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Operators commuting with certain module actions

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Abstract

In this note, we study bounded linear operators associated with unitary representations which commute with certain module actions.

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1 Introduction and preliminaries

Throughout G is a locally compact group with the unit e, a fixed left Haar-measure. The left Haar-integral of a complex-valued Haar-measurable function f on G will be denoted by $\int_G f(x)\,dx$. The convolution product of two complex-valued functions f and g on G is defined as follows.

$$f * g(x) = \int_G f(y)g(y^{-1}x)\,dx,$$

when the integral makes sense. As usual, $L^1(G)$ denotes the group algebra of G as defined in [3]. The notation l_x is the left translation operator by $x \in G$; i.e., $l_x f(y) = f(xy)$ for all complex-valued function f on G. Note that $L^1(G)$ is a left G-module with the action $x \cdot \phi = l_{x^{-1}}\phi$ for all $x \in G$ and $\phi \in L^1(G)$. Let $L^\infty(G)$ is usual Lebesgue space as defined in [3] equipped with the essential supremum $\|\cdot\|_\infty$. Then $L^\infty(G)$ can be identified by the first dual space of $L^1(G)$ under the pairing

$$\langle f, \phi\rangle = \int_G f(x)\phi(x)\,dx \qquad (f \in L^\infty(G),\, \phi \in L^1(G)).$$

Moreover, the dualization of the left G-module action on $L^1(G)$ makes $L^\infty(G)$ as a right G-module as follows

$$\langle f \cdot x, \phi\rangle = \langle f, x \cdot \phi\rangle \quad (f \in L^\infty(G),\, x \in G).$$

We can also consider $L^\infty(G)$ as a right Banach $L^1(G)$-module by the following action.

$$f \cdot \phi = \int_G f(x)\,\phi(x)\,dx \qquad (f \in L^\infty(G),\, \phi \in L^1(G)).$$

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Let also, $LUC(G)$ denote the C^*-algebra of left uniformly continuous functions; i.e., $f \in LUC(G)$ when the map $x \mapsto l_x f$ from G into $L^\infty(G)$ is norm continuous. In recent years, many authors have extensively studied the behavior and relations of G-module and $L^1(G)$-module maps, in the sense of the map commute with the translations, convolutions and conjugations; see for example [5, 7, 8, 9]. Special attention has focused on such operators on $L^\infty(G)$. As known, any bounded linear operator on $L^\infty(G)$ that commutes with convolution from the left also commutes with left translations; see [8]. Here, we study such notions with an emphasis on unitary representations.

All over this paper, (π, H_π) is a unitary representation of a locally compact group G. As mentioned in [1], $Tr(H_\pi)$, all of the trace-class operators on H_π with norm $\|T\|_1 = tr|T|$, takes the role played by $L^1(G)$ in the theory of amenable groups and the left action of G on $L^1(G)$ being replaced by the following left action of G on $Tr(H_\pi)$.

$$x \cdot_\pi S = \pi(x) S \pi(x)^{-1} \qquad \big(x \in G,\ S \in Tr(H_\pi)\big).$$

Moreover, $Tr(H_\pi)$ is an isometric Banach G-module by Lemma 2.1 of [1]. Also, $B(H_\pi)$ is known as the dual space of $Tr(H_\pi)$ by the duality $T(S) = tr(ST)$ for all $T \in B(H_\pi)$ and $S \in Tr(H_\pi)$. Clearly, $T \cdot_\pi x = \pi(x)^{-1} T \pi(x)$ for each $T \in B(H_\pi)$ and $x \in G$. These facts imply that $B(H_\pi)$ is a right Banach $L^1(G)$-module as follows.

$$T \cdot_\pi \phi = \int_G T \cdot_\pi x\, \phi(x)\, dx \qquad (T \in B(H_\pi),\ \phi \in L^1(G)).$$

Since the map $x \mapsto T \cdot_\pi x$ from G into $B(H_\pi)$ is not necessarily norm-continuous, $B(H_\pi)$ is not Banach as a G-module, in general. So, one has considered the set of all $T \in B(H_\pi)$ for which $G \longrightarrow B(H_\pi)$, $x \longmapsto T \cdot_\pi x$ is norm-continuous, $UCB(\pi)$. Elements in $UCB(\pi)$ are called G-continuous operators. Moreover, Cohen's factorization theorem implies that

$$B(H_\pi) \cdot_\pi L^1(G) = UCB(\pi) \cdot_\pi L^1(G) = UCB(\pi).$$

See [1] for more details and the survey article. For any $M \in B(H_\pi)^*$ and $T \in B(H_\pi)$, we can define a complex-valued function MT on G by

$$MT(x) = \langle M,\, T \cdot_\pi x\rangle \quad (x \in G).$$

Obviously, MT is bounded by $\|M\|\|T\|$. Besides,

$$l_x MT = (M)(T \cdot_\pi x) \quad (x \in G).$$

Suppose that $M \in B(H_\pi)^*$. Then the linear operator $\rho_M : UCB(\pi) \longrightarrow LUC(G)$ given by $T \longmapsto MT$ is well-defined due to [2, Lemma 2.2]. Furthermore, let $T \in UCB(\pi)$ and $\phi \in L^1(G)$. Then $\langle MT, \phi\rangle = \langle M, T \cdot_\pi \phi\rangle$ by directly calculation. Therefore, $\rho_M(T \cdot_\pi \phi) = \rho_M(T) \cdot \phi$. Also, $\rho_M(T \cdot_\pi x) = \rho_M(T) \cdot x$ for all $x \in G$. These simple properties of ρ_M are a motivating force for this research. We extend them by the following definition that is the starting point of our path to express the main results in this note.

Definition 1.1. Let (π, H_π) be a unitary representation of a locally compact group G, and let $\gamma : B(H_\pi) \longrightarrow L^\infty(G)$ be a bounded linear operator.

(a) γ is said *to commute with the action as $L^1(G)$-module* if

$$\gamma(T \cdot_\pi \phi) = \gamma(T) \cdot \phi \qquad (T \in B(H_\pi),\ \phi \in L^1(G)). \tag{1.1}$$

(b) γ is said *to commute with the action as G-module* if

$$\gamma(T \cdot_\pi x) = \gamma(T) \cdot x \qquad (T \in B(H_\pi),\ x \in G), \tag{1.2}$$

Suppose that $M \in B(H_\pi)^*$. We do not yet whether $MT \in L^\infty(G)$ for all $T \in B(H_\pi)$ or not. Therefore, we can not define safely the operator ρ_M from $B(H_\pi)$ into $L^\infty(G)$ by $\rho_M(T) = MT$. But as will be seen, there exist such operators. For instance, the map γ_M defined by $\langle \gamma_M(T), \phi\rangle = \langle M, T \cdot_\pi \phi\rangle$ for all $T \in B(H_\pi)$ and $\phi \in L^1(G)$ satisfies in the both of 1.1 and 1.2.

2 The results

We commence the note by the following result that shows 1.1 and 1.2 coincide when the operator γ restricts to $UCB(\pi)$. Before starting, note that for all $M \in UCB(\pi)^*$ and $T \in UCB(\pi)$, we can also define the complex-valued function MT by $\overline{M}T$ on G, where $\overline{M}$ is any Hahn-Banach extension of M. Since the Hahn-Banach extension is not unique, in general, we use again the notation ρ_M instead of $\rho_{\overline{M}}$ for unification.

Theorem 2.1. Let (π, H_π) be a unitary representation of a locally compact group G, and let $\gamma : UCB(\pi) \longrightarrow L^\infty(G)$ be a bounded linear operator. Then each of the following statements implies that the range of γ lies in $LUC(G)$. Also, they are equivalent.

(a) γ commutes with the action as $L^1(G)$-module,
(b) $\gamma = \rho_M$ for some $M \in UCB(\pi)^*$,
(c) γ commutes with action as G-module.

Proof . Let $T \in UCB(\pi)$. If (a) holds, then $\gamma(T) = \gamma(S \cdot_\pi \phi) = \gamma(S) \cdot \phi$ for some $S \in UCB(\pi)$ and $\phi \in L^1(G)$ that yields $\gamma(T) \in LUC(G)$. If (b) holds, then $\gamma(T) = \rho_M(T) = MT \in LUC(G)$. Finally, if (c) holds and $x_\alpha \longrightarrow x$ in G, then

$$\begin{aligned}\|l_{x_\alpha}\gamma(T) - l_x\gamma(T)\|_\infty &= \|\gamma(T) \cdot x_\alpha - \gamma(T) \cdot x\|_\infty \\ &= \|\gamma(T \cdot_\pi x_\alpha) - \gamma(T \cdot_\pi x)\|_\infty \\ &\le \|\gamma\| \, \|T \cdot_\pi x_\alpha - T \cdot_\pi x\| \\ &\longrightarrow 0.\end{aligned}$$

It follows that $\gamma(T) \in LUC(G)$. Now, for equivalency of them, we can confirm (a) and (c) if (b) holds, as noted earlier. Suppose that (a) holds and (ϕ_i) is a bounded approximate identity of $L^1(G)$. Then $(\gamma^*(\phi_i))$ is bounded in $UCB(\pi)^*$, where γ^* is the usual adjoint of γ. Let now $M \in UCB(\pi)^*$ be a weak*-cluster point of $(\gamma^*(\phi_i))$. So, we may assume that $\gamma^*(\phi_i) \longrightarrow M$ in the weak*-topology of $UCB(\pi)^*$. Let $T \in UCB(\pi)$. Then for each $\phi \in L^1(G)$, we have

$$\begin{aligned}\langle \rho_M(T), \phi\rangle &= \langle MT, \phi\rangle = \langle M, T \cdot_\pi \phi\rangle \\ &= \lim_i \langle \gamma^*(\phi_i), T \cdot_\pi \phi\rangle = \lim_i \langle \phi_i, \gamma(T \cdot_\pi \phi)\rangle \\ &= \lim_i \langle \phi_i, \gamma(T) \cdot \phi\rangle = \lim_i \langle \gamma(T) \cdot \phi, \phi_i\rangle \\ &= \lim_i \langle \gamma(T), \phi * \phi_i\rangle = \langle \gamma(T), \phi\rangle.\end{aligned}$$

Therefore, part (b) holds. Now, assume that γ is commuting with the action as G-module. Take $M = \gamma^*(\delta_e) \in UCB(\pi)^*$, where $\delta_e(f) = f(e)$ for all $f \in LUC(G)$. Then for each $T \in UCB(\pi)$ and $x \in G$, we have

$$\begin{aligned}\gamma(T)(x) &= \big(\gamma(T) \cdot x\big)(e) = \langle \delta_e, \gamma(T) \cdot x\rangle \\ &= \langle \delta_e, \gamma(T \cdot_\pi x)\rangle = \langle \gamma^*(\delta_e), T \cdot_\pi x\rangle \\ &= \langle M, T \cdot_\pi x\rangle = MT(x).\end{aligned}$$

It follows that $\gamma(T) = MT = \rho_M(T)$ for all $T \in UCB(\pi)$ and so, $\gamma = \rho_M$. One shows the implication (c) into (b). □

As mentioned earlier, every bounded linear operator on $L^\infty(G)$ commuting with the action as $L^1(G)$-module commute also, with the action as G-module. Here, we have the following result.

Proposition 2.2. Let (π, H_π) be a unitary representation of a locally compact group G, and let γ be a bounded linear operator from $B(H_\pi)$ into $L^\infty(G)$ that is commuting with the action as $L^1(G)$-module. Then γ commutes with the action as G-module.

Proof . Suppose that $T \in B(H_\pi)$, $x \in G$ and $\phi \in L^1(G)$. One can easily check that $(T \cdot_\pi x) \cdot_\pi \phi = T \cdot_\pi (x \cdot \phi)$. Furthermore, let (ϕ_i) be an approximate identity for $L^1(G)$. Then

$$\begin{aligned}
\langle \gamma(T \cdot_\pi x),\, \phi\rangle &= \lim_i \langle \gamma(T \cdot_\pi x),\, \phi_i * \phi\rangle \\
&= \lim_i \langle \gamma(T \cdot_\pi x) \cdot \phi_i,\, \phi\rangle \\
&= \lim_i \langle \gamma\big((T \cdot_\pi x) \cdot_\pi \phi_i\big),\, \phi\rangle \\
&= \lim_i \langle \gamma\big(T \cdot_\pi (x \cdot \phi_i)\big),\, \phi\rangle \\
&= \lim_i \langle \gamma(T) \cdot (x \cdot \phi_i),\, \phi\rangle \\
&= \lim_i \langle \gamma(T),\, (x \cdot \phi_i) * \phi\rangle \\
&= \lim_i \langle \gamma(T),\, x \cdot (\phi_i * \phi)\rangle \\
&= \lim_i \langle \gamma(T) \cdot x,\, \phi_i * \phi\rangle \\
&= \langle \gamma(T) \cdot x,\, \phi\rangle.
\end{aligned}$$

Therefore, γ commutes with the action as G-module. □

It is tempting to know whether the converse of Proposition 2.2 is valid or not. It is known that the converse fails in the same style of operators on $L^\infty(G)$. So, it turns out that the converse fails here, too. It is clear that $B(H_\pi) = UCB(\pi)$ when G is discrete, and so the converse is true by Theorem 2.1. Note that sometimes there are some unitary representations of non-discrete groups such that $B(H_\pi) = UCB(\pi)$. For instance, we have the following example.

Example 2.3. Let $G = (\mathbb{R}, +)$, and let $\pi : G \longrightarrow B\big(L^2(\mathbb{R})\big)$ be the unitary representation given by

$$\big(\pi(x)g\big)(t) = \exp(-ix)g(t)\chi_{(-\infty,\,0)}(t) + \exp(ix)g(t)\chi_{(0,\,+\infty)}(t)$$

for all $x, t \in G$ and $g \in L^2(\mathbb{R})$. Let now $T \in B(L^2(G))$, and $x_\alpha \longrightarrow x$ in G. Then

$$\|T \cdot x_\alpha - T \cdot x\| \le \|T\|(|\exp(-ix_\alpha) - \exp(-ix)| + |\exp(ix_\alpha) - \exp(ix)|) \longrightarrow 0.$$

It follows that $B(L^2(G)) = UCB(\pi)$, whereas G is non-discrete.

Suppose that $(\lambda,\, L^2(G))$ is the left unitary representation of G. We have the following lemma.

Lemma 2.4. Let G be a locally compact group. Then G is discrete if and only if either of the following statements holds.

(a) $L^\infty(G) = LUC(G)$,
(b) $B(H_\pi) = UCB(\pi)$ for all unitary representations $(\pi,\, H_\pi)$ of G,
(c) $B(L^2(G)) = UCB(\lambda)$.

Proof . It is well known that a locally compact group G is discrete if and only if $L^\infty(G) = LUC(G)$. According to [4, Remark 3.11 (i)], an element $f \in L^\infty(G)$ lies in $LUC(G)$ if and only if $T_f \in UCB(\lambda)$, where T_f is the multiplication operator on $L^2(G)$ by f. So, part (c) implies that part (a). The other implications are evident. □

The next example shows that the converse of Proposition 2.2 has been unable to confirm in general. Due to Theorem 2.1 and Lemma 2.4, one can consider a non-discrete group G and the left unitary representation $(\lambda,\, L^2(G))$.

Example 2.5. Let G be either $(\mathbb{R}, +)$ or any infinite compact abelian group. We show that there exists a bounded linear operator γ from $B\big(L^2(G)\big)$ into $L^\infty(G)$ such that γ commutes the action as G-module; whereas, $\gamma(T \cdot_\lambda \phi) \neq \gamma(T) \cdot \phi$, for some $T \in B\big(L^2(G)\big)$ and $\phi \in L^1(G)$. Toward this end, first, recall that for each $f \in L^\infty(G)$, the map $\tau : f \mapsto T_f$ is an isometric embedding of $L^\infty(G)$ into $B\big(L^2(G)\big)$. It is rutin checking that $T_f \cdot_\lambda \phi = T_{f \cdot \phi}$ for each $f \in L^\infty(G)$ and $\phi \in L^1(G)$. On the other hand, G satisfies in conditions of Theorem 4.1 of [9]. So, the following statements hold for some bounded linear operators Ψ on $L^\infty(G)$ such that

(a) Ψ commutes the action as G-module.
(b) each $\Psi(f)$ is a constant function for all $f \in L^\infty(G)$.
(c) $\Psi(f \cdot \phi) \neq \Psi(f) \cdot \phi$ for some $f \in L^\infty(G)$ and some continuous function ϕ with compact support.

Take now, $\gamma = \Psi \circ \tau_l^{-1}$, where τ_l^{-1} is the left inverse of τ. Note that G is non-discrete and so, $B(L^2(G)) \neq UCB(\lambda)$. However, it follows that

$$\gamma(T_f \cdot_\lambda x) = \Psi(f \cdot x) = \Psi(f) \cdot x = \gamma(T_f) \cdot x$$

for each $f \in L^\infty(G)$ and $\phi \in L^1(G)$. Besides,

$$\gamma(T_f \cdot_\lambda \phi) = \Psi(f \cdot \phi) \neq \Psi(f) \cdot \phi = \gamma(T_f) \cdot \phi$$

for each f and ϕ that satisfy part (c) in the above.

Remark 2.6. Extending to Theorem 2.1, we can show that for each bounded linear operator γ from $B(H_\pi)$ into $L^\infty(G)$ the following statements are equivalent.

(a) γ commutes with the action as $L^1(G)$-module,
(b) $\gamma = \gamma_M$ for some $M \in UCB(\pi)^*$.

As seen in Theorem 2.1, when γ restricts to $UCB(\pi)$, the above statements are also equivalent to the following part.

(c) γ commutes with action as G-module.

Moreover, one can readily show that if γ is weak*-weak*-continuous, then all of the above statements are equivalent.

Recall that $LUC(G)^*$ is a Banach algebra endowed with the first Arens product as follows.

$$\langle m \odot n,\, f\rangle = \langle m,\, n \cdot f\rangle \quad \text{and} \quad \langle n \cdot f,\, \phi\rangle = \langle n,\, f \cdot \phi\rangle$$

for all $m,\, n \in LUC(G)^*$, $f \in LUC(G)$ and $\phi \in L^1(G)$. For each $(\pi,\, H_\pi)$ unitary representation of G, we have the bounded bilinear mapping $LUC(G)^* \times UCB(\pi)^* \longrightarrow UCB(\pi)^*$ given by $(m,\, M) \mapsto m \cdot M$, where $\langle m \cdot M,\, T\rangle = \langle m,\, MT\rangle$, which makes $UCB(\pi)^*$ as a left Banach $LUC(G)^*$-module. This fact was proven by Proposition 2.3 of [2]. Now, let $\mathcal{B}(\pi,\, G)$ be the space of all bounded linear operators from $B(H_\pi)$ into $L^\infty(G)$ commuting with the action as $L^1(G)$-module.

Lemma 2.7. Let $(\pi,\, H_\pi)$ be a unitary representation of a locally compact group G. Then

(a) $\mathcal{B}(\pi,\, G)$ is a Banach space with operator norm.
(b) $\mathcal{B}(\pi,\, G)$ is a left Banach $LUC(G)^*$-module by the following action.

$$\langle (m \bullet \gamma)(T),\, \phi\rangle = \langle m,\, \gamma(T) \cdot \phi\rangle.$$

where $m \in LUC(G)^*$, $\gamma \in \mathcal{B}(\pi,\, G)$, $T \in B(H_\pi)$ and $\phi \in L^1(G)$.

Proof . (a). Assume that γ is an element of the norm-cluster of $\mathcal{B}(\pi,\, G)$. Then there exists a net $(\gamma_n) \subseteq \mathcal{B}(\pi,\, G)$ such that converges to γ. So, for each $T \in B(H_\pi)$ and $\phi \in L^1(G)$ with $\|T\| \leq 1$ and $\|\phi\|_1 \leq 1$, we have

$$\|\gamma(T \cdot_\pi \phi) - \gamma(T) \cdot \phi\|_\infty \leq \|\gamma(T \cdot_\pi \phi) - \gamma_n(T \cdot_\pi \phi)\|_\infty + \|\gamma_n(T) \cdot \phi - \gamma(T) \cdot \phi\|_\infty \longrightarrow 0$$

and so, $\gamma(T \cdot_\pi \phi) = \gamma(T) \cdot \phi$. Therefore, for each $T \in B(H_\pi)$ and $\phi \in L^1(G)$,

$$\gamma(\frac{T}{\|T\|} \cdot_\pi \frac{\phi}{\|\phi\|_1}) = \gamma(\frac{T}{\|T\|}) \cdot \frac{\phi}{\|\phi\|_1}.$$

So, since γ and module actions are linear, we have $\gamma(T \cdot_\pi \phi) = \gamma(T) \cdot \phi$. It implies that $\gamma \in \mathcal{B}(\pi,\, G)$ and hence, $\mathcal{B}(\pi,\, G)$ is a closed subspace of $B(B(H_\pi),\, L^\infty(G))$, bounded linear operators from $B(H_\pi)$ into $L^\infty(G)$. Therefore, $\mathcal{B}(\pi,\, G)$ is Banach.

(b). Let $m, n \in LUC(G)^*$, $\gamma \in \mathcal{B}(\pi, G)$, $T \in \mathcal{X}$ and $\phi \in L^1(G)$. It is easily to check that $n \bullet \gamma \in \mathcal{B}(\pi, G)$ and

$$n \cdot \gamma(T \cdot_\pi \phi) = (n \bullet \gamma)(T \cdot_\pi \phi).$$

Then

$$\begin{aligned} \langle \big((m \odot n) \bullet \gamma\big)(T), \phi\rangle &= \langle m \odot n, \gamma(T) \cdot \phi\rangle \\ &= \langle m, n \cdot \gamma(T \cdot_\pi \phi)\rangle \\ &= \langle m, (n \bullet \gamma)(T \cdot_\pi \phi)\rangle \\ &= \langle m, (n \bullet \gamma)(T) \cdot \phi\rangle \\ &= \langle \big(m \bullet (n \bullet \gamma)\big)(T), \phi\rangle. \end{aligned}$$

So, $(m \odot n) \bullet \gamma = m \bullet (n \bullet \gamma)$. Others are evident. □

We end the work with the following result, as one of the important aims of this memoir.

Theorem 2.8. Let (π, H_π) be a unitary representation of a locally compact group G. Then there exists an isometric isomorphism as left Banach $LUC(G)^*$-modules between the dual of $UCB(\pi)$ and $\mathcal{B}(\pi, G)$.

Proof . We define a linear map Θ from $UCB(\pi)^*$ into $\mathcal{B}(\pi, G)$ by $M \mapsto \gamma_M$. Note that Θ is surjective by Remark 2.6. Now, we show that Θ is an isometry. It is clear that $\|\gamma_M\| \leq \|M\|$. To prove the reverse inequality, let (ϕ_i) be an approximate identity of $L^1(G)$ bounded to 1. By a rutin calculation, a bounded linear operator T on H_π lies in $UCB(\pi)$ if and only if

$$\|T \cdot_\pi \phi_i - T\| \longrightarrow 0.$$

So, for each i and $T \in UCB(\pi)$ with $\|T\| \leq 1$, we have

$$\|\gamma_M\| \geq \|\gamma_M(T)\|_\infty \geq |\langle \gamma_M(T), \phi_i\rangle| = |\langle M, T \cdot \phi_i\rangle| \longrightarrow |\langle M, T\rangle|.$$

Consequently, $\|\gamma_M\| \geq \|M\|$ and so, Θ is one-to-one. The proof completes as follows.

$$\begin{aligned} \langle \gamma_{m \cdot M}(T), \phi\rangle = \langle m \cdot M, T \cdot_\pi \phi\rangle &= \langle m, \gamma_M(T) \cdot \phi\rangle \\ &= \langle (m \bullet \gamma_M)(T), \phi\rangle. \end{aligned}$$

It follows that $\Theta(m \cdot M) = m \cdot \Theta(M)$ for all $m \in LUC(G)^*$ and $M \in UCB(\pi)^*$. □

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