nL_2\rho_2l_{F_2}L_{Q_2}\big\|u_n-u\big\|_1+\alpha_nh_n+\alpha_n\big\|e'_n\big\|_2\\
&\leq \big[(1-\alpha_n)+\alpha_nm_2\big]\big\|v_n-v\big\|_2+\alpha_nL_2\rho_2l_{F_2}L_{Q_2}\big\|u_n-u\big\|_1+\alpha_nh_n+\alpha_n\Big\|e'_n\Big\|_1,
\end{aligned}
\tag{4.4}
$$

where

$$
\begin{aligned}
h_n=\Big\|&R^{H_2(A_2,B_2),\varphi_2}_{M_2^n(\cdot,z_2^n),\rho_2}\big[H_2(A_2(g_2-p_2)(v),B_2(g_2-p_2)(v))-\rho_2\varphi_2\circ N_2(v-p_2(v))-\rho_2\varphi_2\circ F_2(Q_2(u),P_2(v))\big]\\
&-R^{H_2(A_2,B_2),\varphi_2}_{M_2(\cdot,z_2),\rho_2}\big[H_2(A_2(g_2-p_2)(v),B_2(g_2-p_2)(v))-\rho_2\varphi_2\circ N_2(v-p_2(v))-\rho_2\varphi_2\circ F_2(Q_2(u),P_2(v))\big]\Big\|_2\\
&\longrightarrow 0,\quad \text{as}\ \ n\to\infty.
\end{aligned}
$$

It follows from (4.3) and (4.4) that

$$
\begin{aligned}
&\big\|u_{n+1}-u\big\|_1+\big\|v_{n+1}-v\big\|_2\\
&\quad\leq \big[1-\alpha_n(1-m_1-L_2\rho_2l_{F_2}L_{Q_2})\big]\big\|u_n-u\big\|_1+\big[1-\alpha_n(1-m_2-L_1\rho_1l_{F_1}L_{Q_1})\big]\big\|v_n-v\big\|_2\\
&\qquad+\alpha_n\big(f_n+h_n+\|e_n\|_1+\|e'_n\|_2\big)\\
&\quad\leq \big[1-\alpha_n(1-k)\big]\big(\big\|u_n-u\big\|_1+\big\|v_n-v\big\|_2\big)+\alpha_n\big(f_n+h_n+\|e_n\|_1+\|e'_n\|_2\big),
\end{aligned}
\tag{4.5}
$$

where $k=\max\{k_1,k_2\}$,

$$
\begin{aligned}
k_i&=m_i+L_j\rho_jl_{F_j}L_{Q_j}<1,\quad m_i=a_i+L_i(b_i+c_i+\rho_id_i),\\
a_i&=\big[1-q_i\sigma_i+q_i\mu_i\big(1+\tau_i^{q_i-1}\big)+c_{q_i}\mu_i^{q_i}\big]^{1/q_i},\\
b_i&=\big[1-q_i(\alpha_i-\beta_i)\mu_i^{q_i}+q_i(\gamma_i+\delta_i)\big(1+\tau_i^{q_i-1}\big)+c_{q_i}\big((\gamma_iL_{A_i})^{q_i}+(\delta_iL_{B_i})^{q_i}\big)\big]^{1/q_i}\\
c_i&=\big[1-\rho_iq_i\zeta_i+\rho_iq_iL_{P_i}L_{F_i}\big(1+\tau_i^{q_i-1}\big)+\rho_i^{q_i}c_{q_i}L_{F_i}^{q_i}L_{P_i}^{q_i}\big]^{1/q_i},\\
d_i&=L_{N_i}\big[1-q_ir_i+q_is_i(1+\tau_i^{q_i-1})+c_{q_i}s_i^{q_i}\big]^{1/q_i},\quad L_i=\frac{\tau_i^{q_i-1}}{(\alpha_i-\beta_i)}
\end{aligned}
$$

and $f_n, h_n \longrightarrow 0$ as $n \to \infty$.
Now, define the norm $\|\cdot\|_\star$ on $X_1 \times X_2$ by

$$\big\|(u,v)\big\|_\star = \big\|u\big\|_1 + \big\|v\big\|_2, \quad \forall\ (u,v) \in X_1 \times X_2.$$

We observe that $(X_1 \times X_2, \|\cdot\|_\star)$ is a Banach space. Hence, it follows from (4.5) that

$$\begin{aligned}\big\|(u_{n+1}, v_{n+1}) - (u,v)\big\|_\star &\le \big[1 - \alpha_n(1-k)\big]\big\|(u_n, v_n) - (u,v)\big\|_\star \\ &\quad + \alpha_n(1-k)\frac{\big(f_n + h_n + \|e_n\|_1 + \|e'_n\|_2\big)}{(1-k)}, \quad k < 1.\end{aligned}$$

If $a_n = \big\|(u_n, v_n) - (u,v)\big\|_\star$, $b_n = \dfrac{d_n}{(1-k)}$, $d_n = \big\{f_n + h_n + \|e_n\|_1 + \|e'_n\|_2\big\}$ and $c_n = \alpha_n(1-k)$, then we have

$$a_{n+1} \le (1 - c_n)a_n + b_n c_n.$$

Using Lemma 2.14, we have $a_n \to 0$ as $n \to \infty$ (since f_n and h_n both tend to 0 as $n \to \infty$). This implies

$$u_n \to u, \quad v_n \to v \quad \text{as} \quad n \to \infty.$$

Thus, the approximate solution (u_n, v_n) generated by Iterative Algorithm 4.1 converges strongly to the solution (u,v) of SGVLIP (2.1).

To demonstrate (b), proceeding as we obtained, (4.1), (4.2) and (4.5), we deduce that

$$\begin{aligned}\big\|\bar{u}_{n+1} - u\big\|_1 &= \Big\|\bar{u}_{n+1} - \Big[(1-\alpha_n)\bar{u}_n + \alpha_n\Big\{\bar{u}_n - (g_1 - p_1)(\bar{u}_n) \\ &\quad + R^{H_1(A_1,B_1),\varphi_1}_{M_1^n(\cdot,\bar{z}_1^n),\rho_1}\,[H_1(A_1(g_1-p_1)(\bar{u}_n), B_1(g_1-p_1)(\bar{u}_n)) - \rho_1\varphi_1 \circ N_1(\bar{u}_n, p_1(\bar{u}_n)) \\ &\quad - \rho_1\varphi_1 \circ F_1(P_1(\bar{u}_n), Q_1(\bar{v}_n))]\Big\} + \alpha_n e_n\Big]\Big\|_1 + \Big\|\Big[(1-\alpha_n)\bar{u}_n + \alpha_n\Big\{\bar{u}_n - (g_1-p_1)(\bar{u}_n) \\ &\quad + R^{H_1(A_1,B_1),\varphi_1}_{M_1^n(\cdot,\bar{z}_1^n),\rho_1}\Big[H_1(A_1(g_1-p_1)(\bar{u}_n), B_1(g_1-p_1)(\bar{u}_n)) - \rho_1\varphi_1 \circ N_1(\bar{u}_n, p_1(\bar{u}_n)) \\ &\quad - \rho_1\varphi_1 \circ F_1(P_1(\bar{u}_n), Q_1(\bar{v}_n))\Big]\Big\} + \alpha_n e_n\Big] - u\Big\|_1 \\ &\le \omega_n + \Big\|\Big[(1-\alpha_n)\bar{u}_n + \alpha_n\Big\{\bar{u}_n - (g_1-p_1)(\bar{u}_n) \\ &\quad + R^{H_1(A_1,B_1),\varphi_1}_{M_1^n(\cdot,\bar{z}_1^n),\rho_1}\Big[H_1(A_1(g_1-p_1)(\bar{u}_n), B_1(g_1-p_1)(\bar{u}_n)) - \rho_1\varphi_1 \circ N_1(\bar{u}_n, p_1(\bar{u}_n)) \\ &\quad - \rho_1\varphi_1 \circ F_1(P_1(\bar{u}_n), Q_1(\bar{v}_n))]\Big\} + \alpha_n e_n\Big\}\Big] - (1-\alpha_n)u - \alpha_n\Big\{u - (g_1-p_1)(u) \\ &\quad + R^{H_1(A_1,B_1),\varphi_1}_{M_1(\cdot,z_1),\rho_1}\Big[H_1(A_1(g_1-p_1)(u), B_1(g_1-p_1)(u)) - \rho_1\varphi_1 \circ N_1(u, p_1(u)) \\ &\quad - \rho_1\varphi_1 \circ F_1(P_1(u), Q_1(v))]\Big\}\Big\|_1 \\ &\le \omega_n + (1-\alpha_n)\Big\|\bar{u}_n - u\Big\|_1 + \alpha_n\Big\|(\bar{u}_n - u) - \Big((g_1-p_1)(\bar{u}_n) - (g_1-p_1)(u)\Big)\Big\|_1 \\ &\quad + \alpha_n\Big\|R^{H_1(A_1,B_1),\varphi_1}_{M_1^n(\cdot,\bar{z}_1^n),\rho_1}\Big[H_1(A_1(g_1-p_1)(\bar{u}_n), B_1(g_1-p_1)(\bar{u}_n)) - \rho_1\varphi_1 \circ N_1(\bar{u}_n, p_1(\bar{u}_n)) \\ &\quad - \rho_1\varphi_1 \circ F_1(P_1(\bar{u}_n), Q_1(\bar{v}_n))]\Big\} \\ &\quad - R^{H_1(A_1,B_1),\varphi_1}_{M_1(\cdot,z_1),\rho_1}\Big[H_1(A_1(g_1-p_1)(u), B_1(g_1-p_1)(u)) - \rho_1\varphi_1 \circ N_1(u, p_1(u)) \\ &\quad - \rho_1\varphi_1 \circ F_1(P_1(u), Q_1(v))]\Big\}\Big\|_1 + \alpha_n\big\|e_n\big\|_1 \\ &\le \omega_n + \big[(1-\alpha_n) + \alpha_n m_1\big]\big\|\bar{u}_n - u\big\|_1 + \alpha_n L_1 \rho_1 l_{F_1} L_{Q_1}\big\|\bar{v}_n - v\big\|_2 + \alpha_n \bar{f}_n + \alpha_n\big\|e_n\big\|_1, \end{aligned} \tag{4.6}$$

where

$$\begin{aligned}\bar{f}_n &= \Big\|R^{H_1(A_1,B_1),\varphi_1}_{M_1^n(\cdot,\bar{z}_1^n),\rho_1}\,[H_1(A_1(g_1-p_1)(u), B_1(g_1-p_1)(u)) - \rho_1\varphi_1 \circ N_1(u, p_1(u)) - \rho_1\varphi_1 \circ F_1(P_1(u), Q_1(v))] \\ &\quad - R^{H_1(A_1,B_1),\varphi_1}_{M_1(\cdot,z_1),\rho_1}\,[H_1(A_1(g_1-p_1)(u), B_1(g_1-p_1)(u)) - \rho_1\varphi_1 \circ N_1(u, p_1(u)) - \rho_1\varphi_1 \circ F_1(P_1(u), Q_1(v))]\Big\|_1\end{aligned}$$

$$\longrightarrow 0, \quad \text{as} \quad n \to \infty.$$

Similarly,

$$\left\|\bar{v}_{n+1} - v\right\|_2 \ \leq \ \omega_n' + [(1-\alpha_n) + \alpha_n m_2]\left\|\bar{v}_n - v\right\|_2 + \alpha_n L_2 \rho_2 l_{F_2} L_{Q_2}\left\|\bar{u}_n - u\right\|_1 + \alpha_n \bar{h}_n + \alpha_n \|e_n\|_2, \tag{4.7}$$

where

$$\begin{aligned}\bar{h}_n = \Big\| & R^{H_2(A_2,B_2),\varphi_2}_{M_2^n(\cdot,\bar{z}_2^n),\rho_2}\big[H_2(A_2(g_2-p_2)(v), B_2(g_2-p_2)(v)) - \rho_2\varphi_2 \circ N_2(v,p_2(v)) - \rho_2\varphi_2 \circ F_2(Q_2(u),P_2(v))\big] \\ & - R^{H_2(A_2,B_2),\varphi_2}_{M_2(\cdot,z_2),\rho_2}\big[H_2(A_2(g_2-p_2)(v), B_2(g_2-p_2)(v)) - \rho_2\varphi_2 \circ N_2(v,p_2(v)) - \rho_2\varphi_2 \circ F_2(Q_2(u),P_2(v))\big]\Big\|_2 \\ & \longrightarrow 0, \quad \text{as} \quad n \to \infty.\end{aligned}$$

It follows from (4.6) and (4.7) that

$$\begin{aligned}\left\|\bar{u}_{n+1} - u\right\|_1 + \left\|\bar{v}_{n+1} - v\right\|_2 &\leq \epsilon_n + \big[1 - \alpha_n(1-k)\big]\big\{\left\|\bar{u}_n - u\right\|_1 + \left\|\bar{v}_n - v\right\|_2\big\} \\ &\quad + \alpha_n(1-k)\frac{\big(\bar{f}_n + \bar{h}_n + \|e_n\|_1 + \|e_n'\|_2\big)}{(1-k)}.\end{aligned} \tag{4.8}$$

This implies that

$$\left\|(\bar{u}_{n+1},\bar{v}_{n+1}) - (u,v)\right\|_\star \leq \epsilon_n + \big[1-\alpha_n(1-k)\big]\big\{\left\|(\bar{u}_n,\bar{v}_n) - (u,v)\right\|_\star\big\} + \alpha_n(1-k)\frac{\big(\bar{f}_n + \bar{h}_n + \|e_n\|_1 + \|e_n'\|_2\big)}{(1-k)}.$$

Suppose that $\lim\limits_{n\to\infty} \epsilon_n = 0$. Further, if $a_n = \left\|(\bar{u}_n,\bar{v}_n) - (u,v)\right\|_\star$, $b_n = \dfrac{d_n}{(1-k)}$, $d_n = \big\{\bar{f}_n + \bar{h}_n + \|e_n\|_1 + \|e_n'\|_2\big\}$ and $c_n = \alpha_n(1-k)$, then we have

$$a_{n+1} \leq (1-c_n)a_n + b_n c_n.$$

Using Lemma 2.14, we have $a_n \to 0$ as $n \to \infty$. This implies $\bar{u}_n \to u$, $\bar{v}_n \to v$ as $n \to \infty$.

Conversely suppose that $\lim\limits_{n\to\infty} (\bar{u}_n,\bar{v}_n) = (u,v)$. Then

$$\begin{aligned}\epsilon_n &= \left\|(\bar{u}_{n+1},\bar{v}_{n+1}) - (u,v)\right\|_\star + \left\|(\omega_n,\omega_n') - (u,v)\right\|_\star \\ &\leq \left\|(\bar{u}_{n+1},\bar{v}_{n+1}) - (u,v)\right\|_\star + \big[1-\alpha_n(1-k)\big]\Big(\left\|\bar{u}_n - u\right\|_1 + \left\|\bar{v}_n - v\right\|_2\Big) \\ &\quad + \alpha_n(1-k)\frac{\big(\bar{f}_n + \bar{h}_n + \|e_n\|_1 + \|e_n'\|_2\big)}{(1-k)} \quad \longrightarrow 0, \quad \text{as} \quad n \to \infty.\end{aligned}$$

Therefore, we have $\lim\limits_{n\to\infty} \epsilon_n = 0$. This completes the proof. □

Remark 4.3. Theorems 3.2 and 4.2 extend, improve and unify many results in the literature, see for example [1]-[3],[10]-[14],[16],[17],[20],[21]. The class of $H(\cdot,\cdot)$-φ-η-accretive operator is much wider and more general than those of (A,η)-accretive operator, (H,η)-monotone operator as already discussed by many researchers in the literature.

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