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A new fractional derivative operator and applications

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Abstract

We introduce a new fractional derivative which obeys classical properties including linearity, product rule, power rule, vanishing derivatives for constant functions, chain rule, quotient rule, Rolle's Theorem and the Mean Value Theorem:

$$D^{\alpha}(f)(t) = \lim_{\epsilon \to 0} \frac{f\left(te^{\frac{1}{\Gamma(1-\alpha)}}e^{-\alpha}\right) - f(t)}{\epsilon},$$

this definition is comfortable with the classical definition of the Caputo Fractional Operator.

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1 Introduction

The derivative of non-integer order has been an interesting subject of research for several centuries. The idea of fractional calculus is as old as traditional calculus. The history of fractional calculus dates back to more than 250 years ago, and the original question which led to the name fractional calculus was: what does $\frac{d^n f}{dx^n}$ mean if $n = \frac{1}{2}$. Since then, several mathematicians contributed to the development of fractional calculus: Riemann, Liouville, Caputo, Grunwald, Letnikov, etc. (see [1], [5]). The authors in [3] and in [5] define new well-behaved simple fractional derivatives called the conformable fractional derivative depending just on the basic limit definition of the derivative. A. Kajouni introduced in [4] the new derivative which is defined by:

$$\left(D^{\alpha} f\right)(t) = \lim_{h \to 0} \frac{f\left(t + he^{(\alpha-1)t}\right) - f(t)}{h}$$

for $t > 0, \alpha \in (0, 1)$. If f is α differentiable in some $(0, a), a > 0$, and $\lim_{t \to 0^+} D^{\alpha}(f)(t)$ exists, then define:

$$D^{\alpha}(f)(0) = \lim_{t \to 0^+} D^{\alpha}(f)(t).$$

Khalil et al. [3] have introduced a new derivative called the conformable fractional derivative of f of order α and is defined by

$$T_{\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon},$$

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for a function $f : [0, +\infty) \longrightarrow \mathbb{R}$ and $0 < \alpha \leq 1$, and the fractional derivative at 0 is defined as $f^{(\alpha)}(0) = \lim_{t\to 0} +T_\alpha(f)(t)$. As a consequence of the above definitions, the authors in [3],[5] showed that the α derivatives obey the product rule and quotient rule and have results similar to Rolle's theorem and the mean value theorem in classical calculus. However, the following are some of the setbacks of the other definitions of D^α:

1) Most of the fractional derivatives except Caputo-type, do not satisfy $D^\alpha(1) = 0$, if α is not a natural number.

2) All fractional derivatives do not satisfy the familiar Product Rule for two functions $D^\alpha(fh) = hD^\alpha(f) + fD^\alpha(h)$.

3) The fractional derivatives do not have a corresponding "calculus".

4) All fractional derivatives do not satisfy the Chain Rule for composite functions $D^\alpha(f \circ h)(t) = D^\alpha(f(h))D^\alpha h(t)$.

5) All fractional derivatives do not satisfy the familiar Quotient Rule for two functions $D^\alpha(f) = \frac{hD^\alpha(f) - fD^\alpha(h)}{h^2}$ with $h \neq 0$.

6) All fractional derivatives do not satisfy the Indices Rule $D^\alpha D^\beta(f) = D^{\alpha+\beta}(f)$.

The object of this paper is to present a new, yet an easy definition of fractional derivative. The new definition seems to be a natural extension of the usual derivative, and it satisfies the properties mentioned above. Our definition coincides with the known fractional derivatives with the classical definition of the Caputo Fractional Operator.

2 New Fractional Derivative

Definition 2.1. Given a function $f : [0, \infty) \to \mathbb{R}$ and then the conformable fractional derivative of f order is defined by :

$$^G D^\alpha f(t) = \lim_{\epsilon\to 0} \frac{f^{(n)}\left(te^{\frac{1}{\Gamma(1-\alpha)}\epsilon t^{n-\alpha}}\right) - f^{(n)}(t)}{\epsilon}$$

Proposition 2.2. If a function $f : [0, \infty) \to \mathbb{R}$ is α-differentiable at $t > 0$ then :

$$^G D^\alpha f(t) = \frac{1}{\Gamma_{(1-\alpha)}} t^{1-\alpha} \frac{df(t)}{dt}$$

Proof .

$$^G D^\alpha(f)(t) = \lim_{\epsilon\to 0} \frac{f\left(te^{\frac{1}{\Gamma(1-\alpha)^{\epsilon t}}\epsilon t^{-\alpha}}\right) - f(t)}{\epsilon} = \lim_{\epsilon\to 0} \frac{f\left(t\left[1 + \frac{1}{\Gamma(1-\alpha)}\epsilon t^{-\alpha} + o(\epsilon)\right]\right) - f(t)}{\epsilon}$$

$$= \lim_{\epsilon\to 0} \frac{f\left(t + \frac{1}{\Gamma(1-\alpha)}\epsilon t^{1-\alpha} + o(\epsilon)\right) - f(t)}{\epsilon} = \lim_{\epsilon\to 0} \frac{f(t+h) - f(t)}{h} \times \frac{\frac{1}{\Gamma(1-\alpha)}\epsilon t^{1-\alpha} + o(\epsilon)}{\epsilon}$$

with $\quad h = \dfrac{1}{\Gamma(1-\alpha)}\epsilon t^{1-\alpha} \quad \epsilon \to 0 \Rightarrow h \to 0$

So the result: $\quad D^G f(t) = \dfrac{1}{\Gamma_{(1-\alpha)}} t^{1-\alpha} \dfrac{df(t)}{dt}$

Definition 2.3. If f is differentiable at a, so we define:

$$^G D^\alpha f(a) = \lim_{t\to a} {}^G D^\alpha f(t).$$

Theorem 2.4. If a function $f : [0, +\infty) \longrightarrow \mathbb{R}$ and α differentiable at $t_0 > 0$, then f is continuous at t_0.

Proof . Since $f\left(t_0 e^{\frac{1}{\Gamma(1-\alpha)}\epsilon t_0^{-\alpha}}\right) - f(t_0) = \frac{f\left(t_0 e^{\frac{1}{\Gamma(1-\alpha)}\epsilon t_0^{-\alpha}}\right) - f(t_0)}{\epsilon} \times \epsilon \longrightarrow {}^G D^\alpha f(t_0) \times 0 = 0$,

$$\lim_{\epsilon\to 0} f\left(te^{\frac{1}{\Gamma(1-\alpha)}}\epsilon_0^{-\alpha}\right) - f(t_0) = \lim_{\epsilon\to 0} f\left(t_0 + \frac{1}{\Gamma(1-\alpha)}\epsilon t_0^{1-\alpha} + o(\epsilon)\right) - f(t_0).$$

Let $h = \frac{1}{\Gamma(1-\alpha)}\epsilon t_0^{1-\alpha}$. Then, $\lim_{h\to 0}[f(t_0 + h) - f(t_0)] = 0$, which implies that $\lim_{h\to 0} f(t_0 + \varepsilon) = f(t_0)$. Hence, f is continuous at t_0. $\square$

Theorem 2.5. Let $0<\alpha\leq 1$ and h,g be α differentiable at a point $t>0$. Then,
(1) ${}^{G}D^{\alpha}(ah+bg)=a\left({}^{G}D^{\alpha}h\right)+b\left({}^{G}D^{\alpha}g\right)$, for all $a,b\in\mathbb{R}$.
(2) ${}^{G}D^{\alpha}(\lambda)=0$, for all constant functions $f(t)=\lambda$.
(3) ${}^{G}D^{\alpha}(hg)=h\left({}^{G}D^{\alpha}g\right)+g\left({}^{G}D^{\alpha}h\right)$.
(4) $\left({}^{G}D^{\alpha}(h/g)\right)=\left(\left({}^{G}D^{\alpha}h\right)g-h\left({}^{G}D^{\alpha}g\right)\right)/g^{2}$.
(5)${}^{G}D^{\alpha}(h\circ g)(t)={}^{G}D^{\alpha}(h)(g(t)){}^{G}D^{\alpha}(g)(t)$.

Proof . Parts (1) through (3) follow directly from the definition. We choose to prove (4) and (5) only since they are crucial. Now, for fixed $t>0$:

$$\begin{aligned}
GD^{\alpha}\left(\frac{h}{g}\right)&=\lim_{\epsilon\to 0}\frac{h\left(te^{\frac{1}{\Gamma(1-\alpha)}\epsilon t^{-\alpha}}\right)}{g\left(te^{\frac{1}{\Gamma(1-\alpha)}\epsilon t^{-\alpha}}\right)}-\frac{h(t)}{g(t)}\\
&=\lim_{\epsilon\to 0}\frac{\frac{h\left(te^{\frac{1}{\Gamma(1-\alpha)}\epsilon t^{-\alpha}}\right)g(t)-g\left(te^{\frac{1}{\Gamma(1-\alpha)}\epsilon t^{-\alpha}}\right)h(t)}{g\left(te^{\frac{1}{\Gamma(1-\alpha)}\epsilon t^{-\alpha}}\right)g(t)}}{\epsilon}\\
&=\lim_{\epsilon\to 0}\frac{g(t)\left(h\left(te^{\frac{1}{\Gamma(1-\alpha)}\epsilon t^{-\alpha}}\right)-h(t)\right)-h(t)\left(g\left(te^{\frac{1}{\Gamma(1-\alpha)}\epsilon t^{-\alpha}}\right)-g(t)\right)}{g\left(te^{\frac{1}{\Gamma(1-\alpha)}\epsilon t^{-\alpha}}\right)g(t)\epsilon}\\
&=\lim_{\epsilon\to 0}\frac{1}{g\left(te^{\frac{1}{\Gamma(1-\alpha)}\epsilon t^{-\alpha}}\right)}\frac{\left(h\left(te^{\frac{1}{\Gamma(1-\alpha)}\epsilon t^{-\alpha}}\right)-h(t)\right)}{\epsilon}-\lim_{\epsilon\to 0}\frac{h(t)\left(g\left(te^{\frac{1}{\Gamma(1-\alpha)}\epsilon t^{-\alpha}}\right)-g(t)\right)}{g\left(te^{\frac{1}{\Gamma(1-\alpha)}\epsilon t^{-\alpha}}\right)g(t)\epsilon}\\
&=\frac{1}{g(t)}{}^{G}D^{\alpha}(h)(t)-\frac{h(t)}{g^{2}(t)}D^{G}(g)(t)=\frac{{}^{G}D^{\alpha}(h)(t)g(t)-h(t){}^{G}D^{\alpha}g(t)}{g^{2}(t)}.
\end{aligned}$$

Next, we prove (5). For this, suppose $u=g(t)$ is $\alpha-$ differentiable at a point a and $y=h(u)$ is α - differentiable at a point $b=g(a)$. Let $\Delta y=h\left(ae^{\frac{1}{\Gamma(1-\alpha)}\epsilon a^{-\alpha}}\right)-h(a)$.

Since $ae^{\epsilon a^{-\alpha}}=a+\epsilon a^{1-\alpha}+O\left(\epsilon^{2}\right)$, we have :

$$\Delta y=\mathcal{D}^{\alpha}f(b)\Delta u+\epsilon_{1}\Delta u$$

where $\epsilon_1\to 0$ as $\Delta u\to 0$. Thus, ϵ_1 is a continuous function of Δu if we define ϵ_1 to be 0 when $\Delta u=0$. Now, if Δt is an increment in t and Δu and Δy (with the possibility of both being equal to 0) are corresponding increments in u and y,respectively. We can then write, using the previous equation :

$$\Delta u=\mathcal{D}^{\alpha}g(a)\Delta t+\epsilon_{2}\Delta t$$

where $\epsilon_2\to 0$ as $\Delta t\to 0$, and

$$\begin{aligned}
\Delta y&=\left[\mathcal{D}^{\alpha}h(b)+\epsilon_{1}\right]\Delta u\\
&=\left[\mathcal{D}^{\alpha}h(b)+\epsilon_{1}\right]\cdot\left[\mathcal{D}^{\alpha}g(a)+\epsilon_{2}\right]\Delta t
\end{aligned}$$

where both $\epsilon_1\to 0$ and $\epsilon_2\to 0$ as $\Delta t\to 0$. Taking $\Delta t=\epsilon$, we now have,

$$\begin{aligned}
\mathcal{D}^{\alpha}(h\circ g)(t)&=\lim_{\epsilon\to 0}\frac{\Delta y}{\epsilon}\\
&=\lim_{\Delta t\to 0}\left[\mathcal{D}^{\alpha}h(b)+\epsilon_{1}\right]\cdot\left[\mathcal{D}^{\alpha}g(a)+\epsilon_{2}\right]\\
&=\mathcal{D}^{\alpha}h(b)\mathcal{D}^{\alpha}g(a)=\mathcal{D}^{\alpha}h(g(a))\mathcal{D}^{\alpha}g(a).
\end{aligned}$$

So. the proof is complete. □

Theorem 2.6. Let $a,n\in R,t\in R$ and $\alpha\in(0,1]$. Then, we have the following results:
(1) $T_{\alpha}\left(t^{p}\right)=\frac{p}{\Gamma(1-\alpha)}t^{p-\alpha}$ for all $p\in\mathbb{R}$.
(2) $T_{\alpha}(1)=0$.

(3) $T_\alpha\left(e^{ct}\right)=\frac{ct^{1-\alpha}}{\Gamma(1-\alpha)}e^{ct}, c\in\mathbb{R}$.
(4) $T_\alpha(\sin bt)=\frac{bx^{1-\alpha}}{\Gamma(1-\alpha)}\cos bt, b\in\mathbb{R}$.
(5) $T_\alpha(\cos bt)=\frac{-bt^{1-\alpha}}{\Gamma(1-\alpha)}\sin bt, b\in\mathbb{R}$.
(6) $T_\alpha\left(\frac{1}{\alpha}t^\alpha\right)=\frac{1}{\Gamma(1-\alpha)}$.
(7) $T_\alpha\left(\sin\frac{1}{\alpha}t^\alpha\right)=\frac{1}{\Gamma(1-\alpha)}\cos\frac{1}{\alpha}t^\alpha$.
(8) $T_\alpha\left(\cos\frac{1}{\alpha}t^\alpha\right)=\frac{-1}{\Gamma(1-\alpha)}\sin\frac{1}{\alpha}t^\alpha$.
(9) $T_\alpha\left(e^{\frac{1}{\alpha}t^\alpha}\right)=\frac{1}{\Gamma(1-\alpha)}e^{\frac{1}{\alpha}t^\alpha}$.

Note: One should notice that a function could be α-differentiable at a point but not differentiable, for example, take $f(t)=2\sqrt{t}$. Then $T_{\frac{1}{2}}(f)(0)=\lim_{t\to 0^+}T_{\frac{1}{2}}(f)(t)=1$, where $T_{\frac{1}{2}}(f)(t)=1$, for $t>0$. But $T_1(f)(0)$ does not exist. This is not the case for the known classical fractional derivatives.

We have verified several results in the case where $\alpha\in(0,1)$. But, what if $\alpha\in(n,n+1]$ for a natural number n? What would be the definition?

Definition 2.7. For $\alpha\in(n,n+1]$, for some $n\in\mathbb{N}$, and f function be an n differentiable at $t>0$. Then the α fractional derivative of f is defined by:

$$\mathcal{D}^\alpha f(t)=\lim_{\epsilon\to 0}\frac{f^{(n)}\left(te^{\frac{1}{\Gamma(1-\alpha)}\epsilon t^{n-\alpha}}\right)-f^{(n)}(t)}{\epsilon}$$

if the limit exists.

Remark 2.8. As a direct consequence of Definition 2.7, we can show that;

$$\left(D^\alpha f\right)(t)=\frac{1}{\Gamma(1-\alpha)}t^{\alpha-n}f^{(n+1)}(t),$$

where $\alpha\in(n,n+1]$ and f is $(n+1)$ differentiable at $t>0$.

Remark 2.9. The preceding definitions of the Riemann-Liouville and Caputo fractional derivative do not allow studying the analysis of the differentiable functions α. However, our definition allows to prove the basic theorems of the analysis such as Rolle's theorem and the mean value theorem.

Theorem 2.10. (Rolle's Theorem). Let $a>0$ and $f:[a,b]\to\mathbb{R}$ be a given function that satisfies:
(i) f is continuous on $[a,b]$,
(ii) f is α-differentiable for some $\alpha\in(0,1)$,
(iii) $f(a)=f(b)$.

Then, there exists $c\in(a,b)$, such that ${}^{G}D^\alpha(f)(c)=0$.

Proof . Since f is continuous on $[a,b]$, and $f(a)=f(b)$, there exists $c\in(a,b)$, which is a point of local extrema. Without loss of the generality, assume c is a point of local minimum. So But, the first limit is non-negative, and the second limit is non-positive. Hence: ${}^{G}D^\alpha(f)(c)=0$. □

Theorem 2.11. (Mean Value Theorem). Let $a>0$ and $f:[a,b]\to\mathbb{R}$ be a function with the properties that:
(1) f is continuous on $[a,b]$,
(2) f is α-differentiable on (a,b) for some $\alpha\in(0,1)$,
Then, there exists $c\in(a,b)$, such that $\mathcal{D}^\alpha(f)(c)=T_\alpha\left(\frac{1}{\alpha}t^\alpha\right)=\frac{1}{\Gamma(1-\alpha)}\frac{f(b)-f(a)}{\frac{1}{\alpha}b^\alpha-\frac{1}{a}a^\alpha}$.

Proof . Consider the function,

$$g(x)=f(x)-f(a)-\frac{f(b)-f(a)}{\frac{1}{\alpha}b^\alpha-\frac{1}{\alpha}a^\alpha}\left(\frac{1}{\alpha}x^\alpha-\frac{1}{\alpha}a^\alpha\right).$$

Then, the function g satisfies the conditions of the fractional Rolle's theorem. Hence, there exists $c \in (a, b)$, such that $\mathcal{D}^{\alpha}(g)(c) = 0$. Using the fact that $T_{\alpha}\left(\frac{1}{\alpha}t^{\alpha}\right) = \frac{1}{\Gamma(1-\alpha)}$, the result follows. $\square$

Along the same lines in basic analysis, one can use the present mean value theorem to prove the following proposition.

Proposition 2.12. Let $f : [a, b] \longrightarrow \mathbb{R}$ be α differentiable for some $\alpha \in]0, 1[$.
(i) If $f^{(a)}$ is bounded on $[a, b]$, where $a > 0$, then f is uniformly continuous on $[a, b]$, and hence, f is bounded.
(ii) If $f^{(a)}$ is bounded on $[a, b]$ and continuous at a, then f is uniformly continuous on $[a, b]$, and hence, f is bounded.

3 Fractional Integral

It is interesting to note that despite the variation in the definitions of fractional derivatives, we can always adapt the same definition of the fractional integral here from the fact that we got similar results in the Fractional Derivative operator, Thus we have the following definition.

Definition 3.1. (Fractional Integral). Let $a \geq 0$ and $t \geq a$. Also, let f be a function defined on $(a, t]$ and $\alpha \in \mathbb{R}$. Then, the α-fractional integral of f is defined by,

$$\mathcal{I}_{a}^{\alpha}(f)(t) = \Gamma(1-\alpha) \int_{a}^{t} \frac{f(x)}{x^{1-\alpha}} dx$$

if the Riemann improper integral exists. It is interesting to observe that the α-fractionalderivative and the α-fractional integral are inverse of each other as given in the next result.

Theorem 3.2. (Inverse property). Let $a \geq 0$, and $\alpha \in (0, 1)$. Also, let f be a continuous function such that $\mathcal{I}_{a}^{\alpha} f$ exists. Then ${}^{G}D^{\alpha}\left(\mathcal{I}_{a}^{\alpha}(f)\right)(t) = f(t), \quad$ for $t \geq a$

Proof . The proof is a direct consequence of the fundamental theorem of calculus. Since f is continuous, $\mathcal{I}_{a}^{\alpha} f$ is clearly differentiable. Therefore, using Proposition 1, we have :

$$\begin{aligned}
{}^{G}D^{\alpha}\left(\mathcal{I}_{a}^{\alpha}(f)\right)(t) &= \frac{t^{\alpha-1}}{\Gamma_{(1-\alpha)}} \frac{d}{dt} \mathcal{I}_{a}^{\alpha}(f)(t), \\
&= \frac{t^{\alpha-1}}{\Gamma_{(1-\alpha)}} \Gamma(1-\alpha) \frac{d}{dt} \int_{a}^{t} \frac{f(x)}{x^{1-\alpha}} dx \\
&= t^{\alpha-1} \frac{f(t)}{t^{1-\alpha}} \\
&= f(t).
\end{aligned}$$

This completes the proof. $\square$

4 Application

The new definition of fractional derivative facilitate the calculations performed to solve the followings differential equations:

Example 1.

$$y^{(1/2)} = -yt^{\frac{1}{2}} - \frac{t^{\frac{3}{2}}}{y}$$

Let us look for a differentiable solution y which verifies this equation. Since ${}^{G}D^{\alpha}y(t) = \frac{1}{\Gamma_{(1-\alpha)}} t^{1-\alpha} \frac{dy(t)}{dt}$, ${}^{G}D^{\frac{1}{2}}y(t) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} t^{\frac{1}{2}} y'(t)$. Thus, the fractional differential equation becomes

$$\frac{1}{\Gamma\left(\frac{1}{2}\right)} t^{\frac{1}{2}} y'(t) = \frac{-y^{2} t^{\frac{1}{2}} - t^{\frac{3}{2}}}{y}.$$

Then we return to a classical equation
$$\frac{1}{\Gamma\left(\frac{1}{2}\right)}y' + y = -\frac{t}{y}.$$

Example 2. Consider now the fractional differential equation:
$$y^{(1/2)} = \frac{y^2 + t^2 y + 2t^4}{t^3 e^t}.$$

As before, the fractional differential equation gives
$$\frac{1}{\Gamma\left(\frac{1}{2}\right)} t^{\frac{1}{2}} y'(t) = \frac{y^2 + t^2 y + 2t^4}{t^3 e^t}.$$

So
$$\frac{e^t}{\Gamma\left(\frac{1}{2}\right)} t^{\frac{1}{2}} y'(t) - \frac{y}{t} = \frac{y^2}{t^3} + 2t$$
and this is a differential equation of Riccati.

Example 3. Consider the following example:
$$y^{(1/2)} + t^{1/2} y = t e^{-t}.$$

We obtain $e^t y^{(1/2)} + t^{t/2} e^t y = t$ and take advantage of the product rule for this fractional derivative (which is not possible with Caputo fractional derivatives)
$$\left(e^t y\right)^{(1/2)} = t + \left(1 - \frac{1}{\Gamma\left(\frac{1}{2}\right)}\right) t^{t/2} e^t y.$$

By using the variable changing $z(t) = e^t y(t)$, similar to Example 1, also we obtained a differential equation of Riccati.

5 Conclusion

In this paper, we have defined an interesting type of fractional derivative operator comfortable with the classical definition of the Caputo Fractional Operator. Further, we have investigated some important properties of the new fractional derivative operator. As an application and justification for our new operator, we illustrate some applications. Using the new fractional operator gives more advantages in fractional calculus, especially in fractional differential equations to describe the systems being studied.

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