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Infinitely many solutions for a nonlinear equation with Hardy potential

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Abstract

In this article, by using critical point theory, we prove the existence of infinitely many weak solutions for a nonlinear problem with Hardy potential. Indeed, intervals of parameters are determined for which the problem admits an unbounded sequence of weak solutions.

Keywords: weak solutions, Navier boundary conditions, p-triharmonic operators
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1 Introduction

In this work, we discuss the existence of infinitely many weak solutions for the following problem

$$\begin{cases} -\Delta_p^3 u + \frac{|u|^{p-2}u}{|x|^{3p}} = \lambda f(x,u) + \mu g(x,u), & x \in \Omega, \\ u = \Delta u = \Delta^2 u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N > 3)$ is a bounded domain with boundary of class C^1, λ is a positive parameter, μ is a non-negative parameter, $f, g \in C^0(\overline{\Omega} \times \mathbb{R})$ and p is a constant with $1 < p < \frac{N}{3}$. The operator $\Delta_p^3 u := div\left(\Delta(|\nabla\Delta u|^{p-2}\nabla\Delta u)\right)$ is the p-triharmonic operator.

In recent years, the study of boundary value problems involving the triharmonic operator has been considered see for instance [9, 11].

Rahal [9] studied the existence of weak solutions to the following nonlinear Navier boundary value problem involving the $p(x)$-Kirchhoff type triharmonic operator

$$\begin{cases} -M\left(\int_\Omega \frac{1}{p(x)}|\nabla\Delta u|^{p(x)}dx\right)\Delta_{p(x)}^3 u = \lambda\zeta(x)|u|^{\alpha(x)-2}u - \lambda\xi(x)|u|^{\beta(x)-2}u, & x \in \Omega, \\ u = \Delta u = \Delta^2 u = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N > 3)$ is a bounded domain with smooth boundary, λ is a positive parameter, $p \in C^0(\overline{\Omega})$ with $1 < p(x) < \frac{N}{3}$ for any $x \in \overline{\Omega}$ and $\zeta, \xi, \alpha, \beta \in C^0(\overline{\Omega})$.

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In [4], presenting a version of the infinitely many critical points theorem of Ricceri (see [10, Theorem 2.5]), the existence of an unbounded sequence of weak solutions for a Strum-Liouville problem, having discontinuous nonlinearities, has been established. In such an approach, an appropriate oscillating behavior of the nonlinear term either at infinity or at zero is required. This type of methodology has been used in several works in order to obtain existence results for different kinds of problems (see, for instance, [1, 2, 3, 5, 6, 7, 12] and references therein).

The rest of this paper is organized as follows. In Section 2, some known definitions and results on Lebesgue and Sobolev spaces, which will be used in sequel, are collected. Moreover, the abstract critical points theorem (Lemma 2.1) is recalled. In Section 3, we state the main result and its proof.

2 Preliminaries

Here and in the sequel Ω is an open bounded subset of $\mathbb{R}^N$ and

$$X := W^{3,p}(\Omega) \cap W_0^{1,p}(\Omega)$$

endowed with the norm

$$\|u\| = \left(\int_\Omega |\nabla \Delta u|^p \, dx \right)^{\frac{1}{p}} \tag{2.1}$$

for $u \in X$.

Corresponding to f and g, we introduce the functions $F, G : \Omega \times \mathbb{R} \to \mathbb{R}$, respectively, as follows

$$F(x,t) := \int_0^t f(x,\xi) \, d\xi, \qquad G(x,t) := \int_0^t g(x,\xi) \, d\xi$$

for all $x \in \Omega$ and $t \in \mathbb{R}$.

For every $u \in X$, let us define $\Phi, \Psi : X \to \mathbb{R}$ by putting

$$\Phi(u) := \frac{\|u\|^p}{p} + \frac{1}{p} \int_\Omega \frac{|u(x)|^p}{|x|^{3p}} dx, \qquad \Psi(u) = \int_\Omega [F(x,u(x)) + \frac{\mu}{\lambda} G(x,u(x)] dx.$$

By standard arguments, we have that Φ and Ψ are Gâteaux differentiable and whose derivative are

$$\Phi'(u)(v) = \int_\Omega |\nabla \Delta u|^{p-2} \nabla \Delta u \cdot \nabla \Delta v dx + \int_\Omega \frac{|u|^{p-2}}{|x|^{3p}} uv dx$$

$$\Psi'(u)(v) = \int_\Omega [f(x,u(x)) + \frac{\mu}{\lambda} g(x,u(x))] v(x) \, dx$$

for any $u, v \in X$. Fixing $q \in [1, p^* := \frac{pN}{N-3p})$, from the Sobolev embedding there exists a positive constant c_q such that $\|u\|_{L^{p^*}(\Omega)} \leq c_q \|u\|$ for $u \in X$. Thus the embedding $X \hookrightarrow L^q(\Omega)$ is compact.

We recall Hardy inequality in X, which says that

$$\int_\Omega \frac{|u(x)|^p}{|x|^{3p}} dx \leq \frac{1}{H} \int_\Omega |\nabla \Delta u(x)|^p dx,$$

where $H = (\frac{[N(p-1)+p](N-p)(N-3p)}{p^3})^p$.

Finally, a weak solution of problem (1.1) is a function $u \in X$ such that

$$\begin{aligned} &\int_\Omega |\nabla \Delta u|^{p-2} \nabla \Delta u \cdot \nabla \Delta v dx + \int_\Omega \frac{|u(x)|^{p-2}}{|x|^{3p}} uv dx \\ &- \lambda \int_\Omega f(x,u(x)) v(x) \, dx - \mu \int_\Omega g(x,u(x)) v(x) \, dx = 0 \end{aligned}$$

for all $v \in X$, it is obvious that our goal is to find critical points of the functional I_λ. For achieving this aim, our main tool is the following critical point theorem of Ricceri [10, Theorem 2.5] (see also [4] for a refined version).

Lemma 2.1. Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\left(\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)},$$

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \qquad \text{and} \qquad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then the following properties hold:

(a) For every $r > \inf_X \Phi$ and every $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional

$$I := \Phi - \lambda\Psi$$

to $\Phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

(b) If $\gamma < +\infty$, then for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either

(b_1) I possesses a global minimum, or

(b_2) there is a sequence $\{u_n\}$ of critical points (local minima) of I such that

$$\lim_{n\to+\infty} \Phi(u_n) = +\infty.$$

3 Main results

In this section, we present our main results.

Theorem 3.1. Assume that

(A_1) $F(x,t) \geq 0$ for every $(x,t) \in \Omega \times [0, +\infty[$;

(A_2) there exists $s > 0$ such that, if we put

$$\alpha := \liminf_{t\to+\infty} \frac{\sup_{\|\xi\|_{L^q(\Omega)}\leq t} \displaystyle\int_\Omega F(x,\xi)\,dx}{t^p},$$

$$\beta := \limsup_{t\to+\infty} \frac{\displaystyle\int_{B(0,\frac{s}{2})} F\left(x, \frac{t}{h}\right) dx}{t^p},$$

where $\alpha < R\beta$, $h > 1$ is a constant, $R = \frac{h^p}{\sigma c_q^p}$ and

$$\sigma = \frac{H+1}{H} \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \int_{\frac{s}{2}}^{s} \left| \frac{128}{3s^4}(N+2)r - \frac{64}{s^3}(N+1) + \frac{32}{3sr^2}(N-1) \right|^p r^{N-1} dr.$$

Then, for each $\lambda \in \left(\frac{1}{pc_q^p R\beta}, \frac{1}{pc_q^p \alpha}\right)$ and for every $g \in C^0(\overline{\Omega} \times \mathbb{R})$ whose potential $G(x,t) := \int_0^t g(x,\xi)\,d\xi$ for all $(x,t) \in \overline{\Omega} \times [0, +\infty[$, is a non-negative function satisfying the condition

$$g_\infty := \limsup_{\xi\to+\infty} \frac{\sup_{\|\xi\|_{L^q(\Omega)}\leq t} \displaystyle\int_\Omega G(x,\xi)\,dx}{t^p} < +\infty, \tag{3.1}$$

if we put

$$\mu_{g,\lambda} := \frac{1}{pc_q^p g_\infty} \left(1 - \lambda p c_q^p \alpha\right),$$

where $\mu_{g,\lambda} = +\infty$ when $g_\infty = 0$, problem (1.1) has an unbounded sequence of weak solutions for every $\mu \in [0, \mu_{g,\lambda})$ in X.

Proof . We want to apply Lemma 2.1(b) with $X = W^{3,p}(\Omega) \cap W_0^{1,p}(\Omega)$ endowed with the norm introduced in (2.1). For fix $\lambda \in (\lambda_1, \lambda_2)$ and $\mu \in (0, \mu_{g,\lambda})$, we take Φ, Ψ as in the previous section. Similar arguments as those used in [1] and assumption (A_2), imply that

$$\gamma \leq \liminf_{n \to +\infty} \varphi(r_n) \leq p c_q^p \Big(\alpha + \frac{\mu}{\lambda} g_\infty\Big) < +\infty, \tag{3.2}$$

and consequently $\lambda < \frac{1}{\gamma}$.

Let λ be fixed. We claim that the functional I is unbounded from below. Since

$$\frac{1}{\lambda} < \frac{ph^p}{\sigma}\beta,$$

there exist a sequence $\{\tau_n\}$ of positive numbers and $\eta > 0$ such that $\lim_{n\to+\infty} \tau_n = +\infty$ and

$$\frac{1}{\lambda} < \eta < \frac{ph^p}{\sigma} \frac{\int_{B(0,\frac{s}{2})} F(x, \frac{\tau_n}{h})\, dx}{\tau_n^p} \tag{3.3}$$

for each $n \in \mathbb{N}$ large enough. For all $n \in \mathbb{N}$ define $w_n \in X$ by

$$w_n(x) := \begin{cases} 0 & \text{if} \quad x \in \overline{\Omega} \setminus B(0,s), \\ \frac{\tau_n}{h}\left(\frac{16}{3s^4}\rho^4 - \frac{64}{3s^3}\rho^3 + \frac{24}{s^2}\rho^2 - \frac{32}{3s}\rho + \frac{8}{3}\right) & \text{if} \quad x \in B(0,s) \setminus B(0,\frac{s}{2}), \\ \frac{\tau_n}{h} & \text{if} \quad x \in B(0,\frac{s}{2}), \end{cases} \tag{3.4}$$

where $\rho = dist(x, 0) = \sqrt{\sum_{i=1}^N (x_i - x_i^0)^2}$. Then, we have

$$\frac{\partial w_n(x)}{\partial x_i} = \begin{cases} 0 & \text{if} \quad x \in \overline{\Omega} \setminus B(0,s) \cap B(0,\frac{s}{2}), \\ \frac{\tau_n}{h}\left(\frac{64}{3s^4}\rho^2 - \frac{64}{s^3}\rho + \frac{48}{s^2} - \frac{32}{3s\rho}\right)x_i & \text{if} \quad x \in B(0,s) \setminus B(0,\frac{s}{2}), \end{cases}$$

$$\frac{\partial^2 w_n(x)}{\partial x_i^2} = \begin{cases} 0 \qquad\qquad \text{if} \quad x \in \overline{\Omega} \setminus B(0,s) \cap B(0,\frac{s}{2}), \\ \frac{\tau_n}{h}\left(\frac{64}{3s^4}(\rho^2 + 2x_i^2) - \frac{64}{s^3}\left(\frac{\rho^2 + x_i^2}{\rho}\right) + \frac{48}{s^2} + \frac{32}{3s}\left(\frac{x_i^2 - \rho^2}{\rho^3}\right)\right) \\ \quad \text{if} \quad x \in B(0,s) \setminus B(0,\frac{s}{2}), \end{cases}$$

$$\sum_{i=1}^N \frac{\partial^2 w_n(x)}{\partial x_i^2} = \begin{cases} 0 \qquad \text{if} \quad x \in \overline{\Omega} \setminus B(0,s) \cap B(0,\frac{s}{2}), \\ \frac{\tau_n}{h}\left(\frac{64\rho^2}{3s^4}(N+2) - \frac{64\rho}{s^3}(N+1) + \frac{48}{s^2}N - \frac{32}{3s\rho}(N-1)\right) \\ \qquad \text{if} \quad x \in B(0,s) \setminus B(0,\frac{s}{2}), \end{cases}$$

$$\frac{\partial \Delta w_n(x)}{\partial x_i} = \begin{cases} 0 \qquad \text{if} \quad x \in \overline{\Omega} \setminus B(0,s) \cap B(0,\frac{s}{2}), \\ \frac{\tau_n}{h}\left(\frac{128}{3s^4}(N+2)x_i - \frac{64}{s^3\rho}(N+1)x_i + \frac{32}{3s\rho^3}(N-1)x_i\right) \\ \qquad \text{if} \quad x \in B(0,s) \setminus B(0,\frac{s}{2}), \end{cases}$$

and

$$|\nabla \Delta w_n(x)| = \frac{\tau_n}{h}\left|\frac{128}{3s^4}(N+2)\rho - \frac{64}{s^3}(N+1) + \frac{32}{3s\rho^2}(N-1)\right|.$$

For any fixed $n \in \mathbb{N}$, one has

$$\begin{aligned} \Phi(w_n) &\leq \frac{H+1}{pH} \int_{B(0,s)\setminus B(0,\frac{s}{2})} |\nabla \Delta w_n(x)|^p dx \\ &= \frac{H+1}{pH}\left(\frac{\tau_n}{h}\right)^p \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \int_{\frac{s}{2}}^s \left|\frac{128}{3s^4}(N+2)r - \frac{64}{s^3}(N+1) + \frac{32}{3sr^2}(N-1)\right|^p r^{N-1} dr = \frac{\sigma \tau_n^p}{ph^p}. \end{aligned} \tag{3.5}$$

On the other hand, bearing (A_1) in mind and since G is non-negative, from the definition of Ψ, we infer

$$\Psi(w_n) = \int_\Omega \left[F(x, w_n(x)) + \frac{\mu}{\lambda} G(x, w_n(x))\right] dx \geq \int_{B(0,\frac{s}{2})} F(x, \frac{\tau_n}{h})\, dx. \tag{3.6}$$

By (3.3), (3.5) and (3.6), we observe that

$$I(w_n) \leq \frac{\sigma \tau_n^p}{ph^p} - \lambda \int_{B(0,\frac{s}{2})} F(x, \frac{\tau_n}{h})\, dx < \frac{\sigma}{ph^p}(1-\lambda\eta)\tau_n^p \tag{3.7}$$

for every $n \in \mathbb{N}$ large enough. Since $\lambda\eta > 1$ and $\lim_{n\to+\infty} \tau_n = +\infty$, we have

$$\lim_{n\to+\infty} I(w_n) = -\infty.$$

Then, the functional I_λ is unbounded from below, and it follows that I_λ has no global minimum. Therefore, by Lemma 2.1(b), there exists a sequence $\{u_n\}$ of critical points of I_λ such that

$$\lim_{n\to+\infty} \|u_n\| = +\infty,$$

and the conclusion is achieved. □

Now, we present the following consequence of Theorem 3.1 with $\mu = 0$.

Theorem 3.2. Let all the assumptions in the Theorem 3.1 hold. Then, for each

$$\lambda \in \left(\frac{1}{pc_q^p R\beta}, \frac{1}{pc_q^p \alpha} \right)$$

the problem

$$\begin{cases} -\Delta_p^3 u + \frac{|u|^{p-2}u}{|x|^{3p}} = \lambda f(x,u), & x \in \Omega, \\ u = \Delta u = \Delta^2 u = 0, & x \in \partial\Omega \end{cases} \tag{3.8}$$

has an unbounded sequence of weak solutions in X.

Here, we point out the following consequence of Theorem 3.1.

Corollary 3.3. Let the assumption (A_1) in the Theorem 3.1 holds. Suppose that

$$\alpha < \frac{1}{pc_q^p}, \qquad \beta > \frac{1}{pc_q^p R}.$$

Then, the problem

$$\begin{cases} -\Delta_{p(x)}^3 u + \frac{|u|^{p-2}u}{|x|^{3p}} = f(x,u), & x \in \Omega, \\ u = \Delta u = \Delta^2 u = 0, & x \in \partial\Omega \end{cases}$$

has an unbounded sequence of weak solutions in X.

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