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L-normal fuzzy filters of a distributive lattice

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Abstract

In this paper, we study the concept of L-normal fuzzy filters of distributive lattices in terms of annulates. We also study the special class of fuzzy filters called L-normal fuzzy filters. Further, a set of equivalent conditions are derived for an L-fuzzy filter to be an L-normal fuzzy filter, which leads to a characterization of disjunctive lattices. We observe that every L-normal fuzzy filter is the intersection of all L-prime normal fuzzy filters containing it. Finally, we study the space of L-prime normal fuzzy filters and investigate some of their properties.

Keywords: Distributive lattice, normal filter, L-fuzzy filter, L-normal fuzzy filter, L-prime normal fuzzy filter
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1 Introduction

In 1970, M. Mandelker [14] introduced the concept of annihilators as a natural generalization of relative pseudo-complement. He also characterized distributive lattices with the help of these annihilators. Latter, properties of annihilators extensively studied by many authors, particularly T.P Speed in the papers [21] and [22]. In 1973, W. Cornish [7], studied the concept of annulets and α-ideals in a distributive lattice with least element "0" and he characterized α-ideals in terms of annulates. In [19], M. S. Rao studied the concept of normal filters of distributive lattices with the help of annihilators. He characterized disjunctive lattice in terms of normal filters.

Fuzzy set theory was guided by the assumption that classical sets were not natural, appropriate or useful notions in describing the real life problems, because every object encountered in this real physical world carries some degree of fuzziness. A lot of work on fuzzy sets has come into being with many applications to various fields such as computer science, artificial intelligence, expert systems, control systems, decision making, medical diagnosis, management science, operations research, pattern recognition, neural network and others (see [9, 15, 24, 26]). Many papers on fuzzy algebra have been published since Rosenfeld [20] introduced the concept of fuzzy subgroup. In particular, Rajesh Kumar, Kumbhojkar and Hadji-Abadi studied the space of prime fuzzy ideals of a ring (see [10, 12, 13]). C. S. S. Raj and K. E. Belleza studied prime fuzzy filter and dual B-filters of a semilattice and B-algebras respectively (see [4, 18]). In [1], B. A. Alaba and W. Z. Norahun introduced the concept of α-fuzzy ideals and the space of prime α-fuzzy ideals in distributive lattices. Latter, B. A. Alaba and T. G. Alemayehu studied e-fuzzy filters and the space of prime e-fuzzy

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filters in MS-algebras [2]. In 2020, W. Z. Norahun introduced the concept of μ-fuzzy filters and the space of prime μ-fuzzy filters in distributive lattices [16].

In this paper, we introduced the concept of L-normal fuzzy filters in a distributive lattice in terms of annihilators and investigate some of its properties. We provide a set of equivalent conditions for an L-fuzzy filter to be an L-normal fuzzy filter, which leads to a characterization of disjunctive lattices. Furthermore, for any L-fuzzy filter of Q, we observe that there exists the smallest L-normal fuzzy filter containing it. We also study the special class of L-fuzzy filters called L-normal fuzzy filters. It is proved that the class of all L-normal fuzzy filters forms a distributive lattice. The concept of an L-prime normal fuzzy filters also studied and we proved that every maximal L-fuzzy filter is an L-normal fuzzy filter. An equivalent condition is derived for the class of L-prime normal fuzzy filters coincides over the class of L-prime fuzzy filters. Moreover, we observe that every L-normal fuzzy filter is the intersection of all L-prime normal fuzzy filters containing it. Finally, we study the space of L-prime normal fuzzy filters in distributive lattice and investigate some of its properties. We prove that the space of L-prime normal fuzzy filters is a T_0 space.

2 Preliminaries

We refer to G. Birkhoff [5] for the elementary properties of lattices.

Definition 2.1. [22] Let Q be a distributive lattice with 0 and $B \subseteq Q$. The annihilator of B is defined as:

$$B^* = \{y \in Q : b \wedge y = 0 \text{ for all } b \in B\}.$$

is an ideal of Q.

If $B = \{b\}$, then $B^* = (b]^*$. An element $a \in Q$ is said to be dense if $(a]^* = \{0\}$ and the set of all dense elements denoted by D.

Definition 2.2. [19] Let Q be a distributive lattice with 0 and F be a filter of Q. The set F^+ defined as:

$$F^+ = \{y \in Q : (y]^* \subseteq (b]^* \text{ for some } b \in F\}$$

is a filter containing F. Let $a \in Q$. Then $[a)^+ = \{y \in Q : (y]^* \subseteq (a]^*\}$.

Definition 2.3. [19] A filter F of Q is said to be a normal filter if $[a)^+ \subseteq F$ for all $a \in F$.

Lemma 2.4. [19] Let Q be a distributive lattice with 0 and $x, y \in Q$. Then

1. $x \leq y \Rightarrow [y)^+ \subseteq [x)^+$.
2. $x \in [y)^+ \Rightarrow [x)^+ \subseteq [y)^+$.
3. $[x)^+ \cap [y)^+ = [x \vee y)^+$.
4. $(x]^* = (x]^* \Leftrightarrow [x)^+ = [y)^+$.

For any $x \in Q$, $[x)^+$ is called a normulate and the set of normulates is denoted by $\mathcal{N}(Q)$. For any two normulates $[x)^+$ and $[y)^+$ there supremum and infimum in $\mathcal{N}$ are

$$[x)^+ \underline{\vee} [y)^+ = [x \wedge y)^+ \text{ and } [x)^+ \cap [y)^+ = [x \vee y)^+$$

respectively. A lattice $(Q, \wedge, \vee)$ is said to be a complete lattice satisfying the infinite meet distributive law if for any $a \in Q$ and $A \subseteq Q$,

$$a \wedge \bigvee_{x \in A} x = \bigwedge_{x \in A} (a \wedge x).$$

Definition 2.5. [8] Let X be a non-empty set and let L be a complete lattice satisfying the infinite meet distributive law. An L-fuzzy subset λ of X is a mapping from X into L.

Note that if L is a unit interval of real numbers, then λ is the usual fuzzy subset of X originally introduced by L. Zadeh [25].

Definition 2.6. [23] An L-fuzzy subset λ of a lattice Q is called an L-fuzzy filter of Q if, for all $x, y \in L$ the following condition satisfies:

1. $\lambda(1) = 1$,
2. $\lambda(x \vee y) \geq \lambda(x) \vee \lambda(y)$,
3. $\lambda(x \wedge y) \geq \lambda(x) \wedge \lambda(y)$.

The binary operation "+" on the set of all L-fuzzy subsets of a distributive lattice Q as:

$$(\lambda + \eta)(x) = Sup\{\lambda(y) \wedge \eta(z) : y, z \in Q,\ y \vee z = x\}$$

If λ and η are L-fuzzy filters of Q, then $\lambda + \eta = \lambda \wedge \eta$.

Lemma 2.7. [17] For any two L-fuzzy subsets λ and η of a distributive lattice Q, we have

$$[\lambda + \eta) = [\lambda) \wedge [\eta).$$

Throughout the rest of this paper Q stands for a distributive lattice with least element 0 unless otherwise mentioned. The set of all fuzzy filters of Q is denoted by $FF(Q)$.

3 L-Normal fuzzy filters

In this section, we introduce the concept of L-normal fuzzy filter in distributive lattice in terms of annihilators and investigate some of its properties. We provide a set of equivalent conditions for an L-fuzzy filter to be an L-normal fuzzy filter and disjunctive lattices characterized in terms of L-normal fuzzy filters. We also prove that the class of L-normal fuzzy filters forms a distributive lattice. Finally, we prove that an L-normal fuzzy filter is the intersection of all L-prime normal fuzzy filters containing it.

Definition 3.1. An L-fuzzy filter λ of Q is called an L-normal fuzzy filter, if:

$$\bigwedge_{a \in (x)^+} \lambda(a) \geq \lambda(x) \text{ for all } x \in Q.$$

Theorem 3.2. An L-fuzzy subset λ of Q is an L-normal fuzzy filter if and only λ_α is a normal filter of Q for all $\alpha \in L$.

Proof . Suppose λ is an L-normal fuzzy filter of Q. Let $\alpha \in L$. Then λ_α is a filter of Q. Let $x \in \lambda_\alpha$. Then $\bigwedge_{a \in (x)^+} \lambda(a) \geq \lambda(x) \geq \alpha$. Which implies that $\lambda(a) \geq \alpha$ for all $a \in (x)^+$. Thus $(x)^+ \subseteq \lambda_\alpha$ and hence λ_α is a normal filter for all $\alpha \in L$. Conversely, suppose that λ_α is a normal filter of Q for all $\alpha \in L$ and $\lambda(x) = \alpha$. Then $(x)^+ \subseteq \lambda_\alpha$. Which implies $a \in \lambda_\alpha$ for all $a \in (x)^+$. Thus $\bigwedge_{a \in (x)^+} \lambda(a) \geq \alpha = \lambda(x)$ and hence λ is an L-normal fuzzy filter of Q. □

Corollary 3.3. A filter F of Q is a normal filter if and only if χ_F is an L-normal fuzzy filter.

Theorem 3.4. For any L-fuzzy filter λ of Q, the L-fuzzy subset $\lambda^\perp$ of Q defined by:

$$\lambda^\perp(x) = Sup\{\lambda(a) : (x]^* = (a]^*, a \in Q\}$$

is an L-fuzzy filter.

Proof . Clearly $\lambda^\perp(1) = 1$. Let $x, y \in Q$. Then

$$\begin{aligned}
\lambda^\perp(x) \wedge \lambda^\perp(y) &= Sup\{\lambda(a) : (x]^* = (a]^*\} \wedge Sup\{\lambda(b) : (y]^* = (b]^*\} \\
&= Sup\{\lambda(a) \wedge \lambda(b) : (x]^* = (a]^*,\ (y]^* = (b]^*\} \\
&= Sup\{\lambda(a \wedge b) : (x]^* = (a]^*,\ (y]^* = (b]^*\}.
\end{aligned}$$

If $(x]^* = (a]^*$ and $(y]^* = (b]^*$, then $(x \wedge y]^* = (a \wedge b]^*$. Using this fact we have

$$\begin{aligned} \lambda^{\perp}(x) \wedge \lambda^{\perp}(y) &\leq Sup\{\lambda(a \wedge b) : (x \wedge y]^* = (a \wedge b]^*\} \\ &\leq Sup\{\lambda(c) : (x \wedge y]^* = (c]^*, c \in L\} \\ &= \lambda^{\perp}(x \wedge y). \end{aligned}$$

On the other hand, $\lambda^{\perp}(x) = Sup\{\lambda(a) : (x]^* = (a]^*\} \leq Sup\{\lambda(a \vee y) : (x \vee y]^* = (a \vee y]^*\} = \lambda^{\perp}(x \vee y)$. Similarly, $\lambda^{\perp}(y) \leq \lambda^{\perp}(x \vee y)$. Thus $\lambda^{\perp}(x \vee y) \geq \lambda^{\perp}(x) \vee \lambda^{\perp}(y)$ and hence $\lambda^{\perp}$ is an L-fuzzy filter of Q. □

Theorem 3.5. For any L-fuzzy filter λ of Q, $\lambda^{\perp}$ is the smallest L-normal fuzzy filter containing λ.

Proof . Clearly $\lambda \subseteq \lambda^{\perp}$. Let $x \in Q$. Then

$$\begin{aligned} \bigwedge_{a \in (x)^+} \lambda^{\perp}(a) &= \bigwedge_{a \in (x)^+} (\bigvee\{\lambda(b) : (a]^* = (b]^*\}) \\ &= \bigwedge_{a \in (x)^+} (\bigvee\{\lambda(b) : (a)^+ = (b)^+ \subseteq (x)^+\}) \\ &\geq \bigwedge_{a \in (x)^+} (\bigvee\{\lambda(b) : (b)^+ = (x)^+\}) \\ &= \bigwedge_{a \in (x)^+} \lambda^{\perp}(x) \\ &= \lambda^{\perp}(x) \end{aligned}$$

Thus $\bigwedge_{a \in (x)^+} \lambda^{\perp}(a) \geq \lambda^{\perp}(x)$ and hence $\lambda^{\perp}$ is an L-normal fuzzy filter.

To show $\lambda^{\perp}$ is the smallest L-normal fuzzy filter, let θ be any L-normal fuzzy filter containing λ. Then

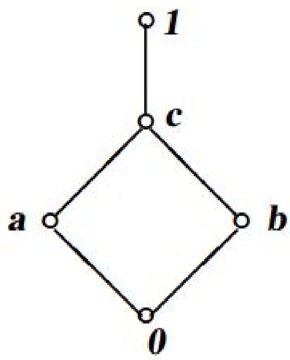
$$\begin{aligned} \lambda^{\perp}(x) &= Sup\{\lambda(a) : (x]^* = (a]^*\} \\ &= Sup\{\lambda(a) : (x)^+ = (a)^+\} \\ &\leq Sup\{\theta(a) : (x)^+ = (a)^+\} \end{aligned}$$

Let $(x)^+ = (a)^+$. Then $x \in (a)^+$ and $a \in (x)^+$. Since θ is an L-normal fuzzy filter, we get $\theta(a) = \theta(x)$. Which implies that $\lambda^{\perp}(x) \leq \theta(x)$. Therefore, $\lambda^{\perp}$ is the smallest L-normal fuzzy filter containing λ. □

Theorem 3.6. An L-fuzzy filter λ of Q is an L-normal fuzzy filter if and only if $\lambda = \lambda^{\perp}$.

Proof . Suppose λ is an L-normal fuzzy filter. Clearly, $\lambda \subseteq \lambda^{\perp}$. Since $\lambda^{\perp}$ is a normal fuzzy filter, by Theorem 3.5, $\lambda^{\perp} \subseteq \lambda$. Thus $\lambda^{\perp} = \lambda$. The converse is trivial. □

Example 3.7. Consider the distributive lattice $Q = \{0, a, b, c, 1\}$ whose Hasse diagram is given below.



Define a fuzzy subsets λ of Q as:

$$\lambda(c) = \lambda(1) = 1,\ \lambda(a) = 0.6,\ \lambda(b) = \lambda(0) = 0.3.$$

Then it can be easily verified that λ is an L-normal fuzzy filter of Q.

Theorem 3.8. Let λ be an L-fuzzy filter of Q. Then

$$[\lambda^{\perp}]_{\alpha} = \bigcup\{\bigcap_{\gamma\in M}[\lambda_{\gamma}]^{\perp} : \alpha \leq \bigvee M, M \subseteq L\}$$

.

Proof . Let $A = \bigcup\{\bigcap_{\gamma\in M}[\lambda_{\gamma}]^{\perp} : \alpha \leq \bigvee M, M \subseteq L\}$ and $B = [\lambda^{\perp}]_{\alpha}$. To show $B \subseteq A$, let $x \in B$. Then $\alpha \leq Sup\{\lambda(a) : (a]^{*} = (x]^{*}\}$. Let us put $M = \{\lambda(a) : (a]^{*} = (x]^{*}\}$. Then $M \subseteq L$ and $\alpha \leq \bigvee M$. If $\beta \in M$, then $\beta = \lambda(b)$ for some $b \in Q$ such that $(b]^{*} = (x]^{*}$. Which implies that $b \in \lambda_{\beta}$ and $(b]^{*} = (x]^{*}$. Thus $x \in (\lambda_{\beta})^{\perp}$ for all $\beta \in M$. Which implies $x \in \bigcap_{\beta\in M}(\lambda_{\beta})^{\perp}$. Hence $B \subseteq A$. Conversely, let $x \in A$. Then there is $M \subseteq L$ with $\alpha \leq \bigvee M$ and $x \in \bigcap_{\beta\in M}(\lambda_{\beta})^{\perp}$. Which implies for each $\beta \in M$ there is $b \in \lambda_{\beta}$ such that $(x]^{*} = (b]^{*}$. We know that $Sup\{\lambda(a) : (x]^{*} = (a]^{*}\} \geq \lambda(a)$. Thus $\lambda^{\perp}(x) \geq \lambda(b)$ for all $b \in \lambda_{\beta}, \beta \in M$. So $\lambda^{\perp}(x) \geq \bigvee M \geq \alpha$ and $x \in B$. Hence $A \subseteq B$ and therefore $A = B$. □

For any L-fuzzy filter λ of Q, the map $\lambda \longrightarrow \lambda^{\perp}$ is a closure operator on $FF(L)$ and the closed elements are L-normal fuzzy filters.

Lemma 3.9. For any L-fuzzy filter λ and γ of Q, we have the following:

1. $\lambda \subseteq \gamma \Rightarrow \lambda^{\perp} \subseteq \gamma^{\perp}$.
2. $\lambda^{\perp\perp} = \lambda^{\perp}$.
3. $(\lambda \cap \gamma)^{\perp} = \lambda^{\perp} \cap \gamma^{\perp}$.
4. $(\lambda \vee \gamma)^{\perp} = (\lambda^{\perp} \vee \gamma^{\perp})^{\perp}$.
5. $\lambda^{\perp} \vee \gamma^{\perp} \subseteq (\lambda \vee \gamma)^{\perp}$.

Proposition 3.10. Every maximal L-fuzzy filter is an L-normal fuzzy filter.

Proof . Let λ be a maximal L-fuzzy filter. Then λ is two valued. Which implies $\lambda^{\perp}$ also two valued and $\lambda \subseteq \lambda^{\perp}$. By the maximality of λ, we get $\lambda = \lambda^{\perp}$. Hence λ is an L-normal fuzzy filter. □

The class of L-normal fuzzy filters is denoted by $FF_N(Q)$.

Lemma 3.11. If $\lambda, \gamma \in FF_N(Q)$, then the supremum of λ and γ is given by:

$$\lambda \underline{\vee} \gamma = (\lambda \vee \gamma)^{\perp}.$$

Proof . Clearly $\lambda \underline{\vee} \gamma$ is an L-normal fuzzy filter of Q. We need to show $(\lambda \vee \gamma)^{\perp}$ is the least upper bound of $\{\lambda,\ \gamma\}$. Since $\lambda,\ \gamma \subseteq \lambda \vee \gamma$, we have $\lambda \vee \gamma \subseteq (\lambda \vee \gamma)^{\perp}$. Thus $(\lambda \vee \gamma)^{\perp}$ is an upper bound of $\{\lambda,\ \gamma\}$. Let η be any upper bound for $\lambda,\ \gamma$ in $FF_N(L)$. Then $(\lambda \vee \gamma)^{\perp} \subseteq \eta$. Thus $(\lambda \vee \gamma)^{\perp}$ is the supremum of $\{\lambda, \gamma\}$. □

Theorem 3.12. The set $FF_N(Q)$ of Q forms a distributive lattice with respect to inclusion ordering of fuzzy sets.

Proof . Clearly $(FF_N(Q), \subseteq)$ is a partially ordered set. For $\lambda, \gamma \in FF_N(Q)$, define

$$\lambda \wedge \gamma = \lambda \cap \gamma \text{ and } \lambda \underline{\vee} \gamma = (\lambda \vee \gamma)^{\perp}.$$

Then clearly $\lambda \wedge \gamma,\ (\lambda \vee \gamma)^{\perp} \in FF_N(Q)$. Hence $(FF_N(Q), \wedge, \underline{\vee})$ is a lattice. We now prove the distributivity. Let $\gamma,\ \nu,\ \lambda \in FF_N(Q)$. Then

$$\begin{aligned} \gamma \underline{\vee} (\nu \cap \lambda) &= (\gamma \vee (\nu \cap \lambda))^{\perp} \\ &= ((\gamma \vee \nu) \cap (\gamma \vee \lambda))^{\perp} \\ &= (\gamma \vee \nu)^{\perp} \cap (\gamma \vee \lambda)^{\perp} \\ &= (\gamma \underline{\vee} \nu) \cap (\gamma \underline{\vee} \lambda) \end{aligned}$$

Therefore, $(FF_N(Q), \wedge, \underline{\vee})$ is a distributive lattice. □

Theorem 3.13. There is an epimorphism from $FF(Q)$ onto $FF_N(Q)$.

Proof . Let $\lambda, \gamma \in FF_N(Q)$. Define a map $f : FF(Q) \longrightarrow FF_N(Q)$ by $f(\lambda) = \lambda^{\perp}$. Then $f(\lambda \cap \gamma) = \lambda^{\perp} \cap \gamma^{\perp} = f(\lambda) \cap f(\gamma)$ and $f(\lambda \vee \gamma) = (\lambda \vee \gamma)^{\perp} = (\lambda^{\perp} \vee \gamma^{\perp})^{\perp} = \lambda^{\perp} \underline{\vee} \gamma^{\perp} = f(\lambda) \underline{\vee} f(\gamma)$. Hence f is a homomorphism. Since $FF_N(Q) \subseteq FF(Q)$, f is an onto homomorphism. □

Theorem 3.14. Let λ be an L-fuzzy filter of Q. Then λ is an L-normal fuzzy filter if and only if for each $x, y \in Q$, $(x]^* = (y]^*$ imply $\lambda(x) = \lambda(y)$.

Corollary 3.15. Let λ be an L-fuzzy filter of Q. Then λ is an L-normal fuzzy filter if and only if for each $x, y \in Q$, $(x)^+ = (y)^+$ imply $\lambda(x) = \lambda(y)$.

Theorem 3.16. Let λ be an L-fuzzy filter of Q and for any chain of L-fuzzy filters $\eta_1 \subseteq \eta_2 \subseteq \eta_3 \subseteq \eta_4, ...$ of Q such that $\lambda \subseteq \eta_1 \subseteq \eta_2 \subseteq \eta_3 \subseteq \eta_4, ... \subseteq \lambda^{\perp}$. Then $\eta_1^{\perp} = \eta_2^{\perp} = \eta_3^{\perp} = \eta_4^{\perp} = ... = \lambda^{\perp}$.

Proof . Suppose λ is an L-fuzzy filter of Q and for any chain of L-fuzzy filters $\eta_1 \subseteq \eta_2 \subseteq \eta_3 \subseteq \eta_4, ...$ of Q such that $\lambda \subseteq \eta_1 \subseteq \eta_2 \subseteq \eta_3 \subseteq \eta_4, ... \subseteq \lambda^{\perp}$. Then $\lambda^{\perp} \subseteq \eta_1^{\perp} \subseteq \eta_2^{\perp} \subseteq \eta_3^{\perp} = \eta_4^{\perp} \subseteq ... \subseteq \lambda^{\perp\perp} = \lambda^{\perp}$. Therefore, $\eta_1^{\perp} = \eta_2^{\perp} = \eta_3^{\perp} = \eta_4^{\perp} = ... = \lambda^{\perp}$. □

Theorem 3.17. The following are equivalent for each non-constant L-normal fuzzy filter λ of Q.

1. For all $\mu, \eta \in FF(Q)$,
$$\mu \cap \eta \subseteq \lambda \Rightarrow \mu \subseteq \lambda \text{ or } \eta \subseteq \lambda.$$
2. For any fuzzy points a_α and b_β of Q,
$$a_\alpha + b_\beta \subseteq \lambda \Rightarrow a_\alpha \subseteq \lambda \text{ or } b_\beta \subseteq \lambda.$$
3. For all $\mu, \eta \in FF_N(Q)$,
$$\mu \cap \eta \subseteq \lambda \Rightarrow \mu \subseteq \lambda \text{ or } \eta \subseteq \lambda.$$

Proof . $1 \Rightarrow 2$: Let $a, b \in Q$ such that $a_\alpha + b_\beta \subseteq \lambda$. Then $[a_\alpha + b_\beta) \subseteq \lambda$ and by Lemma 2.7 $[a_\alpha) \cap [b_\beta) \subseteq \lambda$. By the assumption, $[a_\alpha) \subseteq \lambda$ or $[b_\beta) \subseteq \lambda$. Which implies $a_\alpha \subseteq \lambda$ or $b_\beta \subseteq \lambda$.

$2 \Rightarrow 3$: Let $\mu, \eta \in FF_N(Q)$ such that $\mu \cap \eta \subseteq \lambda$. Suppose $\mu \nsubseteq \lambda$ or $\eta \nsubseteq \lambda$. Then there exist $a, b \in Q$ such that $\mu(a) > \lambda(a)$ and $\eta(b) > \lambda(b)$. Put $\mu(a) = \alpha$ and $\eta(b) = \beta$. Then $a_\alpha \nsubseteq \lambda$ and $b_\beta \nsubseteq \lambda$. Since $a_\alpha + b_\beta \subseteq \mu \cap \eta \subseteq \lambda$, by the assumption, we get that $a_\alpha \subseteq \lambda$ or $b_\beta \subseteq \lambda$. Which is a contradiction. Thus $\mu \subseteq \lambda$ or $\eta \subseteq \lambda$.

$3 \Rightarrow 1$: Suppose that $\mu, \eta \in FF(Q)$ such that $\mu \cap \eta \subseteq \lambda$. Then by Lemma 3.9 we have $\mu^{\perp} \cap \eta^{\perp} \subseteq \lambda$. Since $\mu^{\perp}$ and $\eta^{\perp}$ are L-normal fuzzy filters, by the assumption, $\mu^{\perp} \subseteq \lambda$ or $\eta^{\perp} \subseteq \lambda$. Thus $\mu \subseteq \lambda$ or $\eta \subseteq \lambda$. □

Definition 3.18. A non-constant L-normal fuzzy filter is called L-prime normal fuzzy filter, if any one of the equivalent in Theorem 3.17 holds.

A lattice Q is disjunctive if and only if $(a]^* = (b]^*$ implies $a = b$ for any $a, b \in Q$ [6]. In the following theorem, we establish set of equivalent conditions for an L-fuzzy filter to be an L-normal fuzzy ideal

Theorem 3.19. The following are equivalent.

1. Each L-fuzzy filter is an L-normal fuzzy filter,
2. Each L-prime fuzzy filter is an L-normal fuzzy filter,
3. Q is disjunctive.

Proof . The proof of $1 \Rightarrow 2$ and $3 \Rightarrow 1$ are Straightforward. Now we proceed to prove $2 \Rightarrow 3$. Suppose condition (3) holds and $(a]^* = (b]^*$, $a, b \in Q$. Assume that $a \neq b$. Without loss of generality, we can assume that $(a] \cap [b) = \phi$. Which implies $\chi_{(a]} \cap \chi_{[b)} = 0$ and there exists an L-prime fuzzy filter θ of Q such that $\chi_{[b)} \subseteq \theta$ and $\theta \cap \chi_{(a]} = 0$. Which implies that $\theta(b) = 1$ and $\theta(a) \wedge \chi_{(a]}(a) = 0$. Thus $\theta(a) = 0$. This is a contradiction. Hence Q is disjunctive. □

The above theorem shows that, the class of L-prime normal fuzzy filters of Q coincides with the class of L-prime fuzzy filters of Q if and only if Q is disjunctive. Otherwise, they are different.

Theorem 3.20. Let μ be an L-normal fuzzy filter of Q and η be an L-fuzzy ideal of Q such that $\mu \cap \eta \leq \alpha$, $1 \neq \alpha \in L$. Then there exists an L-prime normal fuzzy filter λ of Q such that $\mu \subseteq \lambda$ and $\lambda \cap \eta \leq \alpha$.

Proof . Consider $\Gamma = \{ \gamma \in FF_N(Q) : \mu \subseteq \gamma \texttt{ and } \eta \cap \gamma \leq \alpha\}$. Since $\mu \in \Gamma$, Γ is nonempty and it forms a poset together with the inclusion ordering of fuzzy sets. Let $\mathcal{P} = \{\mu_i\}_{i\in I}$ be any chain in Γ. Clearly $\bigcup_{i\in I} \mu_i$ is an L-normal fuzzy filter and $(\bigcup_{i\in I} \mu_i) \cap \eta \leq \alpha$. Which implies $\bigcup_{i\in I} \mu_i \in \Gamma$. By applying Zorn's lemma we get a maximal element, let say $\gamma \in \Gamma$; that is γ is an L-normal fuzzy filter of L such that $\mu \subseteq \gamma$ and $\gamma \cap \eta \leq \alpha$.

Now we proceed to show γ is an L-prime fuzzy filter. Assume that γ is not L-prime fuzzy filter. Let $\gamma_1 \cap \gamma_2 \subseteq \gamma$ such that $\gamma_1 \nsubseteq \gamma$ and $\gamma_2 \nsubseteq \gamma$, $\gamma_1,\ \gamma_2 \in FF(Q)$. If we put $\nu_1 = (\gamma_1 \vee \gamma)^{\perp}$ and $\nu_2 = (\gamma_2 \vee \gamma)^{\perp}$, then both ν_1 and ν_2 are L-normal fuzzy filters of L properly containing γ. Since γ is maximal in Γ, we get $\nu_1, \nu_2 \notin \Gamma$. Thus $\gamma_1 \cap \eta \nleq \alpha$ and $\gamma_2 \cap \eta \nleq \alpha$. This implies there exist $x, y \in L$ such that $(\gamma_1 \cap \eta)(x) > \alpha$ and $(\gamma_2 \cap \eta)(y) > \alpha$. Which implies $((\gamma_1 \cap \gamma_2) \cap \eta)(x \wedge y) > \alpha \Rightarrow ((\gamma \vee (\gamma_1 \wedge \gamma_2))(x \wedge y) \wedge \eta(x \wedge y) > \alpha$. This shows that $(\mu \cap \eta)(x \wedge y) > \alpha$. This is a contradiction. Thus γ is L-prime normal fuzzy filter of Q. □

Corollary 3.21. Let λ be an L-normal fuzzy filter of Q, $a \in Q$ and $1 \neq \alpha \in L$. If $\lambda(a) \leq \alpha$, then there exists an L-prime normal fuzzy filter η of Q such that $\lambda \subseteq \eta$ and $\eta(a) \leq \alpha$.

Corollary 3.22. Any L-normal fuzzy filter of Q is the intersection of all L-prime normal fuzzy filters containing it.

Corollary 3.23. The intersection of all L-prime normal fuzzy filters is equal to χ_D.

Proof . Since $\chi_D = \chi_{[1)^+} \subseteq \lambda$, we have $\chi_D \subseteq \lambda$ for all L-normal fuzzy filter. By Corollary 3.22, χ_D is the intersection of all L-prime normal fuzzy filters. □

4 The space of *L*-prime normal fuzzy filters

In this section, we study the space of L-prime normal fuzzy filters in distributive lattice and investigate some of its properties. We prove that the space of L-prime normal fuzzy filters is a T_0 space.

Let us denote SpecQ the set of L-prime normal fuzzy filters of Q. For any L-fuzzy subset η of Q, $\xi(\eta) = \{\lambda \in SpecQ : \eta \nsubseteq \lambda\}$ and $\zeta(\eta) = \{\lambda \in SpecQ : \eta \subseteq \lambda\} = SpecQ - \xi(\eta)$. We let $\lambda_* = \lambda_1$, i.e., $\lambda_* = \{x \in Q : \lambda(x) = 1\}$.

We refer to L. Kelley [11] for topological concepts.

Lemma 4.1. Let λ and γ be L-fuzzy filters of Q. Then

1. $\lambda \subseteq \gamma \Rightarrow \xi(\eta) \subseteq \xi(\nu)$,
2. $\xi(\lambda \vee \gamma) = \xi(\lambda) \cup \xi(\gamma)$,
3. $\xi(\lambda \cap \gamma) = \xi(\lambda) \cap \xi(\gamma)$.

Lemma 4.2. For any L-fuzzy subset λ of Q, $\xi(\lambda) = \xi([\lambda))$.

Proof . Clearly $\xi(\lambda) \subseteq \xi([\lambda))$. To show the other inclusion, let $\nu \in \xi([\lambda))$. Then $[\lambda) \nsubseteq \nu$. We need to show $\lambda \nsubseteq \nu$. Suppose that $\lambda \subseteq \nu$. Then $[\lambda) \subseteq \nu$. Which is a contradiction. Thus $\lambda \nsubseteq \nu$ and hence $\xi(\lambda) = \xi([\lambda))$. □

Theorem 4.3. Let $a, b \in Q$ and $0 \neq \beta \in L$. Then

1. $\xi((a \wedge b)_\beta) = \xi(a_\beta) \cup \xi(b_\beta)$,
2. $\xi((a \vee b)_\beta) = \xi(a_\beta) \cap \xi(b_\beta)$,
3. $\bigcup_{a\in Q,\ \beta\in L} \xi(a_\beta) = SpecQ$.

Proof . (1) Let $\lambda \in \xi((a \wedge b)_\beta)$. Then $\beta > \lambda(a \wedge b) = \lambda(a) \wedge \lambda(b)$. Since λ is prime, $\beta > \lambda(a)$ or $\beta > \lambda(b)$. Which implies that $a_\beta \nsubseteq \lambda$ or $b_\beta \nsubseteq \lambda$. Hence $\lambda \in \xi(a_\beta) \cup \xi(b_\beta)$. Conversely, suppose $\lambda \in \xi(a_\beta) \cup \xi(b_\beta)$. Then $\beta > \lambda(a)$ or $\beta > \lambda(b)$. Which implies $\beta > \lambda(a \wedge b)$. Thus $(a \wedge b)_\beta \nsubseteq \lambda$ and hence $\lambda \in \xi((a \wedge b)_\beta)$.

(2) If $\lambda \in \xi(a_\beta) \cap \xi(b_\beta)$, then $a_\beta \nsubseteq \lambda$ and $b_\beta \nsubseteq \lambda$. Which implies $\beta > \lambda(a)$ and $\beta > \lambda(b)$. This shows that $a, b \notin \lambda_*$. Since λ is prime, λ_* is prime. Which implies $a \vee b \notin \lambda_*$. Thus $(a \vee b)_\beta \nsubseteq \lambda$ and hence $\lambda \in \xi((a \vee b)_\beta)$. Conversely, let $\lambda \in \xi((a \vee b)_\beta)$. Then $(a \vee b)_\beta \nsubseteq \lambda$. Which implies $\beta > \lambda(a) \vee \lambda(b)$. Thus $\beta > \lambda(a)$ and $\beta > \lambda(b)$. This shows that $a_\beta \nsubseteq \lambda$ and $b_\beta \nsubseteq \lambda$. Thus $\lambda \in \xi(a_\beta) \cap \xi(b_\beta)$.

(3) Let $\lambda \in SpecQ$. Then $Im\lambda = \{1, \beta\},\ 0 \neq \beta \in L$. This implies there is $a \in Q$ such that $\lambda(a) = \beta$. Let $\gamma \in L$ such that $\gamma > \beta$. Then $a_\gamma \nsubseteq \lambda$. Thus $SpecQ \subseteq \bigcup_{a\in Q,\ \beta\in L} \xi(a_\beta)$ and hence $SpecQ = \bigcup_{a\in Q,\ \beta\in L} \xi(a_\beta)$. □

Lemma 4.4. Let $0 \neq \beta_1,\ \beta_2 \in L;\ \beta = \beta_1 \wedge \beta_2$ and $a,\ b \in Q$. Then

$$\xi(a_{\beta_1}) \cap \xi(b_{\beta_2}) = \xi((a \vee b)_\beta).$$

Lemma 4.5. Let $\{\eta_j : j \in \Delta\}$ be any family of L-fuzzy filters of Q. Then

$$\bigcap_{j\in\Delta} \zeta(\eta_j) = \zeta([\bigcup_{j\in\Delta} \eta_j)).$$

Proof . Clearly $\zeta([\bigcup_{j\in\Delta}\eta_j)) \subseteq \bigcap_{j\in\Delta}\zeta(\eta_j)$. To show the other inclusion, let $\gamma \in \bigcap_{j\in\Delta}\zeta(\eta_j)$. Then $\eta_j \subseteq \gamma$ for all $j \in \Delta$ and $\gamma(a)$ is an upper bound of $\{\eta_j(a) : j \in \Delta\}$ for all $a \in Q$. Which implies that $[\bigcup_{j\in\Delta}\eta_j) \subseteq \gamma$. Thus $\gamma \in \zeta([\bigcup_{j\in\Delta}\eta_j))$ and hence $\bigcap_{j\in\Delta}\zeta(\eta_j) \subseteq \zeta([\bigcup_{j\in\Delta}\eta_j))$. □

Theorem 4.6. The collection $\mathcal{T} = \{\xi(\lambda) : \lambda \textit{ is an L-fuzzy filter of } Q\}$ is a topology on $SpecQ$.

Proof . Let λ_1, λ_2 be fuzzy subsets of Q such that $\lambda_1(a) = 0$ and $\lambda_2(a) = 1$ for all $a \in Q$. Then $[\lambda_1), \lambda_2$ are fuzzy filters and $[\lambda_1) \subseteq \gamma$ for all $\gamma \in SpecQ$. Thus $\zeta([\lambda_1)) = SpecQ$ and hence $\xi(\lambda_1) = \phi$. Since each $\gamma \in SpecQ$ is non-constant, $\lambda_2 \nsubseteq \gamma$ for all $\gamma \in SpecQ$. Which implies $\xi(\lambda_2) = SpecQ$. Hence $\phi, SpecQ \in \mathcal{T}$. Since $\xi(\lambda_1) \cap \xi(\lambda_2) = \xi(\lambda_1 \cap \lambda_2)$, $\mathcal{T}$ is closed under finite intersection.

Next, Let $\{\lambda_j : j \in \Delta\}$ be subfamily of $SpecQ$. Then by Lemma 4.5, we have $\bigcap_{j\in\Delta}\zeta(\lambda_j) = \zeta([\bigcup_{j\in\Delta}\lambda_j))$. Thus $\bigcup_{j\in\Delta}\xi(\lambda_j) = \xi([\bigcup_{j\in\Delta}\lambda_j))$ and hence $\mathcal{T}$ is closed under arbitrary union. Consequently, $\mathcal{T}$ is a topology on $SpecQ$. The space $(SpecQ, \mathcal{T})$ will be called the space of L-prime normal fuzzy filters in Q. □

Theorem 4.7. The collection $\mathcal{B} = \{\xi(a_\alpha) : a \in Q,\ 0 \neq \alpha \in L\}$ forms base for some topology $SpecQ$.

Proof . To show our claim it is enough to show that, for each open set $\xi(\lambda)$ in $SpecQ$ there is an element in $\mathcal{B}$ contained in $\xi(\lambda)$. Let $\gamma \in \xi(\lambda)$. Then $\lambda \nsubseteq \gamma$ and there exists $a \in Q$ such that $\lambda(a) > \gamma(a)$. Put $\lambda(a) = \beta$. To show $\xi(a_\beta) \subseteq \xi(\lambda)$, let $\theta \in \xi(a_\beta)$. Then $a_\beta \nsubseteq \theta$ and $\lambda(a) > \theta(a)$. Thus $\theta \in \xi(\lambda)$ and $\xi(a_\beta) \subseteq \xi(\lambda)$. □

Theorem 4.8. The space $SpecQ$ is a T_0-space.

Proof . Let $\lambda, \gamma \in SpecQ$ such that $\lambda \neq \gamma$. Then $\gamma \nsubseteq \lambda$ or $\lambda \nsubseteq \gamma$. Assume that $\lambda \nsubseteq \gamma$. Then $\gamma \in \xi(\lambda)$ and $\gamma \notin \xi(\gamma)$. Therefore, $SpecQ$ is a T_0-space. □

Theorem 4.9. For any L-fuzzy filter λ of L, $\xi(\lambda) = \xi(\lambda^\perp)$.

Proof . Let λ be a fuzzy filter of Q. Clearly $\xi(\lambda) \subseteq \xi(\lambda^\perp)$. To show the other inclusion, let $\gamma \in \xi(\lambda^\perp)$. Then $\lambda^\perp \nsubseteq \gamma$. Assume that $\gamma \notin \xi(\lambda)$. Then $\lambda^\perp \subseteq \gamma^\perp = \gamma$. Which is a contradiction. Therefore, $\xi(\lambda) = \xi(\lambda^\perp)$. □

Theorem 4.10. For any family $\mathcal{C} \subseteq SpecQ$, closure of $\mathcal{C}$ is given by:

$$\overline{\mathcal{C}} = \zeta(\bigcap_{\lambda\in\mathcal{C}} \lambda).$$

Proof . Now we proceed to show that $\zeta(\bigcap_{\lambda\in\mathcal{C}}\lambda)$ is the smallest closed set containing $\mathcal{C}$. Let $\eta \in \mathcal{C}$. Then $\bigcap_{\lambda\in\mathcal{C}}\lambda \subseteq \eta$ and $\eta \in \zeta(\bigcap_{\lambda\in\mathcal{C}}\lambda)$. Thus $\mathcal{C} \subseteq \zeta(\bigcap_{\lambda\in\mathcal{C}}\lambda)$. Let $\zeta(\gamma)$ be any closed set in $SpecQ$ containing $\mathcal{C}$. Then $\gamma \subseteq \lambda$ for each $\lambda \in \mathcal{C}$. which implies $\gamma \subseteq \bigcap_{\lambda\in\mathcal{C}}\lambda$ and $\zeta(\bigcap_{\lambda\in\mathcal{C}}\lambda) \subseteq \zeta(\gamma)$. Thus $\zeta(\bigcap_{\lambda\in\mathcal{C}}\lambda)$ is the smallest closed set containing $\mathcal{C}$ and hence $\overline{\mathcal{C}} = \zeta(\bigcap_{\lambda\in\mathcal{C}}\lambda)$. □

Conclusion

In this paper, we introduced the concept of L-normal fuzzy filter in distributive lattice in terms of annulates and investigate some of its properties. We established a set of equivalent conditions for an L-fuzzy filter to be an L-normal fuzzy filter. Disjunctive lattice characterized in terms of L-normal fuzzy filters. We also studied the special class of fuzzy filters called L-normal fuzzy filters and proved that the class of L-normal fuzzy filters forms a distributive lattice. Furthermore, we observed that every L-normal fuzzy filter is the intersection of all L-prime normal fuzzy filters containing it. Finally, we studied the space of L-prime normal fuzzy filters in distributive lattice. Our future work will focus on fuzzy semiprime ideals in $(0, 1)$ distributive lattices.

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