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A NEW METHOD FOR THE GENERALIZED HYERS-ULAM-RASSIAS STABILITY

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Dedicated to the 70th Anniversary of S.M.Ulam's Problem for Approximate Homomorphisms

ABSTRACT. We propose a new method, called the *the weighted space method*, for the study of the generalized Hyers-Ulam-Rassias stability. We use this method for a nonlinear functional equation, for Volterra and Fredholm integral operators.

1. Introduction

The stability problem as started by S.M. Ulam [52] reads as follows: given a group G_1, a metric group G_2 with metric d and a positive number ε, find a positive number δ such that for every $f : G_1 \to G_2$ satisfying

$$d(f(xy), f(x)f(y)) \leq \delta, \quad \forall x, y \in G_1$$

there exists a homomorphism $h : G_1 \to G_2$ with

$$d(f(x), h(x)) \leq \varepsilon, \quad \forall x \in G_1.$$

In 1941, Hyers [22] gave an affirmative answer to the question of Ulam for additive Cauchy equation in Banach spaces.

Let E_1, E_2 be Banach spaces and let $f : E_1 \to E_2$ be a mapping satisfying:

$$\|f(x + y) - f(x) - f(y)\| \leq \delta.$$

for all $x, y \in E_1$ and $\delta > 0$. There exists a unique additive mapping $T : E_1 \to E_2$ which satisfies

$$\|f(x) - T(x)\| \leq \delta, \quad \forall x \in E_1.$$

Hyers proved that the limit

$$T(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for all $x \in E_1$. A generalized solution to Ulam's problem for approximately linear mappings was proved by Th.M. Rassias [42] in 1978. Th.M. Rassias considered

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a mapping $f : E_1 \to E_2$ such that $t \to f(tx)$ is continuous in t for each fixed x. Assume that there exists $\theta \geq 0$ and $0 \leq p < 1$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p) \quad \textit{for any } x, y \in E_1.$$

Then there exists an unique linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p \quad \textit{for any } x, y \in E_1.$$

Thus, the Hyers' Theorem follows as a special case of Th.M.Rassias' Theorem for p=0. Th.M.Rassias' proof of his Theorem [42] applies as well for all real values of p that are strictly less than zero. In 1991, Th.M.Rassias [43] introduced the generalized Hyers sequence.

In 1994, P.Găvruţa [15] provided a generalization of Th.M.Rassias' Theorem for the unbounded Cauchy difference and introduced the concept of generalized Hyers-Ulam-Rassias stability in the spirit of Th.M.Rassias approach.

Theorem 1.1. *Let G and E be an abelian group and a Banach space, respectively, and let $\varphi : G^2 \to [0, \infty)$ be a function satisfying*

$$\Phi(x, y) = \sum_{k=0}^{\infty} 2^{-k-1}\varphi(2^k x, 2^k y) < \infty$$

for all $x, y \in G$. If a function $f : G \to E$ satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x, y)$$

for any $x, y \in G$, then there exists a unique additive function $A : G \to E$ with

$$\|f(x) - A(x)\| \leq \Phi(x, x)$$

for all $x \in G$. If moreover G is a real normed space and $f(tx)$ is continuous in t for each fixed x in G, then A is a linear function.

For a number of generalizations of Hyers' Theorem for the stability of the additive mappings as well as Hyers-Th.M.Rassias' approach for the stability of the linear mapping the reader is referred to [2],[4],[5],[8]-[12],[22],[24]-[28],[30]-[33],[35]-[37],[39],[41],[43]-[51]. Open problems in the field were solved in[14], [16]-[21].

On the other hand, in 1991 J.A.Baker used the Banach fixed point theorem to give Hyers-Ulam stability results for a nonlinear functional equation. Following this idea, V.Radu [40] applied the fixed point alternative theorem for Hyers-Ulam-Rassias stability, D. Miheţ [34] applied the Luxemburg-Jung fixed point theorem in generalized metric spaces to study the Hyers-Ulam stability for two functional equations in a single variable, L.Găvruţa [13] used the Matkowski's fixed point theorem to obtain a new general result concerning the Hyers-Ulam stability of a functional equation in a single variable.

In this paper we give a new method, called *the weighted space method*, for the study of the generalized Hyers-Ulam-Rassias stability. We use this method for a nonlinear functional equation, for Volterra and Fredholm integral operators.

We apply the following theorem on weighted spaces:

Theorem 1.2. *(Banach) Let (X,d) be a complete metric space and $T : X \to X$ a contraction, i.e. there exists $\alpha \in [0,1)$ such that*

$$d(Tx,Ty) \leq \alpha d(x,y), \quad \forall x,y \in X.$$

Then there exists a unique $a \in X$ such that $Ta = a$. Moreover, $a = lim_{n\to\infty} T^n x$, and

$$d(a,x) \leq \frac{1}{1-\alpha} d(x,Tx), \quad for\ any\ x \in X.$$

2. The stability of a nonlinear functional equation

In the following, we consider a nonempty set S, (X,d) a complete metric space, $\eta : S \to S$, $\varphi : S \to (0,\infty)$, $F : S \times X \to X$.

Theorem 2.1. *We suppose that there exists $\alpha \in [0,1)$ so that*

$$\varphi(\eta(x))d(F(x,u(\eta(x))),F(x,v(\eta(x)))) \leq \alpha\varphi(x)d(u(\eta(x)),v(\eta(x))), \quad x \in S \tag{2.1}$$

If $y : S \to X$ is so that:

$$d(y(x),F(x,y(\eta(x)))) \leq \varphi(x), \quad x \in S \tag{2.2}$$

Then there exists a unique $y_0 : S \to X$ such that

$$y_0(x) = F(x,y_0(\eta(x))) \tag{2.3}$$

and

$$d(y(x),y_0(x)) \leq \frac{1}{1-\alpha}\varphi(x), \quad x \in S. \tag{2.4}$$

Proof. We denote

$$Y = \left\{ u : S \to X : \sup_{x\in S} \frac{d(u(x),y(x))}{\varphi(x)} < \infty \right\}$$

Then Y is a complete metric space with the metric

$$\rho(u,v) = \sup_{x\in S} \frac{d(u(x),v(x))}{\varphi(x)}$$

We take

$$(Tu)(x) = F(x,u(\eta(x))), \quad x \in S$$

The condition (2.2) proves that $u \in Y \implies Tu \in Y$. We have

$$\begin{aligned}\rho(Tu,Tv) &= \sup_x \frac{d(F(x),u(\eta(x)),F(x,v(\eta(x))))}{\varphi(x)} \\ &\leq \sup_x \frac{\alpha d(u(\eta(x)),v(\eta(x)))}{\varphi(\eta(x))} \\ &\leq \alpha\rho(u,v)\end{aligned}$$

□

A particular case of this theorem was obtained in [7] using the fixed point alternative theorem.

3. The stability of the Volterra integral operator

We consider $I = [a, b]$, $c \in I$. We denote by $C(I)$ the space of all complex-valued continuous functions on I. Consider the functions $L : I \to [0, \infty)$ to be integrable, $g \in C(I)$, $f : I \times \mathbb{C} \to \mathbb{C}$ and $\varphi : I \to (0, \infty)$ continuous.

Theorem 3.1. *We suppose that:*

there exists a unique $\alpha \in [0, 1)$ *so that*

$$|\int_c^x L(t)\varphi(t)dt| \leq \alpha\varphi(x), \ \ x \in I; \tag{3.1}$$

$$|f(t, u(t)) - f(t, v(t))| \leq L(t)|u(t) - v(t)|, \ \ t \in I, \forall u, v \in C(I) \tag{3.2}$$

If $y \in C(I)$ *is so that*

$$|y(x) - g(x) - \int_c^x f(t, y(t))dt| \leq \varphi(x), \ \ x \in I$$

then there exists a unique $y_0 \in C(I)$:

$$y_0(x) = g(x) + \int_c^x f(t, y_0(t))dt$$

and

$$|y(x) - y_0(x)| \leq \frac{\varphi(x)}{1 - \alpha}, \quad x \in I.$$

Proof. We apply Theorem 1.2 with $X = C(I)$, the metric:

$$d(u, v) = \sup_{x \in I} \frac{|u(x) - v(x)|}{\varphi(x)},$$

and the operator:

$$(Tu)(x) = g(x) + \int_c^x f(t, u(t))dt$$

We have:

$$\begin{aligned} d(Tu, Tv) &= \sup_{x \in I} \frac{|\int_c^x [f(t, u(t)) - f(t, v(t))]dt|}{\varphi(x)} \\ &\leq \sup_{x \in I} \frac{|\int_c^x L(t)|u(t) - v(t)|dt|}{\varphi(x)} \\ &\leq \sup_{t \in I} \frac{|u(t) - v(t)|}{\varphi(t)} \sup_{x \in I} \frac{|\int_c^x L(t)\varphi(t)dt|}{\varphi(x)} \\ &\leq \alpha d(u, v). \end{aligned}$$

□

A particular case of this Theorem was obtained in [29] using the fixed point alternative theorem. See also [6].

4. The stability of the Fredholm operator

We consider $I = [a, b]$, $g \in C(I)$, $\varphi : I \to (0, \infty)$ continuous, $L : I \times I \to [0, \infty)$ integrable $K : I \times I \times \mathbb{C} \to \mathbb{C}$ continuous.

Theorem 4.1. *We suppose that there exists $\beta > 0$:*

$$\int_I L(x,t)\varphi(t)dt \leq \beta\varphi(x), x \in I; \tag{4.1}$$

$$|K(x,t,u(t)) - K(x,t,v(t))| \leq L(x,t)|u(t) - v(t)|, \ u, v \in C(I). \tag{4.2}$$

Let $y \in C(I)$ be so that:

$$|y(x) - g(x) - \lambda \int_I K(x,t,y(t))dt| \leq \varphi(x), \quad x \in I.$$

If $|\lambda| < \frac{1}{\beta}$ then there exists a unique $y_0 \in C(I)$:

$$y_0(x) = g(x) + \lambda \int_I K(x,t,y_0(t))dt$$

and

$$|y(x) - y_0(x)| \leq \frac{\varphi(x)}{1 - |\lambda|\beta}, \quad x \in I.$$

Proof. We apply Theorem 1.2 with $X = C(I)$, the metric:

$$d(u,v) = \sup_{x \in I} \frac{|u(x) - v(x)|}{\varphi(x)}$$

and the operator:

$$(Tu)(x) = g(x) + \lambda \int_I K(x,t,u(t))dt.$$

We have:

$$\begin{aligned} d(Tu,Tv) &= |\lambda| \sup_{x \in I} \frac{|\int_I [K(x,t,u(t)) - K(x,t,v(t))]dt|}{\varphi(x)} \\ &\leq |\lambda| \sup_{x \in I} \frac{\int_I L(x,t)|u(t) - v(t)|dt}{\varphi(x)} \\ &\leq |\lambda| \sup_{t \in I} \frac{|u(t) - v(t)|}{\varphi(t)} \sup_{x \in I} \frac{\int_I L(x,t)\varphi(t)dt}{\varphi(x)} \\ &\leq |\lambda|\beta d(u,v) \end{aligned}$$

□

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