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Hyers-Ulam stability and well-posedness for fixed point problems on quasi b-metric spaces

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Abstract

In this paper, we ensure the existence of a unique fixed point in quasi b-metric spaces for some contraction mappings requiring the concept of Ψ^*-admissibility. The Ulam-Hyers stability and well-posedness for these fixed point results have been studied and investigated. The obtained results generalize and extend many known results in the literature.

Keywords: Quasi b-metric space, Fixed point, Ulam-Hyers stability, Well posedness
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1 Introduction

The Ulam stability is a type of a functional equation stability that has been originated with a question posed by Ulam [26] in 1940 regarding the stability of group homomorphisms. One year later, Hyers [16] provided a partial answer to Ulam's question for Banach spaces, which it subsequently referred to the Ulam-Hyers stability. Several published results on the so-called Hyers–Ulam stability have relaxed the stability conditions. Many mathematicians extended the Hyers results in variant directions. The first authors who studied Hyers-Ulam stability of partial differential equations were Prastaro and Rassias [20]. After that, a few results in this direction were given by other authors, regarding partial differential equations [13, 14]. In 2009, Rus [22] has opened a new direction of study of the Ulam stability using Gronwall type inequalities and Picard operators technique. For furtrher details, see [12, 17, 18]. Another direction of stability research is that in which results regarding fixed point theory are used. Namely, Bota-Boriceanu and Petrusel [7] and Bota et al. [8], have researched and expanded stability of Ulam-Hyers [1, 3, 4, 5, 6, 9, 21, 25].

On the other hand, Czerwik [10] initiated the notion of a b-metric space by changing the triangle inequality with a more generalized inequality involving a coefficient $s \geq 1$. Later, a new space named as a quasi b-metric space was proposed by Felhi et al. [11], which is as a combination of a b-metric space and a quasi metric space. In quasi b-metric spaces, the symmetry property is omitted.

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Definition 1.1. [11] Let $\aleph$ be a nonempty set. Given a real number $s \geq 1$. A function $\hbar : \aleph \times \aleph \longrightarrow [0, \infty)$ is referred to a quasi b-metric function if it meets the following conditions for every $k, l, j \in \aleph$:

(i) $\hbar(k, l) = 0$ if and only if $k = l$;

(ii) $\hbar(k, j) \leq s[\hbar(k, l) + \hbar(l, j)]$.

A pair $(\aleph, \hbar)$ is said to be a quasi b-metric space.

Due to lack of symmetry, we need to give the Cauchyness and the convergence of a sequence in a quasi b-metric space $(\aleph, \hbar)$.

Definition 1.2. [2, 11] Every sequence $\{l_n\}$ in $\aleph$ converges to some $\omega \in \aleph$ if and only if

$$\lim_{n \longrightarrow \infty} \hbar(l_n, \omega) = \lim_{n \longrightarrow \infty} \hbar(\omega, l_n).$$

Definition 1.3. [2, 11] Every sequence $\{l_n\}$ in $\aleph$ is called left-Cauchy (right-Cauchy) if and only if for each $\epsilon > 0$, an integer number $K = K(\epsilon) > 0$ exists such that $\hbar(l_n, l_m) < \epsilon$ for all $n \geq m > K$ ($\hbar(l_n, l_m) < \epsilon$ for all $m \geq n > K$).

Definition 1.4. [2, 11] Every sequence $\{l_n\}$ in $\aleph$ is called Cauchy if and only if for each $\epsilon > 0$, an integer number $K = K(\epsilon) > 0$ exists such that $\hbar(l_n, l_m) < \epsilon$ for all $m, n > K$.

Lemma 1.1. [2, 11] Let $\Bbbk : \aleph \longrightarrow \aleph$ be a continuous mapping at some $u \in \aleph$. Then, for any sequence $\{l_n\} \in \aleph$ converging to u, we have $\Bbbk l_n \longrightarrow \Bbbk u$, i.e.,

$$\lim_{n \longrightarrow \infty} \hbar(\Bbbk l_n, \Bbbk u) = \lim_{n \longrightarrow \infty} \hbar(\Bbbk u, \Bbbk l_n) = 0.$$

Samet et al. [24] proposed the concept of α-admissibility in 2012. Using this concept, they showed that several known published papers are not real generalizations.

Definition 1.5. Let $\aleph$ be a non-empty set and $\alpha : \aleph \times \aleph \longrightarrow [0, \infty)$ be a function. For a given real number $s \geq 1$, the mapping $\Bbbk : \aleph \longrightarrow \aleph$ is named α-admissible, if it meets the condition:

$$k, l \in \aleph, \quad \alpha(k, l) \geq 1 \Longrightarrow \alpha(\Bbbk(k), \Bbbk(l)) \geq 1.$$

The above definition is generalized as follows:

Definition 1.6. Let $\aleph$ be a nonempty set and $\Psi^* : \aleph \times \aleph \longrightarrow [0, \infty)$ be a function. For a given real number $s \geq 1$, the mapping $\Bbbk : \aleph \longrightarrow \aleph$ is called Ψ^*-admissible (or Ψ^* $-b$ -admissible), if it meets the condition:

$$l, k \in \aleph, \quad \Psi^*(l, k) \geq \frac{1}{s^2} \Longrightarrow \Psi^*(\Bbbk(l), \Bbbk(k)) \geq \frac{1}{s^2}.$$

It is clear that every α-admissible mapping is Ψ^*-admissible, but the converse is not true. To illustrate the difference between Ψ^*-admissibility and α-admissibility, we give the following examples.

Example 1.1. Let $\aleph = \Re$ and $s = 2$. Define $\Bbbk : \aleph \to \aleph$ and $\Psi^* : \aleph \times \aleph \to [0, \infty)$ as follows:

$$\Bbbk(l) = -l, \text{ for all } l \in \aleph$$

and

$$\Psi^*(l, k) = \begin{cases} 2 & \text{if } l \geq k \\ \dfrac{1}{2} & \text{otherwise.} \end{cases}$$

Clearly, the mapping $\Bbbk$ is Ψ^*-admissible. While, for $l \geq k$ we have $\Psi^*(l, k) \geq 1$ and $\Psi^*(\Bbbk(l), \Bbbk(k)) = \frac{1}{2} < 1$, so $\Bbbk$ is not α- admissible.

Example 1.2. Let $\aleph = [0,\infty)$ and $s = 2$. Define $\Bbbk : \aleph \to \aleph$ and $\Psi^* : \aleph \times \aleph \to [0,\infty)$ as follows:

$$\Bbbk(l) = \begin{cases} \dfrac{l}{2} & \text{if } l \in [0,2] \\ \ln(l) & \text{otherwise.} \end{cases}$$

and

$$\Psi^*(l,k) = \begin{cases} \frac{(l+k)^2}{2} + \frac{1}{2} & \text{if } l,k \in [0,2] \\ \dfrac{1}{1+\min\{l,k\}} & \text{otherwise.} \end{cases}$$

Clearly, the mapping $\Bbbk$ is Ψ^*-admissible. While, we have $\Psi^*(\frac{1}{2},\frac{1}{2}) \geq 1$ and $\Psi^*(\Bbbk(l),\Bbbk(k)) = \frac{5}{8} < 1$, so $\Bbbk$ is not $\alpha - admissible$.

In the next definition, we generalize the concept of transitivity, which is useful in the sequel.

Definition 1.7. For a nonempty set $\aleph$ and a given real number $s \geq 1$, we say that $\Psi^* : \aleph \times \aleph \longrightarrow [0,\infty)$ is generalized transitive (or a $b-$transitive) function, if it meets the condition:
$l,k,j \in \aleph$, $\Psi^*(l,k) \geq \frac{1}{s^2}$ and $\Psi^*(k,j) \geq \frac{1}{s^2} \Longrightarrow \Psi^*(l,j) \geq \frac{1}{s^2}$.

In this paper, we establish some fixed point results on quasi b-metric spaces for some contraction mappings via the concept of Ψ^*- admissibility. We also study their Ulam-Hyers stability and well-posedness.

2 Main results

For $s \geq 1$, let ω be the class of all functions $\beta : [0,\infty) \longrightarrow [0, \frac{1}{s^2})$ so that for any sequence $\{t_m\}$ of nonnegative real numbers, we have

$$\lim_{m\longrightarrow\infty} \beta(t_m) = \frac{1}{s^2} \Longrightarrow \lim_{m\longrightarrow\infty} t_m = 0.$$

Definition 2.1. Let $(\aleph,\hbar)$ be a quasi b-metric space and $\Bbbk : \aleph \longrightarrow \aleph$ be a self-mapping. We say that $\Bbbk$ is an $\Psi^* - \beta - contraction$ if there are two functions $\Psi^* : \aleph \times \aleph \longrightarrow [0.\infty)$ and $\beta \in \Omega$ such that

$$[\Psi^*(l,k) - \frac{1}{s^2} + \rho_*]^{d\hbar(\Bbbk(l),\Bbbk(k))} \leq \rho^{d\beta(\hbar(l,k))\hbar(l,k)} \tag{2.1}$$

for all $l,k \in \aleph$, where $d \geq 1$ and $1 \leq \rho \leq \rho_*$.

Theorem 2.1. Let $(\aleph,\hbar)$ be a complete quasi b-metric space and $\Bbbk : \aleph \longrightarrow \aleph$ be an $\Psi^* - \beta$- contraction mapping such that

- (i) $\Bbbk$ is Ψ^*-admissible;
- (ii) Ψ^* is generalized transitive;
- (iii) there is $l_0 \in \aleph$ such that $\Psi^*(l_0,\Bbbk(l_0)) \geq \frac{1}{s^2}$ and $\Psi^*(\Bbbk(l_0),l_0) \geq \frac{1}{s^2}$;
- (iv) $\Bbbk$ is continuous.

Then, there exists a fixed point $x^* \in \aleph$ of $\Bbbk$, that is, $x^* = \Bbbk(x^*)$.

Proof . For such $l_0 \in \aleph$ given in condition (iii), define a sequence $\{l_n\}$ by

$$l_n = \Bbbk(l_{n-1}) \;\; \forall n \in \mathbb{N}. \tag{2.2}$$

We assume that $l_n \neq l_{n-1}$ for all $n \in \mathbb{N}$ (Otherwise, if $l_k = l_{k-1}$ for some $k \in \mathbb{N}$, then l_k is a fixed point of $\Bbbk$). Again, from condition (i), we have

$$\Psi^*(l_0,l_1) = \Psi^*(l_0,\Bbbk(l_0)) \geq \frac{1}{s^2}.$$

Also,

$$\Psi^*(l_1, l_2) = \Psi^*(\Bbbk(l_0), \Bbbk(l_1)) \geq \frac{1}{s^2}.$$

By induction, we get $\Psi^*(l_{n-1}, l_n) \geq \frac{1}{s^2}$ and $\Psi^*(l_n, l_{n-1}) \geq \frac{1}{s^2}$ for all $n \in \mathbb{N}$. We have

$$\begin{aligned} \rho^{\hbar(l_n, l_{n+1})} &= \rho^{\hbar(\Bbbk(l_{n-1}), \Bbbk(l_n))} \\ &\leq \rho_*^{\hbar(\Bbbk(l_{n-1}), \Bbbk(l_n))} \\ &\leq [\Psi^*(l_{n-1}, l_n) - \frac{1}{s^2} + \rho_*]^{\hbar(\Bbbk(l_{n-1}), \Bbbk(l_n))}. \end{aligned}$$

Since $\Bbbk$ is an $\Psi^* - \beta$-contraction, we have

$$\rho^{\hbar(l_n, l_{n+1})} \leq \rho^{\beta(\hbar(l_{n-1}, l_n))(\hbar(l_{n-1}, l_n)}.$$

This means that for each $n \in \mathbb{N}$,

$$\hbar(l_n, l_{n+1}) \leq \beta((\hbar(l_{n-1}, l_n))(\hbar(l_{n-1}, l_n) < \frac{1}{s^2}(\hbar(l_{n-1}, l_n). \tag{2.3}$$

We conclude that the real sequence $\{(\hbar(l_{n-1}, l_n)\}$ is strictly decreasing, and so there is $\hbar \geq 0$ such that $(\hbar(l_{n-1}, l_n) \longrightarrow q$ as $n \longrightarrow \infty$. Assume that $\hbar > 0$. Taking limit as $n \longrightarrow \infty$ in (2.3), we obtain that $1 \leq \lim_{n \longrightarrow \infty} \beta(\hbar(l_{n-1}, l_n)) < \frac{1}{s^2}$. It is a contradiction, then $\hbar = 0$, that is, $\lim_{n \longrightarrow \infty} \hbar(l_{n-1}, l_n) = 0$. The same procedure allows us to conclude $\lim_{n \longrightarrow \infty} \hbar(l_n, l_{n-1}) = 0$.

Now, we will prove that $\{l_n\}$ is a Cauchy sequence in $(\aleph, \hbar)$. First, we show that $\{l_n\}$ is a right-Cauchy sequence. We argue by contradiction. Then there exist $\epsilon > 0$ and a subsequence of integers m_j and smallest n_j with $n_j > m_j \geq j$ such that

$$\hbar(l_{m_j}, l_{n_j}) \geq \epsilon \tag{2.4}$$

for all $j \in \mathbb{N}$. Then we get

$$\hbar(l_{m_j}, l_{n_j}) \geq \epsilon, \hbar(l_{m_j}, l_{n_{j-1}}) < \epsilon. \tag{2.5}$$

Thus, we get from triangle inequality,

$$\begin{aligned} \epsilon \leq \hbar(l_{m_j}, l_{n_j}) &\leq s[\hbar(l_{m_j}, l_{n_{j-1}}) + \hbar(l_{n_{j-1}}, l_{n_j})] \\ &\leq s\epsilon + s\hbar(l_{n_{j-1}}, l_{n_j}). \end{aligned}$$

On taking the limit as $j \longrightarrow \infty$, we have

$$\epsilon \leq \lim_{j \longrightarrow \infty} \hbar(l_{m_j}, l_{n_j}) \leq s\epsilon < \infty.$$

Since $n_j > m_j \geq j$ and Ψ^* is generalized transitive, we get $\Psi^*(l_{m_j}, l_{n_j}) \geq \frac{1}{s^2}$. Consider,

$$\begin{aligned} \rho^{\hbar(l_{m_j}, l_{n_j})} &\leq \rho^{s\hbar(l_{m_j}, l_{m_{j+1}}) + s^2\hbar(l_{m_{j+1}}, l_{n_{j+1}}) + s^2\hbar(l_{n_{j+1}}, l_{n_j})} \\ &\leq \rho^{s\hbar(l_{m_j}, l_{m_{j+1}}) + s^2\hbar(\Bbbk(l_{m_j}), \Bbbk(l_{n_j})) + s^2\hbar(l_{n_{j+1}}, l_{n_j})} \\ &\leq \rho^{s\hbar(l_{m_j}, l_{m_{j+1}}) + s^2\hbar(l_{n_{j+1}}, l_{n_j})} \rho_*^{s^2\hbar(\Bbbk(l_{m_j}), \Bbbk(l_{n_j}))} \\ &\leq \rho^{s\hbar(l_{m_j}, l_{m_{j+1}}) + s^2\hbar(l_{n_{j+1}}, l_{n_j})} \rho^{\beta(\hbar(l_{m_j}, l_{n_j}))s^2\hbar(l_{m_j}, l_{n_j})}. \end{aligned}$$

Hence,

$$\hbar(l_{m_j}, l_{n_j}) \leq s\hbar(l_{m_j}, l_{m_{j+1}}) + s^2\hbar(l_{n_{j+1}}, l_{n_j}) + \beta(\hbar(l_{m_j}, l_{n_j}))s^2\hbar(l_{m_j}, l_{n_j}).$$

That is,

$$\frac{\hbar(l_{m_j}, l_{n_j}) - s\hbar(l_{m_j}, l_{m_{j+1}}) - s^2\hbar(l_{n_{j+1}}, l_{n_j})}{s^2\hbar(l_{m_j}, l_{n_j})} \leq \beta(\hbar(l_{m_j}, l_{n_j})) < \frac{1}{s^2}.$$

By taking the limit as $j \longrightarrow \infty$, we get

$$\lim_{j \longrightarrow \infty} \beta(\hbar(l_{m_j}, l_{n_j})) = \frac{1}{s^2}.$$

Since $\beta \in \Omega$, we have $\lim_{j \longrightarrow \infty} \hbar(l_{m_j}, l_{n_j}) = 0$, which is a contradiction. Thus, $\{l_n\}$ is a right-Cauchy sequence in the quasi b-metric space $(\aleph, \hbar)$. Similarly, it is a left-Cauchy sequence in the quasi b-metric space $(\aleph, \hbar)$. That is, $\{l_n\}$ is a Cauchy sequence in the quasi b-metric space $(\aleph, \hbar)$. Since $(\aleph, \hbar)$ is complete, there exists x^* such that $x^* = \lim_{n \longrightarrow \infty} l_n$ and since $\Bbbk$ is continuous,

$$x^* = \lim_{n \longrightarrow \infty} l_n = \lim_{n \longrightarrow \infty} \Bbbk(l_{n+1}) = \Bbbk(\lim_{n \longrightarrow \infty} l_{n+1}) = \Bbbk(x^*).$$

Hence, x^* is a fixed point of $\Bbbk$. □

Theorem 2.2. Let $(\aleph, \hbar)$ be a complete quasi b-metric space and $\Bbbk : \aleph \longrightarrow \aleph$ be an $\Psi^* - \beta$-contraction mapping such that

(i) $\Bbbk$ is Ψ^*-admissible;

(ii) Ψ^* is generalized transitive;

(iii) there exists $l_0 \in \aleph$ such that $\Psi^*(l_0, \Bbbk(l_0)) \geq \frac{1}{s^2}$ and $\Psi^*(\Bbbk(l_0), l_0) \geq \frac{1}{s^2}$;

(iv) if $\{l_n\}$ is a sequence in $\aleph$ such that $\Psi^*(x^*, l_n) \geq \frac{1}{s^2}$ and $\Psi^*(l_n, x^*) \geq \frac{1}{s^2}$ for all $n \in \mathbb{N}$ and $l_n \longrightarrow x \in \aleph$ as $n \longrightarrow \infty$.

Then, there exists a unique fixed point $x^* \in \aleph$ of $\Bbbk$.

Proof . From the proof of Theorem 2.1, the sequence $\{l_n\}$ is Cauchy and converges to some x^* in $(\aleph, \hbar)$. We have $\Psi^*(l_n, x^*) \geq \frac{1}{s^2}$ and $\Psi^*(x^*, l_n) \geq \frac{1}{s^2}$, $\forall n \in \mathbb{N}$. Next,

$$\begin{aligned}
\rho^{\hbar(x^*, \Bbbk(x^*))} &\leq \rho^{s\hbar(x^*, l_{n+1}) + s\hbar(l_{n+1}, \Bbbk(x^*))} \\
&= \rho^{s\hbar(x^*, l_{n+1}) + s\hbar(\Bbbk(l_n), \Bbbk(x^*))} = \rho^{s\hbar(x^*, l_{n+1})} \rho^{s\hbar(\Bbbk(l_n), \Bbbk(x^*))} \\
&\leq \rho^{s\hbar(x^*, l_{n+1})} \rho_*^{s\hbar(\Bbbk(l_n), \Bbbk(x^*))} \\
&\leq \rho^{s\hbar(x^*, l_{n+1})} [\Psi^*(l_n, x^*) - \frac{1}{s^2} + \rho_*]^{s\hbar(\Bbbk(l_n), \Bbbk(x^*))} \\
&\leq \rho^{s\hbar(x^*, l_{n+1})} \rho^{\beta(\hbar(l_n, x^*)) s\hbar(l_n, x^*)} \\
&\leq \rho^{s\hbar(x^*, l_{n+1}) + \beta(\hbar(l_n, x^*)) s\hbar(l_n, x^*)}
\end{aligned}$$

for all $n \in \mathbb{N}$. Then we get

$$\hbar(x^*, \Bbbk(x^*)) \leq s\hbar(x^*, l_{n+1}) + \beta(\hbar(l_n, x^*)) s\hbar(l_n, x^*)$$

for all $n \in \mathbb{N}$. Letting $n \longrightarrow \infty$, we obtain that $\hbar(x^*, \Bbbk(x^*)) = 0$, and so $x^* = \Bbbk(x^*)$. To prove the uniqueness of the fixed point of $\Bbbk$, assume that $y^* \in \aleph$ is another fixed point of $\Bbbk$. We have

$$\begin{aligned}
\rho^{\hbar(x^*, y^*)} &\leq \rho_*^{\hbar(x^*, y^*)} \leq \rho_*^{s\hbar(x^*, l_{n+1}) + s\hbar(l_{n+1}, y^*)} \\
&\leq \rho_*^{s\hbar(\Bbbk(x^*), \Bbbk(l_n))} * \rho_*^{s\hbar(\Bbbk(l_n), \Bbbk(y^*))} \\
&\leq (\Psi^*(x^*, l_n) - \frac{1}{s^2} + \rho_*)^{s\hbar(\Bbbk(x^*), \Bbbk(l_n))} * (\Psi^*(l_n, y^*) - \frac{1}{s^2} + \rho_*)^{s\hbar(\Bbbk(l_n), \Bbbk(y^*))} \\
&\leq \rho^{s\beta(\hbar(x^*, l_n))\hbar(x^*, l_n)} * \rho^{s\beta(\hbar(l_n, y^*))\hbar(l_n, y^*)}.
\end{aligned}$$

Thus,

$$\hbar(x^*, y^*) \leq s\beta(\hbar(x^*, l_n))\hbar(x^*, l_n) + s\beta(\hbar(l_n, y^*))\hbar(l_n, y^*) \leq \frac{1}{s}\hbar(x^*, l_n) + \frac{1}{s}\hbar(l_n, y^*).$$

If we repeat this argument n-times on both $\hbar(x^*, l_n)$ and $\hbar(l_n, y^*)$, we get

$$\hbar(x^*, y^*) \leq (\frac{1}{s})^n \hbar(x^*, l_0) + (\frac{1}{s})^n \hbar(l_0, y^*).$$

By taking limit as $n \longrightarrow \infty$, we get $\hbar(x^*, y^*) \leq 0$. Hence, $\hbar(x^*, y^*) = 0$, then $x^* = y^*$. □

To prove uniqueness of the fixed point that given in Theorem 2.1, we need to add the next hypothesis:

$(C1)$ $\Psi^*(l, k) \geq \frac{1}{s^2}$ or $\Psi^*(k, l) \geq \frac{1}{s^2}$, for all fixed points $l, k \in \aleph$ of $\Bbbk$.

Theorem 2.3. Let $(\aleph, \hbar)$ be a complete quasi b-metric space and $\Bbbk : \aleph \longrightarrow \aleph$ be an $\Psi^* - \beta$- contraction mapping such that

(i) $\Bbbk$ is Ψ^*-admissible;

(ii) Ψ^* is generalized transitive;

(iii) there exists $l_0 \in \aleph$ such that $\Psi^*(l_0, \Bbbk(l_0)) \geq \frac{1}{s^2}$ and $\Psi^*(\Bbbk(l_0), l_0) \geq \frac{1}{s^2}$;

(iv) $(C1)$ holds.

Then, there exists a unique fixed point $x^* \in \aleph$ of $\Bbbk$.

Proof . Following the proof of Theorem 2.1, there exists a fixed point of $\Bbbk$. We claim that the fixed point is unique. Without lose of generality, let x^*, y^* be fixed points of $\Bbbk$ so that $\Psi^*(y^*, x^*) \geq \frac{1}{s^2}$. We have

$$\rho^{\hbar(x^*,y^*)} \leq \rho_*^{\hbar(x^*,y^*)} \leq [\Psi^*(x^*, y^*) - \frac{1}{s^2} + \rho_*]^{\hbar(x^*,y^*)} \leq [\Psi^*(x^*, y^*) - \frac{1}{s^2} + \rho_*]^{\hbar(\Bbbk(x^*),\Bbbk(y^*))}$$

$$\leq [\Psi^*(x^*, y^*) - \frac{1}{s^2} + \rho_*]^{\beta(\hbar((x^*,y^*))\hbar(x^*,y^*)}.$$

It follows that

$$\hbar(x^*, y^*) \leq \beta(\hbar((x^*, y^*))\hbar(x^*, y^*).$$

On contrary, assume that $\neq 0$, then we have

$$1 \leq \beta(\hbar((x^*, y^*)),$$

which is a contradiction. □

3 Application: Ulam-Hyers Stability

Definition 3.1. Let $(\aleph, \hbar)$ be a complete quasi $b$metric space and $\Bbbk : \aleph \longrightarrow \aleph$ be a mapping. The fixed point problem

$$l = \Bbbk(l) \tag{3.1}$$

is called Ulam-Hyers stable if and only if for each $k \in \aleph$ satisfying the inequality

$$\hbar(k, \Bbbk(k)) \leq \epsilon \tag{3.2}$$

and inequality

$$\hbar(\Bbbk(k), k) \leq \epsilon, \tag{3.3}$$

where $\epsilon > 0$, there are a solution $x^* \in \aleph$ of equation (3.1) and a constant $K > 0$ independent of k and x^* such that

$$\hbar(k, x^*) \leq K\epsilon, \tag{3.4}$$

and

$$\hbar(x^*, k) \leq K\epsilon. \tag{3.5}$$

Definition 3.2. Let $(\aleph, \hbar)$ be a complete quasi b-metric space and $\Bbbk : \aleph \longrightarrow \aleph$ be a mapping. The fixed point problem 3.1 is called generalized Ulam-Hyers stable if and only if there exists an increasing function $\Xi : [0, \infty) \longrightarrow [0, \infty)$ continuous at 0 with $\Xi(0) = 0$ such that for all $\epsilon > 0$ and $k \in \aleph$, the inequalities (3.2) and (3.3) hold, there exists a solution $x^* \in \aleph$ of the equation (3.1) such that

$$\hbar(k, x^*) \leq \Xi(\epsilon). \tag{3.6}$$

and

$$\hbar(x^*, k) \leq \Xi(\epsilon). \tag{3.7}$$

Theorem 3.1. Let $(\aleph, \hbar)$ be a complete quasi b-metric space with $s > 1$. Suppose that all the hypotheses of Theorem 2.2 (Theorem 2.3) hold. If $\Psi^*(l, k) \geq \frac{1}{s^2}$ and $\Psi^*(k, l) \geq \frac{1}{s^2}$ for all $l, k \in \aleph$ which are satisfying the inequalities (3.2) and (3.3), then the fixed point of $\Bbbk$ is Ulam-Hyers stable.

Proof . From the proof of Theorem 2.2 (Theorem 2.3), we obtain that $\Bbbk$ has a unique fixed point (say x^*). Let $\epsilon > 0$ and $k \in \aleph$ such that the inequalities (3.2) and (3.3) hold, that is,

$$\hbar(k, \Bbbk(k) \leq \epsilon$$

and

$$\hbar(\Bbbk(k), k) \leq \epsilon.$$

In fact, the fixed point x^* satisfies the inequality (3.2) and the inequality (3.3). From hypotheses, we have $\Psi^*(x^*, k) \geq \frac{1}{s^2}$ and $\Psi^*(k, x^*) \geq \frac{1}{s^2}$. Now, we have

$$\begin{aligned}
\rho^{\hbar(x^*,k)} &= \rho^{\hbar(\Bbbk(x^*),k)} \\
&\leq \rho^{s\hbar(\Bbbk(x^*),\Bbbk(k))+s\hbar(\Bbbk(k),k)} \\
&\leq \rho_*^{s\hbar(\Bbbk(x^*),\Bbbk(k))} * \rho^{s\hbar(\Bbbk(k),k)} \\
&\leq [\Psi^*(x^*, k) - \frac{1}{s^2} + \rho_*]^{s\hbar(\Bbbk(x^*),\Bbbk(k))} * \rho^{s\epsilon} \\
&\leq \rho^{s\beta(\hbar(x^*,k))\hbar(x^*,k)+s\epsilon}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\hbar(x^*, k) &\leq s\beta(\hbar(x^*, k))\hbar(x^*, k) + s\epsilon \\
&\leq \frac{1}{s}\hbar(x^*, k) + s\epsilon.
\end{aligned}$$

This implies that

$$\hbar(x^*, k) \leq \frac{s^2\epsilon}{s-1},$$

where $s > 1$. Consequently, the fixed point problem $\Bbbk$ is Ulam-Hyers stable. □

Theorem 3.2. Let $(\aleph, \hbar)$ be a complete quasi b-metric space. Suppose that all the hypotheses of Theorem 2.2 (Theorem 2.3) hold. Assume that $\beta(0) = 0$ and there is a strictly increasing function $\Psi : [0, \infty) \longrightarrow [0, \infty)$ which is defined by $\Psi(t) = \frac{t - st\beta(t)}{s}$ and onto. If $\Psi^*(l, k) \geq \frac{1}{s^2}$ and $\Psi^*(k, l) \geq \frac{1}{s^2}$ for all $l, k \in \aleph$, satisfying the inequalities (3.2) and (3.3), then the fixed point of $\Bbbk$ is generalized Ulam-Hyers stable.

Proof . From the same process as in the proof of Theorem 3.1 with $s \geq 1$, we obtain that

$$\hbar(x^*, k) \leq s\beta(\hbar(x^*, k))\hbar(x^*, k) + s\epsilon$$

and then

$$\frac{\hbar(x^*, k) - s\beta(\hbar(x^*, k))\hbar(x^*, k)}{s} \leq \epsilon.$$

That is, $\Psi\hbar(x^*, k) \leq \epsilon$. Thus,

$$\hbar(x^*, k) \leq \Psi^{-1}(\epsilon).$$

We can conclude that Ψ^{-1} is increasing, continuous at 0 and $\Psi^{-1}(\{0\}) = 0$. Consequently, the fixed point problem of $\Bbbk$ is generalized Ulam-Hyers stable. □

4 Well-posedness

The concept of well-posedness of a fixed point problem has a great interest for many mathematicians, see [15, 19, 23]. We begin by defining the concept of well-posedness in the context of quasi b-metric spaces as follows:

Definition 4.1. [2] Let $(\aleph, \hbar)$ be a quasi b-metric space and $\Bbbk : \aleph \longrightarrow \aleph$ be a given mapping. Then, the fixed point problem (3.1) is said to be well-posed if:

(1) $\Bbbk$ has a unique fixed point $u \in \aleph$;

(2) for any sequence $\{l_n\} \subseteq X$, if $\lim_{n\longrightarrow\infty} \hbar(\Bbbk l_n, l_n) = \lim_{n\longrightarrow\infty} \hbar(l_n, \Bbbk l_n) = 0$

then, we have $\lim_{n\longrightarrow\infty} \hbar(\Bbbk l_n, u) = \lim_{n\longrightarrow\infty} \hbar(u, \Bbbk l_n) = 0$.

Theorem 4.1. Let $(\aleph, \hbar)$ be a complete quasi b-metric space. Suppose that all the hypotheses of Theorem 2.2 (Theorem 2.3) hold with the next supposition:

- If $\{l_n\} \subseteq X$ is a sequence with $\lim_{n\longrightarrow\infty} \hbar(\Bbbk l_n, l_n) = \lim_{n\longrightarrow\infty} \hbar(l_n, \Bbbk l_n) = 0$, then $\Psi^*(l_n, u) \geq \frac{1}{s^2}$ and $\Psi^*(u, l_n) \geq \frac{1}{s^2}$ for all n, where u is a fixed point of $\Bbbk$.

Then the fixed point equation (3.1) is well-posed.

Proof . By Theorem 2.2 (Theorem 2.3), we have a unique $u \in \aleph$ such that $u = \Bbbk u$. Let $\{l_n\} \subseteq X$ be a sequence with $\lim_{n\longrightarrow\infty} \hbar(\Bbbk l_n, l_n) = \lim_{n\longrightarrow\infty} \hbar(l_n, \Bbbk l_n) = 0$, then we have $\Psi^*(l_n, u) \geq \frac{1}{s^2}$ and $\Psi^*(u, l_n) \geq \frac{1}{s^2}$ for all n. Now, by using the fact that $\Psi^*(l_n, u) \geq \frac{1}{s^2}$, we can write

$$\begin{aligned} \rho^{\hbar(l_n,u)} &\leq \rho^{s\hbar(l_n,\Bbbk(l_n))+s\hbar(\Bbbk(l_n),u)} \\ &\leq \rho^{s\hbar(l_n,\Bbbk(l_n))+s\hbar(\Bbbk(l_n),\Bbbk u)} \\ &\leq \rho^{s\hbar(l_n,\Bbbk(l_n))} * \rho_*^{s\hbar(\Bbbk(l_n),\Bbbk u)} \\ &\leq \rho^{s\hbar(l_n,\Bbbk(l_n))} * [\Psi^*(l_n,u) - \frac{1}{s^2} + \rho_*]^{s\hbar(\Bbbk(l_n),\Bbbk(u))} \\ &\leq \rho^{s\hbar(l_n,\Bbbk(l_n))} * \rho^{s\beta(\hbar(l_n,u))\hbar(l_n,u)}. \end{aligned}$$

That is,

$$\begin{aligned} \hbar(l_n,u) &\leq s\hbar(l_n,\Bbbk(l_n)) + s\beta(\hbar(l_n,u))\hbar(l_n,u) \\ &\leq s\hbar(l_n,\Bbbk(l_n)) + \frac{1}{s}\hbar(l_n,u). \end{aligned}$$

Consequently,

$$\hbar(l_n,u) \leq \frac{s^2}{s-1}\hbar(l_n,\Bbbk(l_n)),$$

for each integer n. Letting $n \longrightarrow \infty$, we get

$$\lim_{n\longrightarrow\infty} \hbar(l_n,u) = 0. \tag{4.1}$$

Again, by the same procedure and using the fact that $\Psi^*(u, l_n) \geq \frac{1}{s^2}$, we can obtain

$$\lim_{n\longrightarrow\infty} \hbar(u,l_n) = 0. \tag{4.2}$$

By (4.1) and (4.2), the fixed point problem (3.1) is well-posed. □

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