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Existence of mild solutions for fractional Schrödinger equations in extended Colombeau algebras

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Abstract

The main crux of this research manuscript is to study the existence and uniqueness of generalized mild solutions for nonlinear Schrödinger equations with singular initial conditions in the extended algebras of generalized functions. The proofs are based on generalized semigroups theory and Grönwall's inequality. As an application, our theoretical results have been illustrated by providing a suitable example.

Keywords: Extended Colombeau algebra, generalized mild solution, generalized function, Schrödinger equation
2020 MSC: 46F10, 46S10

1 Introduction

The wave function of a quantum mechanical system is determined by the Schrödinger equation, which is a partial differential equation. This is a crucial result of quantum mechanics, and its discovery marked an important milestone in the field's development. The Schrödinger equation is the quantum analogue of Newton's second law in conventional physics in terms of idea. When given a set of known initial conditions, Newton's second law offers a mathematical prediction about the path that a given physical system will take over time. In quantum physics, the Schrödinger equation describes the evolution of a wave function over time and is used to characterize an isolated physical system. The equation can be derived from the fact that the time evolution operator has to be unitary, and therefore be generated by the exponential of a self-adjoint operator, which is the quantum hamiltonian. In the beginning of 1980s, Colombeau introduced the algebra of generalized functions $\mathcal{G}$ to handle multiplication distribution problem see [2] and [1]. This algebra is a differential on an inclusive space Schwartz distribution of $\mathcal{D}'$. Moreover, in the algebra $\mathcal{G}$ non-linear operations are more general than multiplication. Therefore, this algebra is more convenient for finding and studying the solutions of nonlinear differential equations with singular data and coefficients. This algebra plays a crucial role in giving the multiplication of the distributions [3] and [10]. As a nonlinear extension of distribution theory to deal with non-linearities and singularities of data and coefficients in the theory of PDEs [10]. This algebra include the space of distributions $\mathcal{D}'$ as a subspace with an embedding realized through convolution with a suitable mollifier. The elements of $\mathcal{G}$ are classes of smooth functions called moderate functions with respect to a set of negligible functions. The reason for introducing this regularity is the possibility to solve nonlinear problems with singularities and derivatives of arbitrary real order.

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In this paper, we investigate the existence and uniqueness of the solution to the Cauchy problem given by:

$$\begin{cases} D^{\alpha}x(t) + Ax(t) = F(t, x(t)), \quad 0 < \alpha < 1, \ t \in [0, \Lambda] \\ \\ x(0) = x_0, \end{cases} \tag{1.1}$$

in the setting of Colombeau algebras, where $x_0 \in \widetilde{\mathbb{R}}$ is a generalized real algebra Banach space, $-A$ be the infinitesimal generator of an analytic generalized semigroup $(T(t))_{t\geq 0}$ of uniformly bounded linear operators on a class of Colombeau algebra. Furthermore, regarding works on the Colombeau semigroup , we refer to [8] and the references therein. Our idea is inspired by the one presented in [14] where the author proved the existence of the Cuachy problem (1.1) under two assumptions concerning the infinitesimal generator $-A$ of an analytic semigroup, in this work we have shown without making any conditions on the generator $-A$, the problem (1.1) has a unique solution in the extension $\mathcal{G}^e(\mathbb{R})^n$ of the Colombeau algebra, then we will apply the results of this work to the fractional problem related with the Schrödinger equation.

$$\begin{cases} \frac{1}{i}\partial_t^{\alpha} u(t, x) - \triangle u(t, x) + v(x)u(t, x) = 0, \\ \\ v(x) = \delta(x), \ u(0, x) = \delta(x), \end{cases} \tag{1.2}$$

where $0 < \alpha < 1$ and δ is the Dirac distribution. The organization of this paper is as follows. In section 2, we recall some fundamental properties of the generalized functions theory. The new notion of generalized semigroup takes place in section 3. Section 4 is consecrated to the proof of existence and uniqueness in Colombeau algebra to the problem given in (1.1). In Section 5 we have introduced an example to illustrate our work.

2 Preliminaries

In this section, we recall some fundamental properties of generalized functions theory in colombeau sense. The regularization methods of Colombeau-type is to model non-smooth objects by approximating nets of any smooth functions, which has a moderate asymptotic bounds and to identify regularizing nets whose differences compared to the moderateness scale are negligible. The equivalence classes of regularization moderates nets with respect a negligible nets are called elements of colombeau generalized functions i.e., sequences of smooth functions satisfying the conditions of asymptotically in the regularization parameter ε. Let $n \in \mathbb{N}^*$, as in [3]. We define the set

$$\mathcal{E}(\mathbb{R}^n) = \left(\mathcal{C}^{\infty}(\mathbb{R}^n)\right)^{(0,1)}.$$

- $\mathcal{E}_M(\Omega)$: Moderate families defined by

$$\forall K \Subset \Omega, \forall \alpha \in \mathbb{N}_0^n, \exists p \geq 0 : \sup_{x\in K} \|\partial^{\alpha} u_{\varepsilon}(x)\| = O_{\varepsilon\to 0}(\varepsilon^{-p}). \tag{2.1}$$

- $\mathcal{E}_M(\Omega)$: Null families defned by

$$\forall K \Subset \Omega, \forall \alpha \in \mathbb{N}_0^n, \forall q \geq 0 : \sup_{x\in K} \|\partial^{\alpha} u_{\varepsilon}(x)\| = O_{\varepsilon\to 0}(\varepsilon^{q}). \tag{2.2}$$

With the following operations $(u_\varepsilon) + (v_\varepsilon) = (u_\varepsilon + v_\varepsilon)$ and $(u_\varepsilon)_\varepsilon \times (v_\varepsilon)_\varepsilon = (u_\varepsilon \times v_\varepsilon)_\varepsilon$. The Colombeau algebra is defined as a factor set

$$\mathcal{G}(\mathbb{R}^n) = \mathcal{E}_M(\mathbb{R}^n)/\mathcal{N}(\mathbb{R}^n).$$

The ring of all generalized real numbers is given by the following set

$$\widetilde{\mathbb{R}} = \mathcal{E}(\mathbb{R})/\mathcal{I}(\mathbb{R}),$$

where

$$\mathcal{E}(\mathbb{R}) = \Big\{(x_\varepsilon)_\varepsilon \in (\mathbb{R})^{(0,1)} / \exists m \in \mathbb{N}, |\ x_\varepsilon| \ = \mathcal{O}_{\varepsilon\to 0}(\varepsilon^{-m})\Big\},$$

and

$$\mathcal{I}(\mathbb{R}) = \Big\{(x_\varepsilon)_\varepsilon \in (\mathbb{R})^{(0,1)} / \forall m \in \mathbb{N}, |\ x_\varepsilon| \ = \mathcal{O}_{\varepsilon\to 0}(\varepsilon^{m})\Big\}.$$

We note that $\widetilde{\mathbb{R}}$ is a ring obtained by factoring moderate families of real numbers with respect to negligible families. The space $\mathcal{E}(\mathbb{R})$ is an algebra, and $\mathcal{I}(\mathbb{R})$ is an ideal of $\mathcal{E}(\mathbb{R})$. Note that $\widetilde{\mathbb{R}}^n \subset \mathcal{G}(\mathbb{R}^n)$. The extended Colombeau algebras of generalized functions $\mathcal{G}^e(\Omega)$ on an open subset Ω of $\mathbb{R}^n$ are defined in the sense of extending entire derivatives to fractional ones, which were first introduced by M. Stojanovic see [13] for more details.
Let $\mathcal{E}(\Omega)$ be the algebra of all nets $(u_\varepsilon)_{\varepsilon>0}$ of real valued functions $u_\varepsilon \in \mathcal{C}^\infty(\Omega)$, the algebra of extended moderate functions is given by

$$\mathcal{E}^e_M(\Omega) = \Big\{(u_\varepsilon)_{\varepsilon>0} \in (\mathcal{E}(\Omega))^{(0,1)} : \forall K \subset\subset \mathbb{R}, \forall\alpha \in R_+ \cup \{0\}, \exists N \in \mathbb{N} \text{ such that } \sup_{x\in K} |\ D^\alpha u_\varepsilon(x)|\ = \mathcal{O}(\varepsilon^{-N}),\ \text{as } \varepsilon \to 0\Big\},$$

and the set of negligible functions is defined this time by

$$\mathcal{N}^e(\Omega) = \Big\{(u_\varepsilon)_{\varepsilon>0} \in (\mathcal{E}(\Omega))^{(0,1)} :\ \forall K \subset\subset \mathbb{R}, \forall\alpha \in R_+ \cup \{0\}, \forall q \in \mathbb{N} \text{ such that } \sup_{x\in K} |\ D^\alpha u_\varepsilon(x)|\ = \mathcal{O}(\varepsilon^{q}),\ \text{as } \varepsilon \to 0\Big\}.$$

Here $D^\alpha,\ m-1 < \alpha < m,\ m \in \mathbb{N}^*$ is the Caputo fractional derivative, for the fractional derivatives and fractional integral we can see [11], [12], [6] and the references therein. $\mathcal{G}^e(\Omega)$ is given by the factor algebras $\mathcal{G}^e(\Omega) = \mathcal{E}^e_M(\Omega)/\mathcal{N}^e(\Omega)$. In [4], a generalized solution to the system of equations (9) is constructed in the context of Colombeau algebras for tempering generalized functions, $\mathcal{G}\tau(\mathbb{R}^n)$ which was firstly introduced by J.F. Colombeau to develop the theory of Fourier transform in the algebra of generalized functions. We start by defining

$$\mathcal{O}_M(\mathbb{R}^n) = \Big\{f \in \mathcal{C}^\infty(\mathbb{R}^n) : \forall\alpha \in \mathbb{N}^n_0,\ \exists N \in \mathbb{N}\ \ : \sup_{x\in\mathbb{R}^n} < x >^{-N} |\ \partial^\alpha f(x)\ | < \infty\Big\},$$

where $< x >^{-N} = (1+ \parallel x \parallel)^N$. The Colombeau algebra of tempered generalized functions is given by

$$\mathcal{G}\tau(\mathbb{R}^n) = \mathcal{E}^e_\tau(\mathbb{R}^n)/\mathcal{N}^e_\tau(\mathbb{R}^n),$$

with

$$\mathcal{E}^e_\tau(\mathbb{R}^n) = \Big\{(u_\varepsilon)_{\varepsilon>0} \in (\mathcal{O}_M(\mathbb{R}^n))^I : \forall\alpha \in R_+ \cup \{0\}, \exists N \geq 0 \text{ such that } \sup_{x\in\mathbb{R}^n} |\ D^\alpha u_\varepsilon(x)|\ = \mathcal{O}(\varepsilon^{-N}),\ \text{as } \varepsilon \to 0\Big\},$$

and

$$\mathcal{N}^e_\tau(\mathbb{R}^n) = \Big\{(u_\varepsilon)_{\varepsilon>0} \in (\mathcal{O}_M(\mathbb{R}^n))^I : \forall\alpha \in R_+ \cup \{0\}, \exists N \geq 0,\ \forall q \geq 0 \text{ such that } \sup_{x\in\mathbb{R}^n} |\ D^\alpha u_\varepsilon(x)|\ = \mathcal{O}(\varepsilon^{q}),\ \text{as } \varepsilon \to 0\Big\},$$

where D^α, $m-1 < \alpha < m$ with $m \in \mathbb{N}^*$is the Caputo fractional derivative. Embedding of the Schwartz distributions space $\mathcal{S}'(\mathbb{R}^n)$ into $\mathcal{G}^e_\tau(\mathbb{R}^n)$ is given by $u \longrightarrow [(u * \phi_\varepsilon)_{\varepsilon\in I}]$, with

$$\phi_\varepsilon(x) = \frac{1}{\varepsilon}\phi(\frac{x}{\varepsilon}),\ \phi \in \mathcal{C}^\infty_0(\mathbb{R}),\ \phi(x) \geq 0,\ \int_{\mathbb{R}} \phi = 1,\ \int_{\mathbb{R}} x^\alpha \phi = 0, \forall\alpha \in \mathbb{N}^n,\ |\ \alpha\ | > 0.$$

2.1 Caputo derivative

The Caputo fractional integral is defined as follows

$$I^\alpha_0 x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds, \tag{2.3}$$

with $m-1 < \alpha < m$ and $m \in \mathbb{N}^*$. Fractional derivative in the Caputo sense of order α of a function x is defined by

$$D^\alpha_0 x(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} x^{(m)}(s) ds, \tag{2.4}$$

where $m-1 < \alpha < m$ and $m \in \mathbb{N}^*$, provided that this integral convergent which is the case when f belongs to the class of absolutely continuous functions see [5].

2.2 Embedding of the Caputo fractional derivative into colombeau algebra

Inspired by the classical theory of Colombeau for integer derivatives. To prove the embedding of the distribution in the extended Colombeau algebra, We have to show that all derivatives are moderate, including fractional derivatives, i.e. we prove that $\tilde{D}^\alpha \omega_\varepsilon(t) = D^\alpha(\omega * \varphi_\varepsilon) * \varphi_\varepsilon(t)$ is moderate. For $\alpha \in]0,1[$ we have

$$\begin{aligned}
\tilde{D}^\alpha \omega_\varepsilon = (\mathrm{D}^\alpha \omega_\varepsilon * \varphi_\varepsilon(t)) &\leq \frac{1}{\Gamma(1-\alpha)} \left(\int_0^t \frac{w'_\varepsilon(s)\mathrm{d}s}{(t-s)^\alpha} \right) * (\varphi_\varepsilon(t)) \\
&\leq \frac{1}{\Gamma(1-\alpha)} \sup_{t\in[0,T)} \left| \int_0^t \frac{w'_\varepsilon(s)\mathrm{d}s}{(t-s)^\alpha} \right| \cdot |\ (\varphi_\varepsilon(t))|_{\ L^1} \\
&\leq \frac{C}{\Gamma(1-\alpha)} \sup_{t\in[0,T)} |\ w'_\varepsilon(t)|\ \frac{T^{1-\alpha}}{1-\alpha} \leq C_{\alpha,T}\varepsilon^{-N},
\end{aligned}$$

Then there exists $N > 0$, such that

$$\tilde{D}^\alpha \omega_\varepsilon \leq C_{\alpha,T}\varepsilon^{-N}, \tag{2.5}$$

hence

$$\left| \left(\tilde{D}^\alpha \omega_\varepsilon\right)' \right| = |\ \mathrm{D}^\alpha \omega_\varepsilon(t) * \varphi'_\varepsilon(t)| \ \leq \frac{C}{\varepsilon} \sup_{t\in[0,T)} |\ \mathrm{D}^\alpha \omega_\varepsilon(t)|\ ,$$

where the last expression is given by (3). Let $0 < \alpha < 1$, for higher fractional derivatives we use the semigroup property of fractional differentiation: $\mathrm{D}^\alpha(\mathrm{D}^\alpha u_\varepsilon) = \mathrm{D}^{\alpha+\alpha} u_\varepsilon$. We have,

$$\mathrm{D}^\alpha\left(\tilde{D}^\alpha \omega_\varepsilon\right) = \mathrm{D}^\alpha(\mathrm{D}^\alpha \omega_\varepsilon * \varphi_\varepsilon) = \mathrm{D}^{\alpha+\alpha}\omega_\varepsilon * \varphi_\varepsilon.$$

Then, there exists $N > 0$ and $t \in [0,T), T > 0$ such that

$$\begin{aligned}
\left(\mathrm{D}^{\alpha+\alpha}\omega_\varepsilon\right) * \varphi_\varepsilon(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{w'_\varepsilon(s)}{(t-s)^{\alpha+\alpha}}\mathrm{d}s \\
&\leq \frac{1}{\Gamma(1-\alpha)} \sup_{t\in[0,T)} |\ w'_\varepsilon(t)|\ \cdot \left| \frac{t^{1-(\alpha+\alpha)}}{1-(\alpha+\alpha)} * \varphi_\varepsilon(t) \right| \\
&\leq \frac{1}{\Gamma(1-\alpha)} \sup_{t\in[0,T)} |\ w'_\varepsilon(t)|\ \frac{C}{\varepsilon}\frac{T^{2-(\alpha+\alpha)}}{2-(\alpha+\alpha)} \leq C_{T,\alpha,\alpha}\varepsilon^{-N}.
\end{aligned}$$

3 Generalized Semigroup

In this section, we will recall the results, concerning generalized semigroup, stated in the paper[9].

Definition 3.1. [9] $\mathcal{SE}_M(\mathbb{R}_+ : \mathcal{L}_c(X))$ is the space of nets $(S_\varepsilon)_\varepsilon$ of strongly continuous mappings $S_\varepsilon : \mathbb{R}_+ \longrightarrow \mathcal{L}_c(X),\ \varepsilon \in (0,1)$ with the property that for every $T > 0$ there exists $a \in \mathbb{R}$ such that,

$$\sup_{t\in[0,T)} |\ S_\varepsilon(t)|\ = O_{\varepsilon\to 0}(\varepsilon^a). \tag{3.1}$$

$\mathcal{SN}(\mathbb{R}_+ : \mathcal{L}_c(X))$ is the space of nets $(N_\varepsilon)_\varepsilon$ of strongly continuous mappings $N_\varepsilon : \mathbb{R}_+ \longrightarrow \mathcal{L}_c(X)$, $\varepsilon \in (0,1)$ with the properties:

For every $b \in \mathbb{R}$ and $T > 0$,

$$\sup_{t\in[0,T)} |\ N_\varepsilon(t)|\ = O_{\varepsilon\to 0}(\varepsilon^b). \tag{3.2}$$

There exists $t_0 > 0$ and $a \in \mathbb{R}$ such that,

$$\sup_{t<t_0} |\ \frac{N_\varepsilon(t)}{t}|\ = O_{\varepsilon\to 0}(\varepsilon^a). \tag{3.3}$$

There exists a net $(H_\varepsilon)_\varepsilon$ in $\mathcal{L}_c(X)$ and $\varepsilon_0 \in (0,1)$ such that,

$$\lim_{t \longrightarrow 0} \frac{N_\varepsilon(t)}{t} = H_\varepsilon x, \quad x \in X, \quad \varepsilon < \varepsilon_0. \tag{3.4}$$

For every $b > 0$,

$$\mid H_\varepsilon \mid \ = O_{\varepsilon \to 0}(\varepsilon^b). \tag{3.5}$$

Remark 3.2. Note that because of (3.1), it is sufficient that (3.2) holds for all $x \in \mathcal{D}$ where $\mathcal{D}$ is a dense subspace of X.

Proposition 3.3. [9] $\mathcal{SE}_M(\mathbb{R}_+ : \mathcal{L}_c(X))$ is algebra with respect to composition and $\mathcal{SN}(\mathbb{R}_+ : \mathcal{L}_c(X))$ is an ideal of $\mathcal{SE}_M(\mathbb{R}_+ : \mathcal{L}_c(X))$.

Now we define Colombeau type algebra as the factor algebra

$$\mathcal{SG}(\mathbb{R}_+ : \mathcal{L}(X)) = \mathcal{SE}_M(\mathbb{R}_+ : \mathcal{L}(X))/\mathcal{SN}(\mathbb{R}_+ : \mathcal{L}(X)). \tag{3.6}$$

Elements of $\mathcal{SG}(\mathbb{R}_+ : \mathcal{L}(X))$ will be denoted by $S = [S_\varepsilon]$, where $(S_\varepsilon)_\varepsilon$ is a representatvie of the above class.

Definition 3.4. [9] $S \in \mathcal{SG}(\mathbb{R}_+ : \mathcal{L}(X))$ is a called a Colombeau C_0-Semigroup if it has a representative $(S_\varepsilon)_\varepsilon$ such that, for some $\varepsilon_0 > 0$, S_ε is a C_0-Semigroup, for every $\varepsilon < \varepsilon_0$.

In the sequel we will use only representatives $(S_\varepsilon)_\varepsilon$ of a Colombeau C_0-semigroup S which are C_0-semigroups, for ε small enough.

Proposition 3.5. [9] Let $(S_\varepsilon)_\varepsilon$ and $(\tilde{S}_\varepsilon)_\varepsilon$ be representatives of a Colombeau C_0-semigroup S, with the infinitesimal generators $A_\varepsilon, \ \varepsilon < \varepsilon_0$, and $\tilde{A}_\varepsilon, \ \varepsilon < \tilde{\varepsilon}_0$, respectively, where ε_0 and $\tilde{\varepsilon}_0$ correspond (in the sense of definition (3.4)) to $(S_\varepsilon)_\varepsilon$ and $(\tilde{S}_\varepsilon)_\varepsilon$, respectively. Then, $D(A_\varepsilon) = D(\tilde{A}_\varepsilon)$ for every $\varepsilon < \bar{\varepsilon} = \min\{\varepsilon_0, \tilde{\varepsilon}_0\}$ and $A_\varepsilon - \tilde{A}_\varepsilon$ can be extended to an element of $\mathcal{L}(X)$, denoted again by $A_\varepsilon - \tilde{A}_\varepsilon$. Moreover, for every $a \in \mathbb{R}$

$$\mid A_\varepsilon - \tilde{A}_\varepsilon \mid \ = O_{\varepsilon \to 0}(\varepsilon^a). \tag{3.7}$$

Now we define the infinitesimal generator of a Colombeau C_0-semigroup S. Denote by $\mathcal{A}$ the set of pairs $((A_\varepsilon)_\varepsilon, (D(A_\varepsilon))_\varepsilon)$ where A_ε is a closed linear operator on X with the dense domain $D(A_\varepsilon) \subset X$ for every $\varepsilon \in (0,1)$. We introduce an equivalence relation in A,

$$((A_\varepsilon)_\varepsilon, (D(A_\varepsilon))_\varepsilon) \sim ((\tilde{A}_\varepsilon)_\varepsilon, (D(\tilde{A}_\varepsilon))_\varepsilon).$$

If there exists $\varepsilon_0 \in (0,1)$ such that $D(A_\varepsilon) = D(\tilde{A}_\varepsilon)$ for every $\varepsilon < \varepsilon_0$. And for every $a \in \mathbb{R}$, there exist $C > 0$ and $\varepsilon_a \leq \varepsilon_0$ such that for $x \in D(A_\varepsilon)$,

$$\mid (A_\varepsilon - \tilde{A}_\varepsilon)x \mid \ \leq C\varepsilon^a \mid x \mid \ , \ x \in D(A_\varepsilon), \varepsilon \leq \varepsilon_a.$$

Since A_ε has a dense domain in X, $R_\varepsilon := A_\varepsilon - \tilde{A}_\varepsilon$ can be extended to be an operator in $\mathcal{L}_c(X)$ satisfying $\mid (A_\varepsilon - \tilde{A}_\varepsilon)x \mid \ = O_{\varepsilon \to 0}(\varepsilon^a)$ for every $a \in \mathbb{R}$, such an operator R_ε is called the zero operator. We denote by A the corresponding element of the quotient space $\mathcal{A}/\sim$. Due to proposition (3.3), the following definition makes sense.

Definition 3.6. $A \in \mathcal{A}/\sim$ is the infinitesimal generator of a Colombeau C_0-semigroup S if there exists a representative $(A_\varepsilon)_\varepsilon$ of A such that A_ε is the infinitesimal generator of S_ε, for ε small enough.

Remark 3.7. [9] Let the assumptions of definition (3.1) holds. Moreover, assume a stronger assumption than (3.1). Then there exist $M > 0$, $a \in \mathbb{R}$ and $\varepsilon_0 \in (0,1)$ such that,

$$\mid S_\varepsilon(t) \mid \ \leq M\varepsilon^a e^{\alpha_\varepsilon t}, \quad \varepsilon < \varepsilon_0, \quad t \geq 0, \tag{3.8}$$

where $0 < \alpha_\varepsilon < \alpha$, for some $\alpha > 0$.

Hence we obtain the corresponding sub algebra of $\mathcal{SG}(\mathbb{R}_+ : \mathcal{L}(X))$. For this we can formulate the Hille-Yosida theorem in a usual way. For the whole algebra of Colombeau C_0-semigroups, $\mathcal{SG}(\mathbb{R}_+ : \mathcal{L}(X))$ the formulation of the Hille-Yosida-Type theorem is an open problem.

4 Generalized mild solutions

Let us consider the Cauchy problem in the framework of Colombeau algebras

$$\begin{cases} D^{\alpha}x(t) + Ax(t) = F(t, x(t)), \\ x(0) = x_0 \in \tilde{\mathbb{R}}. \end{cases} \tag{4.1}$$

where $-A$ is an infinitesemal genrator of a generalized colombeau semigroup $(T(t))_{t\geq 0} = [((T_{\varepsilon}(t))_{t\geq 0})_{\varepsilon}]$, $x \in (\mathcal{G}^e(\mathbb{R}))^n$, $F \in (\mathcal{G}^e(\mathbb{R}))^{n+1}$. The representative form of (4.1) given by

$$\begin{cases} D^{\alpha}x_{\varepsilon}(t) + A_{\varepsilon}x_{\varepsilon}(t) = F_{\varepsilon}(t, x_{\varepsilon}(t)), \\ x_{\varepsilon}(0) = x_{0\varepsilon}. \end{cases} \tag{4.2}$$

According to the defintions (2.3) and (2.4) we write the Cauchy problem in the integral equation

$$\begin{cases} x_{\varepsilon}(t) = x_{0\varepsilon} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[-A_{\varepsilon}x_{\varepsilon}(s) + F_{\varepsilon}(s, x_{\varepsilon}(s))]ds, \\ x_{\varepsilon}(0) = x_{0\varepsilon}. \end{cases} \tag{4.3}$$

The proof of the theorem requires the two lemmas below

Lemma 4.1. If (4.3) holds, then there is a probability density function ϕ_{α} defined on $(0, +\infty)$ such that

$$\begin{cases} x_{\varepsilon}(t) = \int_0^{\infty} \phi_{\alpha}T_{\varepsilon}(t^{\alpha}\xi a)x_{0\varepsilon}d\xi + \alpha\int_0^t\int_0^{\infty} \xi(t-s)^{\alpha-1}\phi_{\alpha}(\xi)T_{\varepsilon}((t-s)^{\alpha}\xi)F_{\varepsilon}(s, x_{\varepsilon}(s))d\xi ds, \\ \\ x_{\varepsilon}(0) = x_{0\varepsilon}. \end{cases} \tag{4.4}$$

Proof . Applying the Laplace transform to the first equation in (4.3),

$$\mathcal{L}x_{\varepsilon}(\lambda) = \frac{1}{\lambda}x_{0\varepsilon} + \frac{1}{\lambda^{\alpha}}A_{\varepsilon}\mathcal{L}(x_{\varepsilon})(\lambda) + \frac{1}{\lambda^{\alpha}}\mathcal{L}(F_{\varepsilon}(., x_{\varepsilon}(.))(\lambda),$$

we have

$$\begin{aligned} x_{\varepsilon}(\lambda) &= \lambda^{\alpha-1}(\lambda^{\alpha-1}I + A_{\varepsilon})^{-1}x_{0\varepsilon} + (\lambda^{\alpha-1}I + A_{\varepsilon})^{-1} \\ &= \lambda^{\alpha-1}(\lambda^{\alpha}I + A_{\varepsilon})^{-1}x_{0\varepsilon} + (\lambda^{\alpha}I + A_{\varepsilon})^{-1}\mathcal{L}(e^{-\lambda s}F_{\varepsilon}(s, x_{\varepsilon}(s)))(\lambda) \\ &= \lambda^{\alpha-1}\int_0^{\infty} e^{-\lambda^{\alpha}s}T\varepsilon(s)x_{0\varepsilon}ds + \int_0^{\infty} e^{-\lambda^{\alpha}s}T_{\varepsilon}(s)\omega(\lambda)ds, \end{aligned} \tag{4.5}$$

where I is the identity operator, and $\omega(\lambda)$ is the Laplace transform of $F_{\varepsilon}(s, x_{\varepsilon}(s))$. Consider the probability density given in [7] by

$$\psi_{\alpha}(\xi) = \frac{1}{\pi}\sum(-1)^{n-1}\xi^{-\alpha n-1}\frac{\Gamma}{\Gamma(n+1)}sin(n\pi\alpha), \;\; \xi \in (0, \infty), \tag{4.6}$$

whose Laplace transform is given by $e^{-\lambda\xi}\psi_{\alpha}(\xi)d\xi = e^{-\lambda^{\alpha}}$, with $\alpha \in (0, 1)$. Then

$$\begin{aligned} x_{0\varepsilon} &= \lambda^{\alpha-1}\int_0^{\infty} e^{-\lambda^{\alpha}s}T_{\varepsilon}(s)x_{0\varepsilon}ds \\ &= \int_0^{\infty} \alpha(\lambda t)^{\alpha-1}e^{-(\lambda t)^{\alpha}}T_{\varepsilon}(t^{\alpha})x_{0\varepsilon}ds \\ &= \int_0^{\infty} \frac{-1}{\lambda}\frac{d}{dt}\Big[e^{-(\lambda t)^{\alpha}}\Big]T_{\varepsilon}(t^{\alpha})x_{0\varepsilon}ds \\ &= \int_0^{\infty}\left[\int_0^{\infty} \xi\psi_{\alpha}(\xi)e^{-(\lambda t\xi)}T_{\varepsilon}(t^{\alpha})x_{0\varepsilon}d\xi\right]dt \\ &= \int_0^{\infty} e^{-\lambda t}\left[\int_0^{\infty} \psi_{\alpha}(\xi)T_{\varepsilon}\Big(\frac{t^{\alpha}}{\xi^{\alpha}}\Big)x_{0\varepsilon}d\xi\right]dt \end{aligned} \tag{4.7}$$

For the second term we have

$$\begin{aligned}\int_0^\infty e^{-\lambda^\alpha s}T\varepsilon(s)\omega(\lambda)ds &= \int_0^\infty\left[\int_0^\infty \alpha t^{\alpha-1}e^{-(\lambda t)^\alpha}T_\varepsilon(t^\alpha)e^{-\lambda s}F_\varepsilon(s,x_\varepsilon(s))ds\right]dt\\ &= \int_0^\infty\int_0^\infty\int_0^\infty \alpha\psi_\alpha(\xi)e^{-\lambda t\xi}T_\varepsilon(t^\alpha)e^{-\lambda s}t^{-\alpha-1}F_\varepsilon(s,x_\varepsilon(s))d\xi\, ds\, dt\\ &= \int_0^\infty e^{-\lambda t}\left[\alpha\int_0^t\int_0^\infty \psi_\alpha(\xi)T_\varepsilon\Big(\frac{(t-s)^\alpha}{\xi^\alpha}\Big)F_\varepsilon(s,x_\varepsilon(s))\frac{(t-s)^\alpha}{\xi^\alpha}d\xi\, ds\right]dt\end{aligned}$$

According to the last equalities we obtain

$$\mathcal{L}x_\varepsilon(\lambda)=\int_0^\infty e^{-\lambda t}\int_0^\infty \phi_\alpha(\xi)T_\varepsilon(t^\alpha\xi)x_{0\varepsilon}d\xi+\alpha\int_0^t\int_0^\infty \xi(t-s)^{\alpha-1}\phi_\alpha(\xi)T_\varepsilon((t-s)^\alpha\xi)F_\varepsilon(s,x_\varepsilon(s))d\xi\, dsdt$$

And the integral solution of (4.2) becomes

$$x_\varepsilon(t)=\int_0^\infty \phi_\alpha(\xi)T_\varepsilon(t^\alpha\xi)x_{0\varepsilon}d\xi+\alpha\int_0^t\int_0^\infty \xi(t-s)^{\alpha-1}\phi_\alpha(\xi)T_\varepsilon((t-s)^\alpha\xi)F_\varepsilon(s,x_\varepsilon(s))d\xi\, ds \tag{4.8}$$

Now we define a representative $(S_\varepsilon^\alpha)_{t\in\mathbb{R}_+}$ by

$$S_\varepsilon^\alpha(t)x_\varepsilon=\alpha\int_0^\infty \xi\phi_\alpha(\xi)T_\varepsilon(t^\alpha\xi)x_\varepsilon d\xi. \tag{4.9}$$

Remark 4.2. It is easy to prove that, the family $((S_\varepsilon^\alpha(t))_{t\geq 0})_\varepsilon$ is a moderate family, then $\overline{(S_\varepsilon^\alpha(t))_{t\geq 0}}$ are generalized operators.

Finally, the integral solution of the Cauchy (4.1) becomes

$$x_\varepsilon(t)=S_\varepsilon^\alpha(t)x_{0\varepsilon}+\int_0^t (t-s)^{\alpha-1}T_\varepsilon^\alpha(t-s)F_\varepsilon(s,x_\varepsilon(s))ds, \tag{4.10}$$

where $(S_\varepsilon^\alpha(t))_{t\in\mathbb{R}_+}$ by

$$S_\varepsilon^\alpha(t)x_\varepsilon=\int_0^\infty \phi_\alpha(\xi)T_\varepsilon(t^\alpha\xi)x_\varepsilon d\xi. \tag{4.11}$$

□

Remark 4.3. The operators $(S^\alpha(t))_{t\geq 0}=\overline{(S(t))_{t\geq 0}}$ called generalized generalized family resolvent.

Lemma 4.4. For any fixed $t\in[0,T]$, $T>0$, $(S_\varepsilon^\alpha(t))_{t\geq 0}$, $(T_\varepsilon^\alpha(t))_{t\geq 0}$ are linear and bounded operators, by report to the variable t, for every $\varepsilon\in(0,1)$.

Proof . We will give the proof for (T_ε^α) because that of (S_ε^α) is similar. For fixed $t>0$, $T_\varepsilon^\alpha(t)$ is linear operator since $T_\varepsilon(t)$ is a linear operator, let $\eta\in[0,1]$ we have

$$\int_0^\infty \frac{1}{\xi^\eta}\psi_\alpha(\xi)d\xi=\frac{\Gamma(1+\eta/\alpha)}{\Gamma(1+\eta)},$$

then we have,

$$\int_0^\infty \xi^\eta\phi_\alpha(\xi)d\xi=\int_0^\infty \frac{1}{\xi^{\alpha\eta}}\psi_\alpha(\xi)d\xi=\frac{\Gamma(1+\eta)}{\Gamma(1+\alpha\eta)}.$$

Since $T_\varepsilon(t)$ has a moderate bounds, then there exists a positive real number a such that

$$\sup_{t\in[0,\Lambda]}\parallel T_\varepsilon(t)\parallel=O(\varepsilon^{-a})\quad\text{as}\quad \varepsilon\to 0, \tag{4.12}$$

then there exists $c>0$ such that for all $t\in[0,T]$, $x_\varepsilon\in\mathcal{E}^e_M(\mathbb{R})$, we have $\parallel T_\varepsilon(t)x_\varepsilon\parallel\leq c\varepsilon^{-a}$. Then

$$\begin{aligned}\Big| T^\alpha_\varepsilon(t)x_\varepsilon\Big| &= \Big|\alpha\int_0^\infty \xi\phi_\alpha(\xi)T_\varepsilon(t^\alpha\xi)x_\varepsilon d\xi\Big|\\ &\leq \sup_{t\in[0,\Lambda]}\Big| T_\varepsilon(t^\alpha\xi)x_\varepsilon\Big| \times\alpha\frac{1}{\Gamma(1+\alpha)},\end{aligned}$$

since $(T_\varepsilon(t))_\varepsilon$ is a representative of the generalized semigroup $(T(t))_t$ by using (4.12) we get,

$$\begin{aligned}\Big| T^\alpha_\varepsilon(t)x_\varepsilon\Big| &\leq \Big(C\,\varepsilon^{-a}\times\alpha\frac{1}{\Gamma(1+\alpha)}\Big)\Big| x_\varepsilon\Big|\\ &= C_\alpha\Big| x_\varepsilon\Big|, \quad \text{where } C_\alpha=C\,\varepsilon^{-a}\times\alpha\frac{1}{\Gamma(1+\alpha)},\end{aligned}$$

which proves that $T^\alpha_\varepsilon(t)$ is a linear and bounded operator by report to t. □

Theorem 4.5. Assume that $F\in(\mathcal{G}^e_\tau(\mathbb{R}))^{n+1}$, and $\mid\nabla_x F\mid\leq C\mid\ln(\varepsilon)\mid$, $\varepsilon\in I=(0,1)$. Then the Cauchy problem (4.1) has a unique solution in the extended Colombeau algebra $(\mathcal{G}^e(\mathbb{R}))^n$.

Proof . For any $\varepsilon\in(0,1)$, $\alpha\in(0,1)$ we have to show that the integral solution (x_ε) given in (4.10) of lemma (4.1) is an element of $\mathcal{E}^e_M(\mathbb{R})$. First we have the estimation

$$\begin{aligned}\Big| x_\varepsilon(t)\Big| &= \Big| S^\alpha_\varepsilon(t)x_{0\varepsilon}+\int_0^t(t-s)^{\alpha-1}T^\alpha_\varepsilon(t-s)F_\varepsilon(s,x_\varepsilon(s))ds\Big|,\\ &\leq \Big| S^\alpha_\varepsilon(t)x_{0\varepsilon}\Big| +\int_0^t\Big|(t-s)^{\alpha-1}T^\alpha_\varepsilon(t-s)F_\varepsilon(s,x_\varepsilon(s))\Big| ds,\\ &\leq \Big| S^\alpha_\varepsilon(t)x_{0\varepsilon}\Big| +\int_0^t(t-s)^{\alpha-1}\Big| T^\alpha_\varepsilon(t-s)F_\varepsilon(s,x_\varepsilon(s))\Big| ds.\end{aligned}$$

The approximation of the first order to F_ε yields

$$F_\varepsilon(t,x(t))=F_\varepsilon(t,0)+\mid\nabla_x F_\varepsilon\mid x_\varepsilon(t)+N_\varepsilon(t), \tag{4.13}$$

where $N_\varepsilon(t)$ is the negligible part of this approximation. By lemma (4.4), and the fact that $(x_\varepsilon)\in\mathcal{E}^e_M(\mathbb{R})$ there are positives constants c c_1, c_2 N_1 and N_2 such that

$$\begin{aligned}\Big| x_\varepsilon(t)\Big| &\leq c\,c_2\varepsilon^{-N_2}+\int_0^t(t-s)^{\alpha-1}\frac{\alpha\,c_1\,\varepsilon^{-N_1}}{\Gamma(1+\alpha)}\Big| F_\varepsilon(s,x_\varepsilon(s))\Big| ds\\ &\leq c\,c_2\varepsilon^{-N_2}+\int_0^t(t-s)^{\alpha-1}\frac{\alpha\,c_1\,\varepsilon^{-N_1}}{\Gamma(1+\alpha)}\Big| F_\varepsilon(s,0)+\mid\nabla_x F_\varepsilon\mid x_\varepsilon(s)+N_\varepsilon(s)\Big| ds.\end{aligned}$$

Using the Gronwall lemma, we obtain

$$\Big| x_\varepsilon(t)\Big| \leq (c\,c_2\varepsilon^{-N_2}+c_T\varepsilon^{-N_1})\exp(-\Lambda\ln\varepsilon).$$

Hence there are positive constants $\tilde{c}$, $\tilde{N}$ such that $\mid x_\varepsilon(t)\mid\leq\tilde{c}\,\varepsilon^{-\tilde{N}}$ which proves the moderateness of the solution. To obtain estimates for higher order derivatives, just differentiate the equation and apply the same inductive arguments, assuming that the lower order terms are known to be mild from the preceding phases. Let us prove the uniqueness of the solution in $(\mathcal{G}^e(\mathbb{R}))^n$, suppose that there are two solutions $x_{1,\varepsilon}$, $x_{2,\varepsilon}$ to the regularized of the problem (4.2) and let v_ε their difference we have

$$v_\varepsilon(t) = \int_0^t(t-s)^{\alpha-1}T^\alpha_\varepsilon(t-s)[F_\varepsilon(s,x_{1,\varepsilon})-F_\varepsilon(s,x_{2,\varepsilon})]ds.$$

Now using the approximation (4.13) of F_ε yields

$$\mid v_\varepsilon(t)\mid \leq \int_0^t\frac{\Lambda^\alpha}{\alpha}\mid T^\alpha_\varepsilon(t-s)[\mid\nabla_x F_\varepsilon\mid(x_{1,\varepsilon}(s)-x_{2,\varepsilon}(s))+N_\varepsilon(s)]\mid ds,$$

by using the boundedness of the linear operator $T_\varepsilon^\alpha(t)$ with $t \geq 0$, Gronwall lemma, the fact that $x_{1,\varepsilon}(s) - x_{2,\varepsilon}(s)$ is of order $\mathcal{O}(\varepsilon^N)$ and the same for raisin for the negligible part N_ε. Then it follows that every $N \geq 0$ we have $| v_\varepsilon(t) | = \mathcal{O}(\varepsilon^N)$ as $\varepsilon \to 0$. which proves the uniqueness of the solution in the algebra $(\mathcal{G}^e(\mathbb{R}))^n$. $\square$

5 Application to Schrödinger equation

Consider the nonlinear Schrödinger equation with singular potential and initial data involving Caputo fractional derivative

$$\begin{cases} \frac{1}{i}\partial_t^\alpha u(t,x) - \triangle u(t,x) + v(x)u(t,x) = 0, \\ v(x) = \delta(x) \ , \ u(0,x) = \delta(x). \end{cases} \tag{5.1}$$

Here $A = -\Delta$. We shall use the regularization for Dirac measure.

$$v_\varepsilon(x) = \delta_\varepsilon(x) = (\phi_\mathcal{E}(x)) = | \ \ln\varepsilon | \ ^{cn}\phi(x| \ \ln\varepsilon | \ ^c),\ c > 0,$$

$x \in \mathbb{R}^n$ and $\int \phi = 1$ with $\phi(x) \geq 0, .$

For the initial data we use

$$u_{0,\varepsilon}(x) = | \ \ln\varepsilon | \ ^{an}\phi(x| \ \ln\varepsilon | \ ^a) \ , \ a > 0$$

$x \in \mathbb{R}^n, \phi \in \int \phi = 1, \phi(x) \geq 0, x \in \mathbb{R}^n.$

5.1 Existence and uniqueness in the Colombeau algebra

Theorem 5.1. The regularized equation of (5.1) is given by

$$\begin{cases} \frac{1}{i}\partial_{\mathrm{t}}^\alpha u_\varepsilon(t,x) - \triangle u_\varepsilon(t,x) + v_\varepsilon(x)u_\varepsilon(t,x) = 0, \\ v_\varepsilon(x) = \delta_\varepsilon(x) \ , \ u_{0,\varepsilon}(x) = \delta_\varepsilon(x), \end{cases} \tag{5.2}$$

where v_ε *and* $u_{0,\varepsilon}$ *are regularized of* v *and* u_0*, respectively. Then, the problem* (5.2) *has a unique solution in* $\mathcal{G}(\mathbb{R}^+ \times \mathbb{R}^n)$.

Proof . By (4.10) the integral solution of the equation (5.2) becomes

$$u_\varepsilon(t,x) = \int_{\mathbb{R}^n} S_\varepsilon^\alpha(t, x-y)u_{0,\varepsilon}(y)dy + \int_0^{\mathrm{t}}\int_{\mathbb{R}^n} S_\varepsilon^\alpha(t-\tau, x-y)v_\varepsilon(y)u_\varepsilon(\tau,y)dyd\tau$$

where $S_\varepsilon^\alpha(t,x)x_\varepsilon = \int_0^\infty \phi_\varepsilon(\xi)S_n\left(t^\alpha\xi\right)x_\varepsilon d\xi$, with $S_n(t,x)$ is the heat kernel given in [8]. Then

$$\| u_\varepsilon(t,\ .)\|_{\ L^\infty(\mathbb{R}^n)} \leq \| S_\varepsilon^\alpha(t, x-.)\|_{\ L^1}\| u_{0,\varepsilon}\|_{\ L^\infty(\mathbb{R}^n)} + \int_0^t \| S_\varepsilon^\alpha(t-\tau, x-.)\|_{\ L^1}\| v_\varepsilon(.)\|_{\ L^\infty(\mathbb{R}^n)}\| u_\varepsilon(\tau,\ .)\|_{\ L^\infty(\mathbb{R}^n)}d\tau.$$

Using lemma (4.4), there is C such that $| \ S_\varepsilon^\alpha | \ \leq C$, we get

$$\| u_\varepsilon(t,\ .)\|_{\ L^\infty(\mathbb{R}^n)} \leq C\| u_{0,\varepsilon}\|_{\ L^\infty(\mathbb{R}^n)} + C\| v_\varepsilon(.)\|_{\ L^\infty(\mathbb{R}^n)}\int_0^t \| u_\varepsilon(\tau,\ .)\|_{\ L^\infty(\mathbb{R}^n)}d\tau.$$

From Gronwall inequality, it follows

$$\| u_\varepsilon(t,\ .)\|_{\ L^\infty(\mathbb{R}^n)} \leq C| \ \ln\varepsilon | \ ^{an}\exp(CT| \ \ln\varepsilon | \ ^{bn}).$$

Then there exist $N > 0$ such that

$$\| u_\varepsilon(t,\ .)\|_{\ L^\infty(\mathbb{R}^n)} \leq C\varepsilon^{-N}.$$

For the first derivative to $x_j, \mathrm{j} \in \{1, ..., \ n\}$ we obtain

$$\partial_{x_j}u_\varepsilon(t,x) = \int_{\mathbb{R}^n} S_\varepsilon^\alpha(t, x-y)\partial_{\mathrm{y}_j}u_{0,\varepsilon}(y)dy + \int_0^{\mathrm{t}}\int_{\mathbb{R}^n} S_\varepsilon^\alpha(t-\tau, x-y)\left(\partial_{\mathrm{y}_j}v_\varepsilon(y)u_\varepsilon(\tau,y) + v_\varepsilon(y)\partial_{\mathrm{y}_j}u_\varepsilon(\tau,y)\right)dyd\tau,$$

so,

$$\|\partial_{x_j} u\mathcal{E}(t,\ .)\|_{L^\infty(\mathbb{R}^n)} \leq \|\ S_\varepsilon^\alpha(t,x-.)\|\ _{L^1}\|\ \partial_{\mathrm{y}_j} u_{0,\varepsilon}\|\ _{L^\infty(\mathbb{R}^n)} + \int_0^t \|\ S_\varepsilon^\alpha(t-\tau,x-.)\|\ _{L^1}(\|\ \partial_{y_i} v_\varepsilon\|\ _{L^\infty(\mathbb{R}^n)}\|\ u_\varepsilon\|\ _{L^\infty}$$
$$+\|\ v_\varepsilon\|\ _{L^\infty(\mathbb{R}^n)}\|\ \partial_{y_i} u_\varepsilon(\tau,\ .)\|\ _{L^\infty(\mathbb{R}^n)})d\tau,$$

which implies

$$\begin{aligned}\|\ \partial_{x_j} u_\varepsilon(t,\ .)\|\ _{L^\infty(\mathbb{R}^n)} &\leq C|\ \ln\varepsilon|\ ^{a(n+1)} + C\int_0^t |\ \ln\varepsilon|\ ^{b(n+1)}\|\ u_\varepsilon\|\ _{L^\infty} + |\ \ln\varepsilon|\ ^{bn}\|\ \partial_{\mathrm{y}_j} u\mathcal{E}(\tau,.)\|\ _{L^\infty(\mathbb{R}^n)}d\tau\\ &\leq C(|\ \ln\varepsilon|\ ^{a(n+1)} + T|\ \ln\varepsilon|\ ^{b(n+1)}\|\ u\mathcal{E}\|\ _{L^\infty}) + C|\ \ln\varepsilon|\ ^{bn}\int_0^{\mathrm{t}} \|\ \partial_{\mathrm{y}_j} u\mathcal{E}(\tau,.)\|\ _{L^\infty(\mathbb{R}^n)}d\tau.\end{aligned}$$

Using Gronwall inequality

$$\|\ \partial_{x_j} u_\varepsilon(t,\ .)\|\ _{L^\infty(\mathbb{R}^n)} \leq C(|\ \ln\varepsilon|\ ^{a(n+1)} + T|\ \ln\varepsilon|\ ^{b(n+1)}\|\ u\mathcal{E}\|\ _{L^\infty})\exp(CT|\ \ln\varepsilon|\ ^{bn}),$$

the previous step ensure there exist $N>0$ such that

$$\|\ \partial_{x_j} u_\varepsilon(t,\ .)\|\ _{L^\infty(\mathbb{R}^n)} \leq C\varepsilon^{-N}.$$

For the second derivative for $y_{\mathrm{i}}, i\in\{1,...,\ n\}$ we obtain

$$\partial_{x_i}\partial_{x_j} u\mathcal{E}(t,x) = \int_{\mathbb{R}^n} S_\varepsilon^\alpha(t,x-y)(\partial_{\mathrm{y}_i}\partial_{\mathrm{y}_j} u_{0,\varepsilon}(y)dy + \int_0^t\int_{\mathbb{R}^n} S_\varepsilon^\alpha(t-\tau,x-y)(\partial_{y_i}\partial_{y_j} v_\varepsilon(y)u_\varepsilon(\tau,y)$$
$$+\partial_{y_j} v_\varepsilon(y)\partial_{y_i} u_\varepsilon(\tau,y) + \partial_{y_i} v_\varepsilon(y)\partial_{y_j} u_\varepsilon(\tau,y) + v_\varepsilon(y)\partial_{y_i}\partial_{y_j} u_\varepsilon(\tau,y))dyd\tau,$$

thus,

$$\begin{aligned}\|\ \partial_{x_i}\partial_{x_j} u_\varepsilon(t,\ .)\|\ _{L^\infty(\mathbb{R}^n)} \leq&\|\ S_\varepsilon^\alpha(t,x-.)\|\ _{L^1}\|\ \partial_{y_i}\partial_{y_j} u_{0,\varepsilon}(.)\|\ _{L^\infty(\mathbb{R}^n)}\\ &+\int_0^t \|\ S_\varepsilon^\alpha(t-\tau,x-.)\|\ _{L^1}(\|\ \partial_{y_j}\partial_{y_j} v_\varepsilon(.)\|\ _{L^\infty(\mathbb{R}^n)}\|\ u_\varepsilon\|\ _{L^\infty}\\ &+\|\ \partial_{y_j} v_\varepsilon(.)\|\ _{L^\infty(\mathbb{R}^n)}\|\ \partial_{y_j} u_\varepsilon\|\ _{L^\infty} + \|\ \partial_{y_i} v_\varepsilon(.)\|\ _{L^\infty(\mathbb{R}^n)}\|\ \partial_{y_j} u_\varepsilon\|\ _{L^\infty}\\ &+\|\ v_\varepsilon(.)\|\ _{L^\infty}\|\ \partial_{y_i}\partial_{y_j} u_\varepsilon(\tau,\ .)\|\ _{L^\infty(\mathbb{R}^n)})d\tau.\end{aligned}$$

We obtain

$$\begin{aligned}\|\ \partial_{x_i}\partial_{x_j} u_\varepsilon(t,\ .)\|\ _{L^\infty(\mathbb{R}^n)} \leq& C(|\ \ln\varepsilon|\ ^{a(n+2)} + |\ \ln\varepsilon|\ ^{b(n+1)}\|\ u_\varepsilon\|\ _{L^\infty} + |\ \ln\varepsilon|\ ^{b(n+1)}\|\ \partial_{y_i} u_\varepsilon\|\ _{L^\infty}\\ &+|\ \ln\varepsilon|\ ^{b(n+1)}\|\ \partial_{y_j} u_\varepsilon\|\ _{L^\infty}) +\ C|\ln\varepsilon|^{bn}\int_0^{\mathrm{t}} \|\partial_{\mathrm{y}_j}\partial_{\mathrm{y}_j} u_\varepsilon(\tau,\ .)\|\ _{L^\infty(\mathbb{R}^n)}d\tau.\end{aligned}$$

Gronwall inequality gives

$$\begin{aligned}\|\ \partial_{x_j}\partial_{x_j} u_\varepsilon(t,\ .)\|\ _{L^\infty(\mathbb{R}^n)} \leq& C(|\ \ln\varepsilon|\ ^{a(n+2)} + |\ \ln\varepsilon|\ ^{b(n+1)}\|\ u\mathcal{E}\|\ _{L^\infty} + |\ \ln\varepsilon|\ ^{b(n+1)}\|\ \partial_{y_i} u_\varepsilon\|\ _{L^\infty}\\ &+|\ \ln\varepsilon|\ ^{b(n+1)}\|\ \partial_{y_j} u_\varepsilon\|\ _{L^\infty})\exp(CT|\ \ln\varepsilon|\ ^{bn}).\end{aligned}$$

Then, there exist $N>0$ such that

$$\|\ \partial_{x_j}\partial_{x_j} u_\varepsilon(t,\ .)\|\ _{L^\infty(\mathbb{R}^n)} \leq C\varepsilon^{-N}.$$

Let us prove the uniqueness. Suppose that there exist two solutions $u_{1,\varepsilon}(t,.), u_{2,\varepsilon}(t,\ .)$ to the problem (4.2), we get

$$\frac{1}{i}\partial_t^\alpha(u_{1,\varepsilon}(t,x)-u_{2,\varepsilon}(t,x)) - \triangle(u_{1,\varepsilon}(t,x)-u_{2,\varepsilon}(t,x)) + v_\varepsilon(x)(u_{1,\varepsilon}(t,x)-u_{2,\varepsilon}(t,x)) = N_\varepsilon(t,x),$$

$$u_{1,\varepsilon}(0,x)-u_{2,\varepsilon}(0,x) = N_{0,\varepsilon}(x),$$

where $N_\varepsilon(t,x) \in \mathcal{N}(\mathbb{R}^+ \times \mathbb{R}^n), N_{0,\varepsilon}(x) \in \mathcal{N}(\mathbb{R}^n)$. Then

$$u_{1,\varepsilon}(t,x) - u_{2,\varepsilon}(t,x) = \int_{\mathbb{R}^n} S_\varepsilon^\alpha(t,x-y)N_{0,\varepsilon}(y)dy + \int_0^t \int_{\mathbb{R}^n} S_\varepsilon^\alpha(t-\tau,x-y)v_\varepsilon(y)(u_{1,\varepsilon}(\tau,y) - u_{2,\varepsilon}(\tau,y))dyd\tau$$
$$+ \int_0^t \int_{\mathbb{R}^n} S_\varepsilon^\alpha(t-\tau,x-y)N_\varepsilon(\tau,y)dyd\tau$$

which leads to

$$\| u_{1,\varepsilon}(t,\ .) - u_{2,\varepsilon}(t,\ .)\|_{L^\infty(\mathbb{R}^n)} \leq \| S_\varepsilon^\alpha(t,x-.)\|_{L^1} \| N_{0,\varepsilon}(.)\|_{L^\infty(\mathbb{R}^n)} + \| S_\varepsilon^\alpha(t,x-.)\|_{L^1}$$
$$\times \int_0^{\mathrm{t}} \| v_\varepsilon(.)\|_{L^\infty(\mathbb{R}^n)} \| u_{1\varepsilon}(\tau,\ .) - u_{2\varepsilon}(\tau,\ .)\|_{L^\infty(\mathbb{R}^n)} d\tau$$
$$+ \ \| S_\varepsilon^\alpha(t,x-.)\|_{L^1} \| N_\varepsilon(\tau,.)\|_{L^\infty}.$$

Therefore,

$$\| u_{1,\varepsilon}(t,\ .) - u_{2,\varepsilon}(t,\ .)\|_{L^\infty(\mathbb{R}^n)} \leq C(\| N_{0,\varepsilon}(.)\|_{L^\infty(\mathbb{R}^n)} + \| N_\varepsilon(\tau,.)\|_{L^\infty})$$
$$+ C\| v_\varepsilon(.)\|_{L^\infty(\mathbb{R}^n)} \int_0^t \| u_{1,\varepsilon}(\tau,\ .) - u_{2,\varepsilon}(\tau,\ .)\|_{L^\infty(\mathbb{R}^n)} d\tau.$$

Gronwall inequality gives

$$\| u_{1,\varepsilon}(t,\ .) - u_{2,\varepsilon}(t,\ .)\|_{L^\infty(\mathbb{R}^n)} \leq C(\| N_{0,\varepsilon}(.)\|_{L^\infty(\mathbb{R}^n)} + \| N_\varepsilon(\tau,.)\|_{L^\infty}) \exp(CT\| v_\varepsilon(.)\|_{L^\infty(\mathbb{R}^n)}).$$

Which proves that

$$\| u_{1,\varepsilon}(t,\ .) - u_{2,\varepsilon}(t,\ .)\|_{L^\infty(\mathbb{R}^n)} \leq C\varepsilon^q,\ \forall q \in \mathbb{N}.$$

Hence the problem (4.2) has a unique solution in $\mathcal{G}(\mathbb{R}^+ \times \mathbb{R}^n)$. □

5.2 Existence and uniqueness in the extension of Colombeau algebra

We prove the existence and uniqueness result for nonlinear Schrödinger equation with singular potential, initial data and an equation controlled by the fractional derivative of delta distribution in a framework of the extended algebra of generalized functions. It means that we prove moderation and negligibility for entire derivatives and fractional derivatives to the spatial variable x.

Theorem 5.2. The problem (5.1) have the following regularization:

$$\begin{cases} \frac{1}{i}\partial_{\mathrm{t}}^\alpha u_\varepsilon(t,x) - \triangle u_\varepsilon(t,x) + v_\varepsilon(x)u_\varepsilon(t,x) = 0, \\ v_\varepsilon(x) = \delta_\varepsilon(x)\ ,\ u_{0,\varepsilon}(x) = \delta_\varepsilon(x), \end{cases}$$

where v_ε and $u_{0,\varepsilon}$ are regularized of v and u_0, respectively. Then, this problem (5.1) has a unique solution in $\mathcal{G}^e(\mathbb{R}^+ \times \mathbb{R}^n)$.

Proof . We shall prove only the fractional part since the entire part is already proved in the theorem (5.1). Consider fractional derivative D^β with $0 < \beta < 1$. Without loss of generality, the same holds for $m-1 < \beta < m$, $m \in \mathrm{N}^*$. Take the fractional derivative to the spatial variable to equation (5.1), we have

$$D^\beta(u_\varepsilon(t,x)) = \int_{\mathbb{R}^n} S_\varepsilon^\alpha(t,x-y)D^\beta u_{0,\varepsilon}(y)dy + \int_0^{\mathrm{t}} \int_{\mathbb{R}^n} S_\varepsilon^\alpha(t-\tau,x-y)D^\beta v_\varepsilon(y)u_\varepsilon(\tau,y)dyd\tau$$
$$+ \int_0^{\mathrm{t}} \int_{\mathbb{R}^n} S_\varepsilon^\alpha(t-\tau,x-y)v_\varepsilon(y)D^\beta u_\varepsilon(\tau,y)dyd\tau.$$

Then

$$\| D^\beta(u_\varepsilon(t,.))\|_{L^\infty(\mathbb{R}^n)} \leq \| S_\varepsilon^\alpha(t,x-.)\|_{L^1} \| D^\beta u_{0,\varepsilon}(.)\|_{L^\infty(\mathbb{R}^n)}$$
$$+ \ \| S_\varepsilon^\alpha(t-\tau,x-.)\|_{L^1} \int_0^{\mathrm{t}} \| D^\beta v_{\mathcal{E}}(.)\|_{L^\infty(\mathbb{R}^n)} \| u_{\mathcal{E}}(\tau,\ .)\|_{L^\infty(\mathbb{R}^n)} d\tau$$
$$+ \ \| S_\varepsilon^\alpha(t-\tau,x-.)\|_{L^1} \int_0^{\mathrm{t}} \| v_\varepsilon(y)\|_{L^\infty(\mathbb{R}^n)} \| D^\beta u_{\mathcal{E}}(\tau,\ .)\|_{L^\infty(\mathbb{R}^n)} d\tau,$$

thus

$$\| D^{\beta}(u_{\varepsilon}(t,.)) \|_{L^{\infty}(\mathbb{R}^n)} \leq C(\| D^{\beta}u_{0,\varepsilon}(.)\|_{L^{\infty}(\mathbb{R}^n)} + T\| D^{\beta}v_{\varepsilon}(.)\|_{L^{\infty}(\mathbb{R}^n)}\| u_{\varepsilon}(\tau, .)\|_{L^{\infty}}) + C\| v_{\varepsilon}(.)\|_{L^{\infty}(\mathbb{R}^n)}\int_0^t \| D^{\beta}u_{\varepsilon}(\tau, .)\|_{L^{\infty}(\mathbb{R}^n)}d\tau.$$

Apply the Gronwall inequality

$$\|D^{\beta}(u_{\varepsilon}(t,.)\|_{L^{\infty}(\mathbb{R}^n)} \leq C\|D^{\beta}u_{0,\varepsilon}(.)\|_{L^{\infty}(\mathbb{R}^n)} + C\Lambda\| D^{\beta}v_{\varepsilon}(.)\|_{L^{\infty}(\mathbb{R}^n)}\| u_{\varepsilon}\|_{L^{\infty}})\exp(CT\| v_{\varepsilon}(.)\|_{L^{\infty}(\mathbb{R}^n)}.$$

By Theorem (5.1) we get

$$\| D^{\beta}(u_{\varepsilon}(t,.)\|_{L^{\infty}(\mathbb{R}^n)} \leq C(C_{\alpha,\Lambda}|\ln\varepsilon|^{a(n+2)} + TC_{\alpha,\Lambda}|\ln\varepsilon|^{b(n+1)}\|u_{\varepsilon}\|_{L^{\infty}})\exp(C\Lambda\| v_{\varepsilon}(.)\|_{L^{\infty}(\mathbb{R}^n)}).$$

Then there exist $N > 0$ such that

$$\|D^{\beta}(u_{\varepsilon}(t,.)\|_{L^{\infty}(\mathbb{R}^n)} \leq C\varepsilon^{-N}.$$

It follows a moderation for the fractional derivatives in the space $\mathcal{G}^e(\mathbb{R}^+\times\mathbb{R}^n)$. For uniqueness take $D^{\beta}, 0 < \alpha < 1$, to equation ((5.2)).

$$\begin{aligned} D^{\beta}(u_{1,\varepsilon}(t,x) - u_{2,\varepsilon}(t,x)) = &\int_{\mathbb{R}^n} S_{\varepsilon}^{\alpha}(t,x-y)D^{\beta}N_{0,\varepsilon}(y)dy + \int_0^t\int_{\mathbb{R}^n} S_{\varepsilon}^{\alpha}(t-\tau,x-y)D^{\beta}v_{\varepsilon}(y)(u_{1\varepsilon}(\tau,y) - u_{2\varepsilon}(\tau,y))dyd\tau \\ &+ \int_0^t\int_{\mathbb{R}^n} S_{\varepsilon}^{\alpha}(t-\tau,x-y)v_{\varepsilon}(y)D^{\beta}(u_{1\varepsilon}(\tau,y) - u_{2\varepsilon}(\tau,y))dyd\tau \\ &+ \int_0^t\int_{\mathbb{R}^n} S_{\varepsilon}^{\alpha}(t-\tau,x-y)D^{\beta}N_{\varepsilon}(\tau,y)dyd\tau, \end{aligned}$$

we get

$$\begin{aligned} \| D^{\beta}(u_{1,\varepsilon}(t,.) - u_{2,\varepsilon}(t,.))\|_{L^{\infty}(\mathbb{R}^n)} \leq &\| S_{\varepsilon}^{\alpha}(t,x-.)\|_{L^1}\| D^{\beta}N_{0,\varepsilon}(.)\|_{L^{\infty}(\mathbb{R}^n)} + \| S_{\varepsilon}^{\alpha}(t-\tau,x-.)\|_{L^1} \\ &\times\int_0^t \| v_{\varepsilon}(.)\|_{L^{\infty}(\mathbb{R}^n)}\| D^{\beta}(u_{1\varepsilon}(\tau,.) - u_{2\varepsilon}(\tau,.))\|_{L^{\infty}(\mathbb{R}^n)}d\tau \\ &+ \| S_{\varepsilon}^{\alpha}(t-\tau,x-.)\|_{L^1}\times\int_0^t \| v_{\varepsilon}(.)\|_{L^{\infty}(\mathbb{R}^n)}\| D^{\beta}(u_{1\varepsilon}(\tau,.) - u_{2\varepsilon}(\tau,.)\|_{L^{\infty}(\mathbb{R}^n)}d\tau \\ &+ \| S_{\varepsilon}^{\alpha}(t-\tau,x-.)\|_{L^1}\int_0^t \| D^{\beta}N_{\varepsilon}(\tau, .)\|_{L^{\infty}(\mathbb{R}^n)}d\tau, \end{aligned}$$

then

$$\begin{aligned} | D^{\beta}(u_{1,\varepsilon}(t,.) - u_{2,\varepsilon}(t,.))|_{L^{\infty}(\mathbb{R}^n)} \leq &C(| (N_{0,\varepsilon}|_{L^{\infty}(\mathbb{R}^n)} + \Lambda\| D^{\beta}v_{\varepsilon}(.)\|_{L^{\infty}(\mathbb{R}^n)}\| u_{1,\varepsilon}(\tau, .) - u_{2,\varepsilon}(\tau, .)\|_{L^{\infty}}d\tau \\ &+ \| D^{\beta}N_{\varepsilon}(\tau,.)\|_{L^{\infty}}) + C\| v_{\varepsilon}(.)\|_{L^{\infty}(\mathbb{R}^n)}\times\int_0^t \| D^{\beta}(u_{1,\varepsilon}(\tau, .) - u_{2,\varepsilon}(\tau,.))\|_{L^{\infty}(\mathbb{R}^n)}d\tau. \end{aligned}$$

Gronwall inequality implies

$$\| D^{\beta}(u_{1,\varepsilon}(t, .) - u_{2,\varepsilon}(t,.))\|_{L^{\infty}(\mathbb{R}^n)} \leq C(\| N_{0,\varepsilon}\|_{L^{\infty}(\mathbb{R}^n)} + \Lambda\| D^{\beta}v_{\varepsilon}(.)\|_{L^{\infty}(\mathbb{R}^n)}\| u_{1,\varepsilon} - u_{2,\varepsilon}\|_{L^{\infty}}d\tau + \| D^{\beta}N_{\varepsilon}\|_{L^{\infty}})\exp(CT\| v_{\varepsilon}\|_{L^{\infty}(\mathbb{R}^n)}).$$

Finally, using theorem (5.1)

$$\| D^{\beta}(u_{1,\varepsilon}(t,.) - u_{2,\varepsilon}(t,.))\|_{L^{\infty}(\mathbb{R}^n)} \leq C\varepsilon^q,\ \forall q\in\mathbb{N}.$$

□

5.3 Association

Let w_1 be a solution to the problem

$$\frac{1}{i}\partial_{\mathrm{t}}^{\alpha} w_1(t,x) - \triangle w_1(t,x) = 0,$$
$$w_1(0,x) = \delta(x),$$

and w_2 be a solution of the problem

$$\frac{1}{i}\partial_t^{\alpha} w_2(t,x) - \triangle w_2(t,x) + v(x) w_{2,\varepsilon}(t,x) = 0,$$
$$v(x) = \delta(x)\ ,\ w_2(0,x) = 0.$$

Proposition 5.3. The generalized solution u of problem (5.1) is associated with $w_1 + w_2$.

Proof . Let $w_{1,\varepsilon}$ be the classical solution of

$$\frac{1}{i}\partial_{\mathrm{t}} w_{1,\varepsilon}(t,x) - \triangle w_{1,\varepsilon}(t,x) = 0,$$
$$w_{1,\varepsilon}(0,x) = \delta_\varepsilon(x),$$

$w_{2,\varepsilon}$ be the classical solution of

$$\frac{1}{i}\partial_{\mathrm{t}}^{\alpha} w_{2,\varepsilon}(t,x) - \triangle w_2(t,\ x) + v_\varepsilon(x)(w_{2,\varepsilon}(t,x) + m(t,x)) = 0,$$
$$v_\varepsilon(x) = \delta(x)\ ,\ w_{2,\varepsilon}(0,x) = 0.$$

Then

$$\frac{1}{i}\partial_t^{\alpha}(u_\varepsilon(t,x) - w_{1,\varepsilon}(t,x) - w_{2,\varepsilon}(t,x)) - \triangle(u_\varepsilon(t,x) - w_{1,\varepsilon}(t,x) - w_{2,\varepsilon}(t,x)) + v_\varepsilon(x)(u_\varepsilon(t,x) - w_{2,\varepsilon}(t,x) - m(t,x)) = 0,$$
$$u_\varepsilon(0,x) - w_{1,\varepsilon}(0,x) - w_{2,\varepsilon}(0,x) = 0,$$

Hence,

$$\begin{aligned} u_\varepsilon(t,x) - w_{1,\varepsilon}(t,x) - w_{2,\varepsilon}(t,x) &= \int_0^t \int_{\mathbb{R}^n} S_\varepsilon^{\alpha}(t-\tau, x-y) v_\varepsilon(y)(u_\varepsilon(\tau,y) - w_{2,\varepsilon}(\tau,y) - m(\tau,y)) dy d\tau \\ &= \int_0^t \int_{\mathbb{R}^n} S_\varepsilon^{\alpha}(t-\tau, x-y) v_\varepsilon(y)(u_\varepsilon(\tau,y) - w_{1,\varepsilon}(\tau,y) - w_{2,\varepsilon}(\tau,y)) dy d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}^n} S_\varepsilon^{\alpha}(t-\tau, x-y) v_\varepsilon(y)(w_{1,\varepsilon}(\tau,y) - m(\tau,y)) dy d\tau, \end{aligned}$$

which implies,

$$\| u_\varepsilon(t,\ .) - w_{1,\varepsilon}(t,\ .) - w_{2,\varepsilon}(t,\ .)\|_{\ L^\infty(\mathbb{R}^n)} \le \int_0^t \| S_\varepsilon^{\alpha}(t-\tau, x-.)\|_{\ L^1} \| v_\varepsilon(.)\|_{\ L^\infty(\mathbb{R}^n)} \| (w_{1,\varepsilon}(\tau,\ .) - m(\tau,\ .)\|_{\ L^\infty(\mathbb{R}^n)} d\tau$$
$$+\ \int_0^{\mathrm{t}} \| S_\varepsilon^{\alpha}(t-\tau, x-.)\|_{\ L^1} \| v_\varepsilon(.)\|_{\ L^\infty(\mathbb{R}^n)} \| u_\varepsilon(\tau,.) - w_{1,\varepsilon}(\tau,.) - w_{2,\varepsilon}(\tau,.)\|_{\ L^\infty(\mathbb{R}^n)} d\tau.$$

Then,

$$\| u_\varepsilon(t,\ .) - w_{1,\varepsilon}(t,\ .) - w_{2,\varepsilon}(t,\ .)\|_{\ L^\infty(\mathbb{R}^n)} \le C \| v_\varepsilon(.)\|_{\ L^\infty(\mathbb{R}^n)} \times \int_0^{\mathrm{t}} \| (w_{1,\varepsilon}(\tau,\ .) - m(\tau,\ .)\|_{\ L^\infty(\mathbb{R}^n)} d\tau$$
$$+\ C \| v_\varepsilon(.)\|_{\ L^\infty(\mathbb{R}^n)} \int_0^{\mathrm{t}} \| u_\varepsilon(\tau,\ .) - w_{1,\varepsilon}(\tau,\ .) - w_{2,\varepsilon}(\tau,\ .)\|_{\ L^\infty(\mathbb{R}^n)} d\tau.$$

With Gronwall inequality

$$\| u_\varepsilon(t,\ .) - w_{1,\varepsilon}(t,\ .) - w_{2,\varepsilon}(t,\ .)\|_{\ L^\infty(\mathbb{R}^n)} \le [C \| v_\varepsilon(.)\|_{\ L^\infty(\mathbb{R}^n)} \times \int_0^{\mathrm{t}} \| (w_{1,\varepsilon}(\tau,\ .) - m(\tau,\ .)\|_{\ L^\infty(\mathbb{R}^n)} d\tau]$$
$$\times \exp(\mathrm{CT}) \| \mathrm{v}(.)\|_{\ L^\infty(\mathbb{R}^n)}.$$

By passage to the limit, we have $u \approx w_1 + w_2$, which completes the proof of the proposition. □

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