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New accelerated iterative algorithm for (λ, ρ)-quasi firmly nonexpansive multiValued mappings

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Abstract

In this paper, introduced a new accelerated iterative algorithm in (λ, ρ) -quasi firmly nonexpansive multi-valued mappings in modular function spaces and present some results for convergence to a fixed point in this mapping, we use faster convergence theorem to comparison our iteration with some other iterations and introduced numerical example. As an application, we have referred to previous work by other researchers.

Keywords: Multivalued mappings, quasi firmly nonexpansive mappings, modular function spaces, iterative scheme, fixed point
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1 Introduction

For nearly a century, there have been tremendous into the existence of fixed points and it is applications depending on contraction mapping, (quasi) non expansive mapping, etc. as indicated in the sources below and others, see [9, 12]. In this context, results have been given in the standard spaces within previous research. As known, the modular function spaces are extensions of Riesz, Orlicz and Lebesgue where the basic concept of modular space introduced by Nakano [16] and corresponding modular linear spaces were constructed by Musielak and Orlicz [15]. Later, many researcher provided various studies in several fields, including approximating fixed point, see [1, 17]. Abed and AbdulSada studied two common fixed point about the dual of modular function space in ρ- nonexpansive mapping, and prove some results in weak and strong converge [2], Khan extend the idea λ-firmly nonexpansive mapping from Banach spaces to (λ, ρ)-firmly nonexpansive in modular function spaces, and introduced iterative scheme [13]. The (λ, ρ)- quasi firmly nonexpansive mapping in modular spaces introduced by Panwar and discussed some results for fixed point in these mapping [11]. The concept of normalized duality mapping discussed by Abed and Abduljabbar, in addition to approximating fixed point for convex modular spaces [3]. Finally Okeke, Bishop and Khan [18] proved some interesting theorems for ρ-quasi-nonexpansive mappings using the Picard-Krasnoselskii hybrid iterative processes and applied these results to solve the following problem in differential equations by using the same technique in [12], Theorem 5.28:

Let $\rho \in \mathfrak{R}$ consider the following initial value problem $u : [0, A] \longrightarrow E$ where $C \in E_p$, $u\,(0) = f$ and $u'\,(t) + (I - T) = 0$, where $f \in E,\ A > 0$, and $T : E \longrightarrow E$, ρ_ρ^T is ρ- quasi nonexpansive mapping and it solved throughout the following theorem.

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Theorem 1.1 ([18] Theorem 27). Let $\rho \in \Re$ be separable, and $E \subset E_\rho$ be nonempty, convex, ρ- bounded, ρ-closed set with Vitali property, $T : E \longrightarrow \rho_\rho(E)$ be a multivalued mapping such that ρ_ρ^T is ρ- quasi nonexpansive mapping, let one fixed $f \in E$, define sequence of function $u_n : [0, A] \longrightarrow E$ by the following inductive formula

$u_0\,(t) = f$

$u_{n+1}\,(t) = e^{-1}f + \int_0^t e^{s-t}T\,(u_n\,(s))\,\mathrm{ds}$
then for every $t \in [0, A]$ there exists $u(t) \in C$ such that $\rho(u_n\,(t) - u\,(t)) \longrightarrow 0$ and the function $u : [0, A] \longrightarrow E$ is solation to initial value problem, moreover

$\rho(f - u_n\,(t)) \leq K^{n+1}(A)\delta_\rho(E).$

Now, let $T : E \longrightarrow 2^E$, and E nonempty convex subset of L_p sequence, here, we introduced the sequence $\{f_n\}$ by the following algorithm.

$$\begin{aligned} &f_1 \in E \\ &h_n = (1-\beta_n)f_n + \beta_n u_n \\ &g_n = v_n \\ &J_n = (1-\alpha_n)\,g_n + \alpha_n w_n \\ &f_{n+1} = m_n, n \in N \end{aligned} \tag{1.1}$$

where $\{\alpha_n\}$and $\{\beta_n\}$ in (0,1), $u_n \in P_\rho^T(f_n),\ v_n \in P_\rho^T(h_n),\ \ w_n \in P_\rho^T(g_n)$ and $m_n \in P_\rho^T(J_n)$,

This paper concludes three convergence main results, comparison result and illustrative example to comparison between algorithm 1.1 and the following two well-known 1.2 and 1.3

$$\begin{aligned} &f_{n+1} \in P_\rho^T g_n \\ &g_n = (1-\lambda)\,f_n + \lambda P_\rho^T(v_n) \quad n \in N \end{aligned} \tag{1.2}$$

where $\{\lambda\} \subset (0, 1)$, $v_n \in P_\rho^T(f_n)$ [17].

$$\begin{aligned} &f_0 \in D \\ &g_n = (1-\beta_n)\,f_n + \beta_n u_n \\ &f_{n+1} = (1-\alpha_n)u_n + \alpha_n v_n,\ n \in N \end{aligned} \tag{1.3}$$

where $\{\alpha_n\}$and $\{\beta_n\}$ in (0,1) , $u_n \in P_\rho^T(f_n),\ v_n \in P_\rho^T(g_n)$ [6].

2 Preliminearies

This section is included with the basis required. Let Ω be a nonempty set and Σ be a nontrivial σ-algebra of subsets of L_p .let ρ be a nontrivial ring subsets of Ω, which means that ρ is closed with respect to forming finite union, and countable intersections and differences, Assume further that $E \cap A \in \rho$ for any $E \in \rho$ and $A \in \Sigma$, let us assume that there exists an increasing sequence of sets $K_n \in \rho$ such that $\Omega = \bigcup K_n$. Throughout this paper, $E :=$ the linear space of all simple functions with supports from ρ, $M_\infty :=$ we denote the space of all extended measurable functions, $1_A :=$ the characteristic function of the set A [17].
$f : \Omega \longrightarrow [-\infty, \infty]$ such that there exists a sequence $\{g_n\} \subset E$, $|g_n| \leq |f|$and $g_n(w) \longrightarrow f$ for all $w \in \Omega$, By 1_A we denote the characteristic function of the set A [5,10].

Definition 2.1 ([17]). Let $\rho : M_\infty \longrightarrow [0, \infty]$ be a nontrivial, convex, and even function. We say that ρ is a regular convex function pseudo modular if:

(a) $\rho\,(0) = 0$
(b) ρ is monotone, that is, $|f(w)| \leq |g(w)|$ for all $w \in \Omega$ implies $\rho(f) \leq \rho(g)$, where $f, g \in M_\infty$
(c) $\rho$is orthogonally sub additive , that is, $\rho\,(f_{1_{A\cup B}}) \leq \rho\,(f_{1_A}) + \rho\,(f_{1_B})$ for any $A, B \in \Sigma$ such that $A \cap B$ nonempty , where $f \in M_\infty$.
(d) ρ has the Fatou property: $|f_n(w)| \uparrow |f(w)|$ for all $w \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$,where $f \in M_\infty$.
(e) ρ is order continuous in E , that is, $g_n \in E$ and $|g_n(w)| \downarrow 0$ implies $\rho(g_n) \downarrow 0$.

we define $M = \{f \in M_\infty : \ |f(w)| < \infty, \ \rho - a.e\}$, where each $f \in M$ is actually an equivalence class of functions equal $\rho - a.e.$ rather than an individual function.

Definition 2.2 ([10]). Let $\rho : M \longrightarrow [0,\infty]$ possesses the following properties

1. $\rho(0) = 0$ iff , $f = 0, \ \rho - a.e$
2. $\rho(\alpha f) = \rho(f)$, for every scalar α.
3. $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$ for every $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$.

ρ is called a convex modular.

Definition 2.3 ([13]). If ρ is convex modular in X, then is called modular function spaces

$$L_p = \{f \in M : \ \rho(\lambda f) \longrightarrow 0 \ as \ \lambda \longrightarrow 0\}$$

The modular spaces L_p can be equipped with an F-norm define by

$$\|f\|_\rho = \inf\{\alpha > 0 \ : \ \rho\left(\frac{f}{\alpha}\right) \le \alpha\}$$

If ρ is convex modular F-norm is define

$$\|f\|_\rho = \inf\{\alpha > 0 \ : \ \rho\left(\frac{f}{\alpha}\right) \le 1\}$$

F-norm is called Luxemburg norm.

Also we define $L_\rho^0 = \{f \in L_\rho, \ \rho(f,.)$ is order continuous$\}$ and define the liner space $E_\rho = \{f \in L_\rho : \ \lambda f \in L_\rho^0$ for every $\lambda > 0$ $\}$

Definition 2.4 ([2, 3]). Let $\rho \in \mathfrak{R}$

1. We say that $\{f_n\}$is ρ-convergent to f if $\rho(f_n - f) \longrightarrow 0$
2. A sequence $\{f_n\}$ is ρ-Cauchy sequence if $\rho(f_n - f_m) \longrightarrow 0$ as $n, m \to \infty$
3. A set $B \subset L_p$ is called ρ-closed if for any $f_n \in L_p$ the convergence $\rho(f_n - f) \longrightarrow 0$and f belongs to B.
4. A set $B \subset L_p$ is called ρ-bounded if ρ- diameter is finite. ρ- diameter define as $\mathfrak{H}_p(B) = \sup\{\rho(f-g), f \in B, g \in B\} < \infty$.
5. A set $B \subset L_p$ is called strongly ρ-bounded if there exists $\beta > 1$ such that $M_p(B) = \sup\{\rho(\beta(f-g)), f \in B, g \in B\} < \infty$.
6. A set $B \subset L_p$ is called ρ-compact if every $f_n \ \in B$, there exists a subsequence $\{f_{n_k}\}$ and f in B $\rho(f_{n_k} - f) \to 0$.
7. A set $B \subset L_p$ is called $\rho - a.e$, closed if every $f_n \ \in B$, which$\rho - a.e$, converges to some f, then f in B.
8. A set $B \subset L_p$ is called $\rho - a.e$, -compact if every $f_n \ \in B$, there exists a subsequence $\{f_{n_k}\}$ $\rho - a.e$ -converges to some f in B.
9. Let f in L_p and $B \subset L_p$, the ρ-distance between f and B is defined as $\text{dist}_p(f, B) = \inf\{\rho(f-g), g \in B\}$.

Definition 2.5 ([9]). Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ by two iterative scheme sequence converging to the same fixed point s, and let $\lim_{n\longrightarrow\infty} \frac{\rho(a_n - s)}{\rho(b_n - s)} = L$, then

1. if $L = 0$ then $\{a_n\}_{n=1}^\infty$ converges faster than $\{b_n\}_{n=1}^\infty$ to fixed point s.
2. if $1 < L < \infty$ then $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ have the same rate of

Definition 2.6 ([13]). Let $\rho \in \mathfrak{R}$ then ρ has Δ_2-condition if $\sup_{n\ge1} \rho(2f_n, D_k) \longrightarrow 0$ as $k \longrightarrow \infty$ and $D_k \longrightarrow \varnothing$, and $\sup_{n\ge1} \rho(f_n, D_k) \longrightarrow 0$
ρ is regular convex function modular if $\rho(f) = 0$ then $f = 0, \ a - e$ the class of all nonzero regular convex function in modular Ω is denoted by $\mathfrak{R}$

Note that, $L_\rho = E_\rho$ if ρ is satisfied Δ_2-condition and convex.
Note that, modular converge and F-norm converge are equivalent if and only if ρ is satisfied Δ_2-condition

Definition 2.7 ([11]). Let ρ be a nonzero regular convex function modular defined on Ω let $r > 0,\ \epsilon > 0$ define $D(r, \epsilon) = \{(f, g) : f, g \in L_P,\ \rho f \leq r,\ \rho f - g \geq \epsilon r\}$
Let $\xi_1 (r, \epsilon) = \inf \left\{1 - \frac{1}{r}\rho(\frac{f+g}{2})\ : (f, g) \in D(r, \epsilon)\right\}$ if $D(r, \epsilon) \neq \varnothing$ and $\xi_1 (r, \epsilon) = 1$,If $D(r, \epsilon) = \varnothing$, said to be ρ satisfy (UC1) if for every r> 0, $\epsilon > 0$ $\xi_1 (r, \epsilon) > 0$ then $D(r, \epsilon) \neq \varnothing$.

Definition 2.8 ([13]). $E \subset L_p$, let $T : E \longrightarrow 2^E$said to be satisfy condition (I) if there exists no decreasing function $\varnothing : [0, \infty) \longrightarrow [0, \infty)$ with $\varnothing (0) = 0,\ \varnothing(r) > 0$ for all $r \in [0, \infty]$ such that $\rho(f - Tf) \geq \varnothing(\text{dist}_\rho (f, F_p (t)))$ for all $f \in E$.

Definition 2.9 ([14, 7]). A set $E \subset L_p$ is called ρ- proximinal if for each $f \in L_p$ there exists an element g in E such that

$\rho (f - g) = \text{dist}_p (f, E) = inf\{\ \rho (f - h) : h\ in\ E\}$.

$P_p(E)$:= the family of nonempty ρ-proximinal, ρ-bounded subsets of E

$C_p (E)$:=the family of nonempty ρ-closed, ρ-bounded subsets of E

$H_p(.,.)$:= ρ- Hausdorff distance on $C_p(E)$, where

$H_p (A, B) = \max\left\{\ \sup_{f\in A}\ \text{dist}_p\ (f, B)\ ,\ \ \sup_{g\in B}\ \text{dist}_p\ (g, A)\right\}\ A, B\ \in C_p(L_p)$

and $\text{dist}_p\ (f, B) = inf\{\ \rho (f - h) : h\ in\ B\}$

Lemma 2.10 ([11]). Let $\rho \in \mathfrak{R}$ satisfy (UUC1) and let $\{t_n\}$ in (0,1) be bounded away from 0 and 1, if there exists $m > 0$ such that

$$\lim sup_{n\longrightarrow\infty}\rho (f_n) \leq m,\ \lim sup_{n\longrightarrow\infty}\rho(g_n) \leq m$$

And $\lim_{n\longrightarrow\infty} \rho (t_n f_n + (1 - t_n) g_n) = m$, then $\lim_{n\longrightarrow\infty}\rho (f_n - g_n) = 0$

Lemma 2.11 ([14]). Let $\rho\in \mathcal{R}$ and satisfy $A, B \in P_p(L_p)$ for each f in A there exists g in B such that $\rho(f - g) \leq H_p(A, B)$.

Definition 2.12 ([14]). Let $T : E \longrightarrow 2^E$ is multivalued mapping said to be ρ- quasi nonexpansive mapping if for $s \in F_p(T)$ is the set of fixed point of T in modular spaces

$$H_p (Tf, s) \leq \rho (f - s)$$

said to be ρ-contraction mapping if there exists constant $0 \leq k < 1$

$$H_p(Tf - Tg) \leq k\rho (f - g)$$

for all f, g in E.

Definition 2.13 ([18]). Let $T : E \longrightarrow 2^E$ be a multivalued mapping, a sequence $\{f_n\}$ in E is said to be Fajer monotone if $\rho(f_{n+1} - s) \leq \rho(f_n - s)$ for all s fixed point.

3 Convergence Results

Begin this section with the following definition and useful Lemma.

Definition 3.1. Let $\subset L_p$, let $T : E \longrightarrow 2^E$ is multivalued mapping said to be said to be (λ, ρ)- quasi firmly nonexpansive mapping if for λ in (0,1) and $s \in F_p(T)$ is the set of fixed point of T in modular spaces

$H_p(Tf, s) \leq \rho[(1 - \lambda) (f - s) + \lambda (u - s)]$ where $u \in Tf$

Lemma 3.2. Every(λ,ρ)- quasi firmly nonexpansive mapping is ρ- quasi nonexpansive mapping

Proof . $H_p(Tf,s) \leq \rho\left[(1-\lambda)(f-s)+\lambda(u-s)\right]$, $u \in Tf$
By convexity of ρ, Lemma 2.11, and Definitions 2.12, 3.1, we get

$$\begin{aligned} H_p(Tf,s) &\leq (1-\lambda)\rho(f-s)+\lambda\rho(u-s) \\ &\leq (1-\lambda)\rho(f-s)+\lambda H_p(Tf,s) \end{aligned}$$

Hence $H_p(Tf,s) \leq \rho(f-s)$ □

Theorem 3.3. Let $\rho \in \mathfrak{R}$ satisfy (UUC1) and Δ_2-condition , let E be nonempty ρ-bounded, ρ-closed and convex $E \subset L_p$ and $T: E \longrightarrow 2^E$, be (λ,ρ)- quasi firmly nonexpansive multivalued mapping, let $\{f_n\}$ in E define by 1.1, then $\{f_n\}$ is Fajer monotone

Proof . $s \in F_p(T)$, by 1.1, convexity of ρ, Definitions 2.12, 3.1, Lemmas 2.11, 3.2 implies that

$$\rho(f_{n+1}-s) = \rho(m_n-s) \leq H_p(P_p^T(J_n),(s)) \leq \rho(J_n-s) \tag{3.1}$$

$$\begin{aligned} \text{And}\quad \rho(J_n-s) &\leq \rho\left((1-\alpha_n)g_n+\alpha_n w_n\right)-s) \\ &\leq ((1-\alpha_n)\rho(g_n-s)+\alpha_n\rho(w_n-s)) \\ &\leq (1-\alpha_n)\rho(g_n-s)+\alpha_n H_p(P_p^T(g_n),(s)) \\ &\leq \rho(g_n-s) \end{aligned} \tag{3.2}$$

$$\text{Similarity,}\quad \rho(g_n-s) = \rho(v_n-s) \leq H_p(P_p^T(h_n),(s)) \leq \rho(h_n-s) \tag{3.3}$$

$$\begin{aligned} \text{Similarity,}\quad \rho(h_n-s) &= \rho(\beta_n u_n+(1-\beta_n)f_n-s) \\ &\leq \beta_n H_p(P_p^T(f_n),(s))+(1-\beta_n)\rho(f_n-s) \\ &\leq \rho(f_n-s) \end{aligned} \tag{3.4}$$

By 3.1, 3.2, 3.3, 3.4 and Definition 2.13 $\{f_n\}$ is Fajer monotone □

Theorem 3.4. Let $\rho \in \mathfrak{R}$ satisfy (UUC1) and Δ_2-condition , let E be nonempty ρ-bounded, ρ-closed and convex $E \subset L_p$ and $T: E \longrightarrow 2^E$, be (λ,ρ)- quasi firmly nonexpansive multivalued mapping, let $\{f_n\}$ in E define by 1.1, then $\lim_{n\longrightarrow\infty}\rho(f_n-s)$ exists for all s is fixed point.

Proof . By 3.1, 3.2, 3.3 and 3.4 so, $\rho(f_{n+1}-s) \leq \rho(f_n-s)$ this prove is complet. □

Theorem 3.5. Let $\rho \in \mathfrak{R}$ satisfy (UUC1) and Δ_2-condition , let E be nonempty ρ-bounded, ρ-closed and convex $E \subset L_p$ and $T: E \longrightarrow 2^E$, be (λ,ρ)- quasi firmly nonexpansive multivalued mapping, let $\{f_n\}$ in E define by 1.1 then $\lim_{n\longrightarrow\infty}\ \mathrm{dist}_\rho\rho\left(f_n, P_p^T(f_n)\right) = 0$

Proof . By Theorem 3.4 $\lim_{n\longrightarrow\infty}\rho(f_n-s)$ exists

$$\text{Let}\qquad \lim_{n\longrightarrow\infty}\rho(f_n-s) = k, \quad \text{where } k \geq 0 \tag{3.5}$$

By 3.2, 3.3 and 3.4 the following hold

$$\rho(h_n-s) \leq \rho(f_n-s) \Rightarrow\ \lim_{n\longrightarrow\infty}\rho(h_n-s) \leq k \tag{3.6}$$

$$\lim_{n\longrightarrow\infty}\rho(g_n-s) \leq k \tag{3.7}$$

$$\lim_{n\longrightarrow\infty}\rho(J_n-s) \leq k \tag{3.8}$$

$$\begin{aligned} &\rho(v_n-s) \leq H_p(P_p^T(h_n),P_p^T(s)) \leq \rho(h_n-s) \leq \rho(f_n-s) \\ &\lim_{n\longrightarrow\infty}\rho(v_n-s) \leq \lim_{n\longrightarrow\infty}\rho(f_n-s) \leq k \end{aligned} \tag{3.9}$$

$$\begin{aligned} &\rho(u_n-s) \leq H_p\left(P_p^T(f_n),P_p^T(s)\right) \leq \rho(f_n-s), \\ \text{then } &\lim_{n\longrightarrow\infty}\rho(u_n-s) \leq k \end{aligned} \tag{3.10}$$

$$\rho(w_n - s) \le H_p(P_p^T(g_n), P_p^T(s)) \le \rho(g_n - s) \le \rho(f_n - s)$$

$$\text{then} \lim_{n\longrightarrow\infty} \rho(w_n - s) \le k \tag{3.11}$$

$$\rho(m_n - s) \le H_p(P_p^T(J_n), P_p^T(s)) \le \rho(J_n - s) \le \rho(f_n - s)$$

$$\text{then} \lim_{n\longrightarrow\infty} \rho(m_n - s) \le k \tag{3.12}$$

$$\text{Let} \lim_{n\longrightarrow\infty} \alpha_n = \alpha$$

$$\rho(f_{n+1} - s) = \rho(m_n - s) \le H_p(P_p^T(J_n), P_p^T(s)) \le \rho(J_n - s) \le \rho(\alpha_n w_n + (1-\alpha_n) g_n - s)$$

$$\le \alpha_n \rho(w_n - s) + (1-\alpha_n)\rho(g_n - s).$$

$$\text{so,} \lim_{n\longrightarrow\infty} \inf \rho(f_{n+1} - s) \le \lim_{n\longrightarrow\infty} inf[\alpha_n \rho(w_n - s) + (1-\alpha_n)\rho(g_n - s)]$$

$$\text{then,} \quad k \le \lim_{n\longrightarrow\infty} inf \alpha_n \rho(w_n - s) + (1-\alpha) k \Rightarrow \alpha k \le \alpha \lim_{n\longrightarrow\infty} inf \rho(w_n - s)$$

$$\text{hence,} \quad k \le \lim_{n\longrightarrow\infty} inf \rho(w_n - s) \tag{3.13}$$

By 3.11 and 3.13,

$$\lim_{n\longrightarrow\infty} \rho(w_n - s) = k \tag{3.14}$$

$$\rho(w_n - s) \le H_p(P_p^T(g_n), P_p^T(s)) \le \rho(g_n - s)$$

$$\text{then,} \quad k \le \rho(g_n - s) \tag{3.15}$$

By 3.7 and 3.15,

$$\lim_{n\longrightarrow\infty} \rho(g_n - s) = k \tag{3.16}$$

Since,

$$\rho(g_n - s) = \rho(v_n - s), so, \lim_{n\longrightarrow\infty} \rho(v_n - s) = k \tag{3.17}$$

$$\rho(v_n - s) \le H_p(P_p^T(h_n), P_p^T(s)) \le \rho(h_n - s) \Rightarrow \lim_{n\longrightarrow\infty} \rho(v_n - s) \le \lim_{n\longrightarrow\infty} \rho(h_n - s)$$

$$\text{so,} \quad k \le \lim_{n\longrightarrow\infty} \rho(h_n - s) \tag{3.18}$$

By 3.6 and 3.18, then

$$\lim_{n\longrightarrow\infty} \rho(h_n - s) = k \tag{3.19}$$

By 3.19,

$$\lim_{n\longrightarrow\infty} \rho(h_n - s) = k \Rightarrow \lim_{n\longrightarrow\infty} \rho(\beta_n u_n + (1-\beta_n) f_n - s) = k$$

$$\lim_{n\longrightarrow\infty} \rho(\beta_n (u_n - s) + (1-\beta_n)(f_n - s) = k \tag{3.20}$$

By 3.5, 3.10, 3.20 and Lemma 2.11,

$$\lim_{n\longrightarrow\infty} \rho(f_n - u_n) = 0.$$

Then $u_n \in P_p^T(f_n)$. Since $\text{dist}_\rho \rho\left(f_n, P_p^T(f_n)\right) \le \lim_{n\longrightarrow\infty} \rho(f_n - u_n)$, $\lim_{n\longrightarrow\infty}$ $\text{dist}_\rho \rho\left(f_n, P_p^T(f_n)\right) = 0$. This completes the proof. □

Theorem 3.6. Let $\rho \in \Re$ satisfy (UUC1) and Δ_2-condition , let E be nonempty ρ-bounded, ρ-closed, ρ-compact and convex $E \subset L_p$ and $T : E \longrightarrow 2^E$, be (λ, ρ)- quasi firmly nonexpansive multivalued mapping, and T satisfied condition (I), let $\{f_n\}$ in E define by 1.1 then f_n converge to fixed point s of T.

Proof . By Theorem 3.4 $\lim_{n\longrightarrow\infty} \rho(f_n - s)$ exists for all s is fixed point, if $\lim_{n\longrightarrow\infty} \rho(f_n - s) = 0$, nothing to prove, if $\lim_{n\longrightarrow\infty} \rho(f_n - s) = k, \ k \ge 0$

Since $\rho(f_{n+1} - s) \leq \rho(f_n - s)$,

$$\text{dist}_\rho(f_{n+1}, F_p(T)) \leq \text{dist}_\rho(f_n, F_p(T)).$$

So $\lim_{n\longrightarrow\infty} \text{dist}_\rho(f_n, F_p(T))$ exists, by applying condition (I) and Theorem 3.5

$$\lim_{n\longrightarrow\infty} \varnothing(\text{dist}_\rho(f_n, F_p(T)) \leq \lim_{n\longrightarrow\infty} \text{dist}_\rho \rho\left(f_n, P_p^T(f_n)\right) = 0.$$

Since $\varnothing(0) = 0$, we have

$$\lim_{n\longrightarrow\infty} \text{dist}_\rho(f_n, F_p(T)) = 0.$$

By Theorem 3.4 $\lim_{n\longrightarrow\infty} \rho(f_n - s)$ exists, then $\lim_{n\longrightarrow\infty} \rho(f_n - F_p(T))$ exists and $s \in F_p(T)$. Suppose that f_{n_k} is a subsequence of f_n, and u_k is a sequence in $F_p(T)$. Then $\rho(f_{n_k} - u_k) \leq \frac{1}{2^k}$, because $\liminf_{n\longrightarrow\infty} \text{dist}_p(f_n, F_p(T)) = 0$. So

$$\rho(f_{n_{+1}} - u_k) \leq \rho(f_n - u_k) \leq \frac{1}{2^k}.$$

Thus,

$$\rho(u_{k+1} - u_k) \leq \rho(u_{k+1} - f_{n+1}) + \rho(f_{n+1} - u_k) \leq \frac{1}{2^{k+1}} + \frac{1}{2^k} \leq \frac{1}{2^{k-1}}.$$

This implies that

$$\rho(u_{k+1} - u_k) \longrightarrow 0$$

as $k \longrightarrow \infty$. Hence, u_k is a ρ-Cauchy in $F_p(T)$. Since Δ_2 condition implies that ρ-cauchy$\Longleftrightarrow$ ρ-converge. So, u_k is ρ-converges to $F_p(T)$, then $\rho(u_k - s) \longrightarrow 0$. Now, we havw

$$\rho(f_{n_k} - s) \leq \rho(f_{n_k} - u_k) + \rho(u_k - s).$$

Hence, f_n converges to fixed point s in $F_p(T)$. □

4 Faster Convergence Results

In this section, we will prove that the iterative scheme in equation 1.1 is faster than iterative schemes in 1.2 and 1.3, in contraction mapping, and prove the iterative scheme in 1.2 is faster than iterative scheme in 1.3 through the following theorem.

Theorem 4.1. Let $\rho \in \Re$ satisfy (UUC1), let E be nonempty ρ-bounded, ρ-closed and convex $E \subset L_p$ and $T : E \longrightarrow 2^E$, be contraction multivalued mapping, let α_n and β_n in (0,1), consider iterative scheme, defined by 1.1,1.2 and 1.3 respectively then

1. the iterative scheme in 1.1 converges to fixed point s faster than 1.2 and 1.3
2. the iterative scheme in 1.2 converges to fixed point s faster than 1.3.

Proof . Firstly the iterative scheme in 1.1

By 1.1, convexity of ρ, Lemma 2.11, 3.2, and Definition 2.12, 3.1

$\rho(f_{n+1} - s) = \rho(m_n - s) \leq H_p(P_p^T(J_n), P_p^T(s)) \leq K\rho(J_n - s)$

$\leq k((1-\alpha_n)\rho(g_n - s) + \alpha_n H_p(P_p^T(g_n), P_p^T(s))$

$= k((1-\alpha_n) + k\alpha_n)\rho(g_n - s))$

$\leq k((1-\alpha_n) + k\alpha_n)H_p(P_p^T(h_n), P_p^T(s)))$

$\leq k(k(1-\alpha_n) + k^2\alpha_n)\rho((1-\beta_n)f_n + \beta_n u_n - s))$

$\leq k(k(1-\alpha_n) + k^2\alpha_n)((1-\beta_n)\rho(f_n - s) + \beta_n H_p(P_p^T(f_n), P_p^T(s)))$

$\leq k(k(1-\alpha_n)(1-\beta_n) + k^2\alpha_n(1-\beta_n) + k^2(1-\alpha_n)\beta_n + k^3\alpha_n\beta_n)\rho(f_n - s))$

assume $\alpha_n = \beta_n = \gamma$

$$\rho(f_n - s) \leq k^n(k(1-\gamma)^2 + 2k^2\gamma(1-\gamma) + k^3\gamma^2)^n \rho(f_0 - s)) \tag{4.1}$$

By the same way, the iterative scheme in 1.2, and assume $\lambda_n = \gamma$

$$\rho(f_{n+1}-s) \leq k^n((1-\gamma)+k\gamma)^n \rho(f_0-s) \tag{4.2}$$

By the same way, the iterative scheme in 1.3, and assume $\alpha_n = \beta_n = \gamma$

$$\rho(f_{n+1}-s) \leq k^n[(1-\gamma^2)+k\gamma^2]^n \rho(f_0-s) \tag{4.3}$$

1. By 4.1 and 4.2 $\lim_{n\longrightarrow\infty} \frac{\rho(f_n-s)\ in\ 1.1}{\rho(f_n-s) in\ 1.2} = 0$ then 1.1 converges to fixed point s faster than 1.2.
   and by 4.1 and 4.3 $\lim_{n\longrightarrow\infty} \frac{\rho(f_n-s)\ in\ 1.1}{\rho(f_n-s)\ in\ 1.3} = 0$ then 1.1 converges to fixed point s faster than 1.3
2. By 4.2 and 4.3 $\lim_{n\longrightarrow\infty} \frac{\rho(f_n-s)\ in\ 1.2}{\rho(f_n-s)\ in\ 1.3} = 0$ then 1.2 converges to fixed point s faster than 1.3.

□

The results of the above theorem will be confirmed by the following example

Example 4.2. The set of real number $\Re$ by the space $\rho(f) = |f|$, ρ is satisfy (UUC1) and Δ_2-condition, $E = [0, 2]$ define $T : E \longrightarrow E$ a mapping, $\varnothing : [0, \infty) \longrightarrow [0, \infty)$, $\varnothing(r) = \frac{r}{8}$ and

$$Tf = \begin{cases} 1 & if\ f \in [0,1] \\ \frac{f+3}{4} & if\ f \in [1,2] \end{cases}, \qquad F_p(T) = \{1\}.$$

To prove $\rho(f - Tf) \geq \varnothing(\mathrm{dist}_p(f, F_p(T))$ for all $f\ in\ E$. If $f \in [0,1]$, then $\rho(f - Tf) = \rho(f-1) = |f-1| = f-1$, while

$$\varnothing(\mathrm{dist}_p(f, F_p(T)) = \varnothing(\mathrm{dist}_p(f, \{1\}) = \phi[\rho(f-1)] = \frac{f-1}{8}.$$

If $f \in [1,2]$, then $\rho(f - Tf) = \rho\left(f - \frac{f+3}{4}\right) = \frac{3f+3}{4}$, while

$$\varnothing(\mathrm{dist}_p(f, F_p(T)) = \varnothing(\mathrm{dist}_p(f, \{1\}) = \phi[\rho(f-1)] = \frac{f-1}{8}.$$

Now, prove T is (λ,ρ)-quasi firmly nonexpansive mapping. If $f \in [0,1]$, then $\rho(Tf - s) = \rho(1-s) = \rho\left(\frac{4}{4}(1-s)\right) = \rho(\frac{3}{4}(1-s) + \frac{1}{4}(1-s))$, T is (λ,ρ)-quasi nonexpansive mapping when $\lambda = \frac{1}{4}$.

If $f \in [1,2]$, then $\rho(Tf - s) = \rho\left(\frac{f+3}{4} - \frac{s+3}{4}\right) = \left|\frac{1}{4}(f-s)\right| \leq \left|\frac{13}{16}(f-s)\right| \leq \rho(\frac{3}{4}(f-s) + \frac{1}{4}\left(\frac{1}{4}(f-s)\right))$, T is (λ,ρ)-quasi nonexpansive mapping when $\lambda = \frac{1}{4}$.

We will comparison the numerical results of first, second and third equations, as shown in the tables 1, 2

Table 1: shown f_n in 1.1, 1.2 and 1.3 where $\alpha_n = \beta_n = \lambda_n = 0.5,\ with\ f_1 = 2$

step	f_n in 1.1	f_n in 1.2	f_n in 1.3
1	2	2	2
2	1.024414063	1.15625	1.203125
3	1.000596046	1.024414063	1.041259765
4	1.000014552	1.003814697	1.00838089
5	1.000000355	1.000596046	1.001702368
6	1.000000008	1.000093137	1.000345793
7	1	1.000014552	1.000070239
8	1	1.000002274	1.000014267
9	1	1.000000355	1.000002897
10	1	1.000000055	1.000000588
11	1	1.000000008	1.000000119
12	1	1.000000001	1.000000024
13	1	1	1.000000004
14	1	1	1

5 Conclusion

In this paper, the concept of (λ,ρ)-quasi firmly multivalued nonexpansive mappings were introduced in modular function spaces and some results to approximate the fixed points of these mappings on a faster iterative algorithm were proved. Through Example 4.2 it was noted that Tables 1 and 2, show that $\{f_n\}$ ρ-converges to 1, the fixed point of T on 6th iteration and 7th iteration. $\{f_n\}$ ρ-converges faster to 1 if α_n, β_n and λ_n near the fixed point. As an application, it is possible to adopt Theorem 1.1 and algorithm 15 which was considered in [18] as a special case of algorithm 1.

Table 2: shown f_n in 1.1, 1.2 and 1.3 where $\alpha_n = \beta_n = \lambda_n = 0.9$, *with* $f_1 = 2$

step	f_n in 1.1	f_n in 1.2	f_n in 1.3
1	2	2	2
2	1.006601563	1.08125	1.098125
3	1.00004358	1.0006601563	1.009628515
4	1.000000288	1.000536377	1.000944798
5	1.000000002	1.000043580	1.000092708
6	1	1.000003541	1.000009097
7	1	1.0000002880	1.000000892
8	1	1.0000000023	1.000000087
9	1	1.000000002	1.000000008
10	1	1	1.0000000001
11	1	1	1

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