Int. J. Nonlinear Anal. Appl. 14 (2023) 7, 321–326
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.27406.3589

Characterization of Bloch type spaces and symmetric lifting operator

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(Communicated by Ali Jabbari)

Abstract

In this paper we first investigate the characterization of Bloch space in terms of pseudo-hyperbolic metric. Then the action of symmetric lifting operator on weighted Bloch spaces will be studied.

Keywords: Symmetric lifting operator, Bloch spaces, Pseudo-hyperbolic metric, hyperbolic metric
2020 MSC: 32A18

1 Introduction and Preliminaries

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of holomorphic functions on $\mathbb{D}$. For $\alpha > 0$, the weighted Banach space of analytic functions H_α^∞ consists of all $f \in H(\mathbb{D})$ for which

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f(z)| < \infty.$$

The Bloch space $\mathcal{B}^\alpha$ is the set of all holomorphic functions f on $\mathbb{D}$ such that

$$b(f) = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f'(z)| < \infty.$$

The above statement is just a semi norm on $\mathcal{B}^\alpha$ and it can be changed to norm with

$$||f||_{\mathcal{B}^\alpha} = |f(0)| + b(f).$$

If $\alpha > 1$, then $\mathcal{B}^\alpha = H_{\alpha-1}^\infty$. For $z, w \in \mathbb{D}$ write

$$\rho(z,w) = \left|\frac{z-w}{1-\bar{z}w}\right|.$$

This is called the pseudo-hyperbolic metric on $\mathbb{D}$. The pseudo-hyperbolic disk centered at z with radius r is denoted by

$$D(z,r) = \{w \in \mathbb{D} : \rho(z,w) < r\}.$$

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Received: June 2022 *Accepted:* August 2022

Also the Bergman metric (hyperbolic metric) defined by

$$\beta(z,w) = \frac{1}{2}\log\frac{1+\rho(z,w)}{1-\rho(z,w)}.$$

It is easy to see that $\rho(z,w) \le \beta(z,w)$. The classical Bloch space $\mathcal{B} = \mathcal{B}^1$ has been characterized by some methods. For example Zhu proved that $f \in \mathcal{B}$ if and only if there exists a positive constant C such that

$$|f(z) - f(w)| \le C\beta(z,w), \tag{1.1}$$

see [16]. Or some papers is devoted to characterize $\mathcal{B}$ in terms of $\frac{f(z)-f(w)}{z-w}$ and double integral of $\frac{f(z)-f(w)}{z-w}$, see for example [4, 5, 14].

Let $\mathbb{D}^2 = \mathbb{D}\times\mathbb{D}$ denote the bidisk in $\mathbb{C}^2$ and $H(\mathbb{D}^2)$ is the space of holomorphic functions on $\mathbb{D}^2$. The Bloch space $\mathcal{B}(\mathbb{D}^2)$ is the space of all $f \in H(\mathbb{D}^2)$ satisfying

$$||f||_{\mathcal{B}(\mathbb{D}^2)} = |f(0)| + \sup_{(z,w)\in\mathbb{D}^2}\left((1-|z|^2)\left|\frac{\partial f}{\partial z}\right| + (1-|w|^2)\left|\frac{\partial f}{\partial w}\right|\right) < \infty,$$

where the derivatives are computed in (z,w). The norm in the Bloch space can be written in the other ways even in the general case $\mathbb{D}^n$. But it was proved in [11] and [12] that all the other norms is equivalent with $||.||_{\mathcal{B}(\mathbb{D}^2)}$. For some characterization of the norm in Bloch space $\mathcal{B}(\mathbb{D}^2)$ one can refer to [9] and [15].

Similar to $\mathcal{B}(\mathbb{D}^2)$ we can define Bloch type space $\mathcal{B}^\alpha(\mathbb{D}^2)$, $\alpha > 0$, which is the space of $f \in H(\mathbb{D}^2)$ satisfying

$$||f||_{\mathcal{B}^\alpha(\mathbb{D}^2)} = |f(0)| + \sup_{(z,w)\in\mathbb{D}^2}\left((1-|z|^2)^\alpha\left|\frac{\partial f}{\partial z}\right| + (1-|w|^2)^\alpha\left|\frac{\partial f}{\partial w}\right|\right) < \infty,$$

For $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space $A^p_\alpha(\mathbb{D})$ consists of all holomorphic functions in $L^p(\mathbb{D}, dA_\alpha)$, that is

$$A^p_\alpha(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : ||f||^p_{\alpha,p} = \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) < \infty \right\},$$

where

$$dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$$

and dA is the normalized are measure on $\mathbb{D}$. Furthermore, we define $A^p_\alpha(\mathbb{D}^2)$ as the space of all $f \in H(\mathbb{D}^2)$ for which

$$\int_{\mathbb{D}}\int_{\mathbb{D}} |f(z,w)|^p dA_\alpha(z) dA_\alpha(w) < \infty.$$

The operator we use here is the well-known symmetric lifting operator which lifts analytic functions from $\mathbb{D}$ to $\mathbb{D}^2$ and defined by

$$L : H(\mathbb{D}) \to H(\mathbb{D}^2)$$

$$L(f)(z,w) = \frac{f(z) - f(w)}{z - w}.$$

The has been an interest for characterizing the function spaces, means that find necessary and sufficient conditions for a function to belong to a function space. In [13], Wulan and Zhu characterized weighted Bergman spaces in terms of Euclidean, pseudo-hyperbolic and Bergman metric using Lipschitz type condition and then studied symmetric lifting operator on them. Double integral characterization for Bergman spaces in the unit ball were give in papers like [2, 6, 7].

In this paper we give some characterization as Lipschitz condition for Bloch type spaces and use them to investigate the symmetric lifting operator.

Here is some lemmas we need for proving the main results.

Lemma 1.1. ([3] or [16]) For any fixed $r \in (0,1)$ there exists a positive constant C such that

$$C^{-1} \le \frac{1-|z|^2}{|1-\bar{z}w|} \le C$$

whenever $\rho(z,w) \le r$. Consequently, there exists a positive constant C such that

$$C^{-1} \le \frac{1-|z|^2}{1-|w|^2} \le C$$

whenever $\rho(z,w) \le r$.

The following lemma is well known in the literature.

Lemma 1.2. [8] If $f \in \mathcal{B}^\alpha$, then

$$|f(z)| \le C||f||_{\mathcal{B}^\alpha} \begin{cases} 1, & 0<\alpha<1; \\ \log 2(1-|z|^2)^{-1}, & \alpha = 1; \\ (1-|z|^2)^{1-\alpha}, & \alpha > 1 \end{cases} \tag{1.2}$$

where C is a positive constant.

Lemma 1.3. For any fixed $z \in \mathbb{D}$ we have

$$\lim_{w\to z} \frac{\beta(z,w)}{|z-w|} = \lim_{w\to z} \frac{\rho(z,w)}{|z-w|} = \frac{1}{1-|z|^2}.$$

2 Characterization for Bloch space $\mathcal{B}^\alpha$

In this section some characterizations for $\mathcal{B}$ in terms of pseudo-hyperbolic metric will be proved.

Theorem 2.1. Let $0<\alpha<1$. If $f \in \mathcal{B}^\alpha$, then there exists a positive constant C such that

$$|f(z) - f(w)| \le C\rho(z,w),$$

for all z and w in $\mathbb{D}$.

Proof . Assume $f \in \mathcal{B}^\alpha$. We fix a radius $r \in (0,1)$ and two points z and w in $\mathbb{D}$ with $\rho(z,w) < r$. It is easy to see from the fundamental theorem of calculus that

$$f(z) - f(w) = (z-w)\int_0^1 f'(tz + (1-t)w)dt.$$

Since $D(z,r)$ is a convex set, for all $t \in [0,1]$ we have

$$tz + (1-t)w \in D(z,r).$$

Hence

$$|f(z) - f(w)| \le |z-w| \sup\{|f'(u)| : \ u \in D(z,r)\}.$$

From Lemma 1.1, there exists a positive constant C such that

$$\begin{aligned} |f(z) - f(w)| \le & C\rho(z,w) \sup\{(1-|u|^2)|f'(u)| : \ u \in D(z,r)\} \\ \le & C\rho(z,w) \sup\{(1-|u|^2)|f'(u)| : \ u \in \mathbb{D}\} \\ \le & C\rho(z,w) \sup\{(1-|u|^2)^\alpha|f'(u)| : \ u \in \mathbb{D}\} \\ \le & C\rho(z,w)||f||_{\mathcal{B}^\alpha}, \end{aligned}$$

for z and w in $\mathbb{D}$ with $\rho(z,w) < r$. If $\rho(z,w) \ge r$ then

$$\begin{aligned} |f(z) - f(w)| \le & |f(z)| + |f(w)| \\ \le & \frac{\rho(z,w)}{r}(|f(z)| + |f(w)|) \\ \le & 2C\rho(z,w)||f||_{\mathcal{B}^\alpha}. \end{aligned}$$

The last line of the above comes from Lemma 1.2. □

In Theorem 2.1, we just prove a necessary condition and the converse can not be exists because in that case we have a similar result for $\mathcal{B}$. But, there may be other if and only if characterization using pseudo-hyperbolic metric. Since $\rho(z,w) \le \beta(z,w)$, then we can replace the Bergman metric $\beta(z,w)$ with $\rho(z,w)$. In general, we have the following theorem.

Theorem 2.2. Suppose that $\alpha > 0$. Let $f \in \mathcal{B}^\alpha$. Then there exists a continuous function g with $\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|g(z)| < \infty$ such that

$$|f(z) - f(w)| \leq \rho(z,w)(g(z) + g(w)),$$

for all z and w in $\mathbb{D}$.

Proof . **Case** $0 < \alpha < 1$: The result can be obtained from Theorem 2.1.

Case $\alpha = 1$: Suppose that $f \in \mathcal{B}^\alpha$. We fix a radius $r \in (0,1)$ and two points z and w in $\mathbb{D}$ with $\rho(z,w) < r$. Similar to the proof of Theorem 2.1, we have

$$|f(z) - f(w)| \leq C\rho(z,w)\sup\{(1-|u|^2)|f'(u)| : \ u \in D(z,r)\}.$$

Write

$$h(z) = C\sup\{(1-|u|^2)|f'(u)| : \ u \in D(z,r)\}.$$

Then

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|h(z)| < \infty$$

and

$$|f(z) - f(w)| \leq \rho(z,w)(h(z) + h(w)).$$

If $\rho(z,w) \geq r$. Then

$$\begin{aligned}|f(z) - f(w)| \leq& |f(z)| + |f(w)| \\ \leq& \frac{\rho(z,w)}{r}(|f(z)| + |f(w)|)\end{aligned}$$

Set $g(z) = \frac{|f(z)|}{r}$. We get

$$\begin{aligned}\sup_{z\in\mathbb{D}}(1-|z|^2)|g(z)| \leq& \frac{1}{r}\sup_{z\in\mathbb{D}}(1-|z|^2)|f(z)| \\ \leq& \frac{C}{r}\sup_{z\in\mathbb{D}}(1-|z|^2)\log 2(1-|z|^2)^{-1}||f||_{\mathcal{B}} \\ \leq& \frac{C}{r},\end{aligned}$$

where the last line is due to the boundedness of the function $(1-|z|^2)\log(1-|z|^2)^{-1}$.

Case $\alpha > 1$: Suppose that $f \in \mathcal{B}^\alpha$. We fix a radius $r \in (0,1)$ and two points z and w in $\mathbb{D}$ with $\rho(z,w) < r$. Similar to the proof of Theorem 2.1, we have

$$|f(z) - f(w)| \leq C\rho(z,w)\sup\{(1-|u|^2)|f'(u)| : \ u \in D(z,r)\}.$$

Write

$$h(z) = C\sup\{(1-|u|^2)|f'(u)| : \ u \in D(z,r)\}.$$

Then

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|h(z)| \leq \sup_{z\in\mathbb{D}} C(1-|z|^2)|h_1(z)| < \infty,$$

where

$$h_1(z) = C\sup\{(1-|u|^2)^\alpha|f'(u)| : \ u \in D(z,r)\}.$$

If $\rho(z,w) \geq r$. Then

$$\begin{aligned}|f(z) - f(w)| \leq& |f(z)| + |f(w)| \\ \leq& \frac{\rho(z,w)}{r}(|f(z)| + |f(w)|)\end{aligned}$$

Set $g(z) = \frac{|f(z)|}{r}$. Recall that for $\alpha > 1$, $\mathcal{B}^\alpha = H^\infty_{\alpha-1}$. So

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|g(z)| \leq \frac{1}{r}\sup_{z\in\mathbb{D}}(1-|z|^2)(1-|z|^2)^{\alpha-1}|f(z)| < \infty.$$

□

3 Symmetric lifting operator on $\mathcal{B}^\alpha$

In this section we show an application of the results of the previous section to the problem of lifting analytic functions from the unit disk to the bidisk.

Theorem 3.1. Suppose $0 < \alpha \leq 1$ and $\gamma \geq 2$. Then the symmetric lifting operator L maps $\mathcal{B}^\alpha$ boundedly into $\mathcal{B}^\gamma(\mathbb{D}^2)$.

Proof . Suppose that $f \in \mathcal{B}^\alpha$. We need to prove that

$$\sup_{(z,w)\in\mathbb{D}^2}\left((1-|z|^2)^\gamma\left|\frac{\partial Lf}{\partial z}\right| + (1-|w|^2)^\gamma\left|\frac{\partial Lf}{\partial w}\right|\right) < \infty.$$

Note that

$$\frac{\partial Lf}{\partial z}(z,w) = \frac{f'(z)(z-w)-(f(z)-f(w))}{(z-w)^2}.$$

Due to the similarity, we just prove that

$$\sup_{(z,w)\in\mathbb{D}^2}(1-|z|^2)^\gamma\left|\frac{f'(z)(z-w)-(f(z)-f(w))}{(z-w)^2}\right| < \infty. \tag{3.1}$$

We divide the statement into two part. We get

$$\begin{aligned}\sup_{(z,w)\in\mathbb{D}^2}(1-|z|^2)^\gamma\left|\frac{f'(z)(z-w)}{(z-w)^2}\right| &\leq \sup_{(z,w)\in\mathbb{D}^2}(1-|z|^2)^\gamma\frac{|f'(z)|}{|z-w|}\\ &\leq \sup_{(z,w)\in\mathbb{D}^2}(1-|z|^2)^{\gamma-1-\alpha}||f||_{\mathcal{B}^\alpha}\\ &<\infty.\end{aligned}$$

On the other hand, for $0 < \alpha < 1$, Theorem 2.1 implies that for some positive constant C

$$\begin{aligned}\sup_{(z,w)\in\mathbb{D}^2}(1-|z|^2)^\gamma\left|\frac{f(z)-f(w)}{(z-w)^2}\right| &\leq C\sup_{(z,w)\in\mathbb{D}^2}(1-|z|^2)^\gamma\frac{\rho(z,w)}{|z-w|^2}\\ &\leq C\sup_{(z,w)\in\mathbb{D}^2}(1-|z|^2)^{\gamma-2}\\ &<\infty.\end{aligned}$$

For $\alpha = 1$ we use the Lipschitz characterization for $\mathcal{B}$ which mentioned in the introduction. So

$$\begin{aligned}\sup_{(z,w)\in\mathbb{D}^2}(1-|z|^2)^\gamma\left|\frac{f(z)-f(w)}{(z-w)^2}\right| &\leq C\sup_{(z,w)\in\mathbb{D}^2}(1-|z|^2)^\gamma\frac{\beta(z,w)}{|z-w|^2}\\ &\leq C\sup_{(z,w)\in\mathbb{D}^2}(1-|z|^2)^{\gamma-2}\\ &<\infty.\end{aligned}$$

Now, we have the equation (3.1). So the symmetric lifting operator maps $\mathcal{B}^\alpha$ into $\mathcal{B}^\gamma(\mathbb{D}^2)$. The boundedness comes from an application of closed graph theorem. □

We can consider the operator $L : \mathcal{B}^\alpha \to A^p_\alpha(\mathbb{D}^2)$ and obtain some result like the results of [13] in some conditions. For example, if $p = 1$ then one can prove that $L : \mathcal{B}^\alpha \to A_\alpha(\mathbb{D}^2)$ is bounded for $\alpha > 0$.

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