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Existence and stability results for a class of nonlinear fractional q-integro-differential equation

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Abstract

This paper deals with the stability results for the solution of a fractional q–integro-differential problem with integral conditions. Using Krasnoselskii's, and Banach's fixed point theorems, we prove the existence and uniqueness of results. Based on the results obtained, conditions are provided that ensure the generalized Ulam stability of the original system on a time scale. The results are illustrated by the examples under the numerical technique.

Keywords: q–integro-differential problem, Krasnoselskii's theorem, stability
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1 Introduction and formulation of the problem

It is interesting to study solution to fractional q-integro-differential problem with integral conditions, which will allow a generalized stability [9, 11, 3, 12]. The authors in [1], considered the problem for the system (1.1) and we generalized the system in the q-fractional differential equation which it is not explicitly presented, and therefore it makes sense to consider for $\mathfrak{t} \in \overline{\mathbb{I}} := [0.1]$, $\sigma, \nu \in \mathbb{I} := (0, 1)$, the problem for the system

$$^{C}\mathbb{D}_q^{\sigma+\nu}[y](\mathfrak{t}) = h_1(\mathfrak{t}, y(\mathfrak{t})) + \mathbb{I}_q^{\sigma}[h_2](\mathfrak{t}, y(\mathfrak{t})) + \int_0^{\mathfrak{t}} \Theta(\mathfrak{t}, \xi, y(\xi))\, \mathrm{d}\xi, \tag{1.1}$$

under boundary condition

$$y(0) = \eta \int_0^{\tau^*} y(\xi)\, \mathrm{d}\xi, \qquad \forall \tau^* \in \mathbb{I},$$

where η is a real constant, $^{C}\mathbb{D}_q^{\sigma+\nu}$ is the Caputo fractional q-derivative of order $\sigma + \nu$, $\mathbb{I}_q^{\sigma}$ denotes the left sided Riemann–Liouville fractional q-integral of order σ and $h_i : \overline{\mathbb{I}} \times \mathfrak{H} \to \mathfrak{H}$ $(i = 1, 2)$, $\Theta : \overline{\mathbb{I}}^2 \times \mathfrak{H} \to \mathfrak{H}$, are an appropriate functions satisfying some conditions which will be stated later. $\mathfrak{H}$ here is a Banach space equipped with the norm $\|.\|$.

Here we focused our study on the question of existence and uniqueness in Sec. 3. And Sec. 4 is devoted to show a generalized stability. Note that this representation also allows us to generalize the results obtained recently in the literature. The paper is ended by two examples illustrating our results in Sec. 5. Finally, we will give some suggestions to the reader in the conclusion section 6

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2 Notations and notions preliminaries

We recall some essential preliminaries that are used for the results of the subsequent sections. Let $\mathfrak{t}_0 \in \mathbb{R}$ and $q \in \mathbb{I}$. The time scale $\mathbb{T}_{t_0}$ is defined by

$$\mathbb{T}_{t_0} = \{0\} \cup \{\mathfrak{t} : \mathfrak{t} = \mathfrak{t}_0 q^n, \ \forall n \in \mathbb{N}\}.$$

If there is no confusion concerning $\mathfrak{t}_0$ we shall denote $\mathbb{T}_{\mathfrak{t}_0}$ by $\mathbb{T}$. Let $s \in \mathbb{R}$. Define $[s]_q = (1-q^s)/(1-q)$ [8]. The q-factorial function $(y-z)_q^{(n)}$ is defined by

$$(y-z)_q^{(n)} = \prod_{k=0}^{n-1}(y - zq^k), \qquad n \in \mathbb{N}_0, \tag{2.1}$$

and $(y-z)_q^{(0)} = 1$, where $y, z \in \mathbb{R}$ and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ ([2]). Also, we have

$$(y-z)_q^{(\sigma)} = y^\sigma \prod_{k=0}^{\infty} \frac{y - zq^k}{y - zq^{\sigma+k}}, \qquad \sigma \in \mathbb{R}, s \neq 0. \tag{2.2}$$

In the paper [4], the authors proved

$$(y-z)_q^{(\sigma+\nu)} = (y-z)_q^{(\sigma)}(y - q^\sigma z)_q^{(\nu)}$$

and

$$(sy - sz)_q^{(\sigma)} = s^\sigma (y-z)_q^{(\sigma)}.$$

If $z = 0$, then it is clear that $y^{(\sigma)} = y^\sigma$. The q-Gamma function is given by [8]

$$\Gamma_q(y) = (1-q)^{1-y}(1-q)_q^{(y-1)}, \qquad (y \in \mathbb{R}\backslash\{\cdots, -2, -1, 0\}).$$

In fact, by using (2.2), we have

$$\Gamma_q(y) = (1-q)^{1-y} \prod_{k=0}^{\infty} \frac{1-q^{k+1}}{1-q^{y+k-1}}. \tag{2.3}$$

Algorithm 1 shows the MATLAB lines for calculation of $\Gamma_q(y)$ which we tend n to infinity in it [7, 13].

Algorithm 1: MATLAB lines for calculation $\Gamma_q(x)$.

```
function p = qGamma(q,x,n)
s=1;
for k=0:n
        s=s*(1-q^(k+1))/(1-q^(x+k-1));
end;
p = s*(1-q)^(1-x);
end
```

Note that, $\Gamma_q(y+1) = [y]_q \Gamma_q(y)$ [4, Lemma 1]. For any positive numbers σ and ν, the q-beta function define by

$$B_q(\sigma, \nu) = \int_0^1 (1-\xi)_q^{(\sigma-1)} \xi^{\nu-1} \, \mathrm{d}_q\xi = \frac{\Gamma_q(\sigma)\Gamma_q(\nu)}{\Gamma_q(\sigma+\nu)}. \tag{2.4}$$

For a function $w : \mathbb{T} \to \mathbb{R}$, the q-derivative of w, is

$$\mathbb{D}_q[y](\mathfrak{t}) = \left(\frac{\mathrm{d}}{\mathrm{d}\mathfrak{t}}\right)_q y(\mathfrak{t}) = \frac{y(q\mathfrak{t}) - y(\mathfrak{t})}{\mathfrak{t}(1-q)}, \tag{2.5}$$

for all $\mathfrak{t} \in \mathbb{T} \setminus \{0\}$, and ([2])

$$\mathbb{D}_q[y](0) = \lim_{\mathfrak{t} \to 0} \mathbb{D}_q[y](\mathfrak{t}).$$

Also, the higher order q-derivative of the function y is defined by

$$\mathbb{D}_q^n[y](\mathfrak{t}) = \mathbb{D}_q\left[\mathbb{D}_q^{n-1}[y]\right](\mathfrak{t}), \qquad \forall n \geq 1,$$

where $\mathbb{D}_q^0[y](\mathfrak{t}) = y(\mathfrak{t})$ [2]. In fact

$$\mathbb{D}_q^n[y](\mathfrak{t}) = \frac{1}{\mathfrak{t}^n(1-q)^n}\sum_{k=0}^{n}\frac{(1-q^{-n})_q^{(k)}}{(1-q)_q^{(k)}}q^k y(\mathfrak{t}q^k), \tag{2.6}$$

for $\mathfrak{t} \in \mathbb{T}\setminus\{0\}$ [3].

Remark 2.1. By using Eq. (2.1), we can change Eq. (2.6) as follows:

$$\mathbb{D}_q^n[y](\mathfrak{t}) = \frac{1}{\mathfrak{t}^n(1-q)^n}\sum_{k=0}^{n}\prod_{i=0}^{k-1}\frac{1-q^{i-n}}{1-q^{i+1}}\,q^k y(\mathfrak{t}q^k). \tag{2.7}$$

The q-integral of the function y is defined by

$$\mathbb{I}_q[y](\mathfrak{t}) = \int_0^{\mathfrak{t}} y(\xi)\,\mathrm{d}_q\xi = \mathfrak{t}(1-q)\sum_{k=0}^{\infty}q^k y(\mathfrak{t}q^k), \tag{2.8}$$

for $0 \leq \mathfrak{t} \leq b$, provided the series is absolutely converges [2]. By using the Algorithm 2, we can obtain the numerical results of $\mathbb{I}_q[y](\mathfrak{t})$ when $n \to \infty$.

Algorithm 2: MATLAB lines for calculation $\mathcal{I}_q[w](t)$.

```
function p = Iq(q,x,n,fun)
s=1;
for k=0:n
        s=s+q^k*eval(subs(fun,x*q^k));
end;
p=x*(1-q)*s;
end
```

If s in $[0, b]$, then

$$\int_s^b y(\xi)\,\mathrm{d}_q\xi = \mathbb{I}_q[y](b) - \mathbb{I}_q[y](s) = (1-q)\sum_{k=0}^{\infty}q^k\left[by(bq^k) - sy(sq^k)\right],$$

whenever the series exists. The operator $\mathbb{I}_q^n$ is given by $\mathbb{I}_q^0[y](\mathfrak{t}) = y(\mathfrak{t})$ and

$$\mathbb{I}_q^n[y](\mathfrak{t}) = \mathbb{I}_q\left[\mathbb{I}_q^{n-1}[y]\right](\mathfrak{t}),$$

for $n \geq 1$ and $y \in C([0, b])$ [2]. It has been proved that

$$\mathbb{D}_q\left[\mathbb{I}_q[y]\right](\mathfrak{t}) = y(\mathfrak{t}), \qquad \mathbb{I}_q\left[\mathbb{D}_q[y]\right](\mathfrak{t}) = y(\mathfrak{t}) - y(0),$$

whenever the function y is continuous at $\mathfrak{t} = 0$ [2]. The fractional Riemann–Liouville type q-integral of the function y is defined by

$$\mathbb{I}_q^\sigma[y](\mathfrak{t}) = \int_0^{\mathfrak{t}}(\mathfrak{t}-\xi)_q^{(\sigma-1)}\frac{y(\xi)}{\Gamma_q(\sigma)}\,\mathrm{d}_q\xi, \qquad \mathbb{I}_q^0[y](\mathfrak{t}) = y(\mathfrak{t}), \tag{2.9}$$

for $\mathfrak{t} \in [0, 1]$ and $\sigma > 0$ [3, 6].

Remark 2.2. By using Eqs. (2.2), (2.3) and (2.8), we obtain

$$\begin{aligned}\int_0^{\mathfrak{t}}(\mathfrak{t}-\xi)_q^{(\sigma-1)}\frac{y(\xi)}{\Gamma_q(\sigma)}\,\mathrm{d}_q\xi &= \frac{1}{\Gamma_q(\sigma)}\int_0^{\mathfrak{t}}\mathfrak{t}^{\sigma-1}\prod_{i=0}^{\infty}\frac{\mathfrak{t}-\xi q^i}{\mathfrak{t}-\xi q^{\sigma+i-1}}y(\xi)\,\mathrm{d}_q\xi \\ &= \mathfrak{t}^\sigma(1-q)^\sigma\prod_{i=0}^{\infty}\frac{1-q^{\sigma+i-1}}{1-q^{i+1}}\sum_{k=0}^{\infty}q^k\prod_{i=0}^{\infty}\frac{1-q^{k+i}}{1-q^{\sigma+k+i-1}}y(\mathfrak{t}q^k).\end{aligned}$$

Therefore,

$$\mathbb{I}_q^{\sigma}[y](\mathfrak{t}) = \mathfrak{t}^{\sigma}(1-q)^{\sigma} \lim_{n\to\infty} \sum_{k=0}^{n} q^k \prod_{i=0}^{n} \frac{\left(1-q^{\sigma+i-1}\right)\left(1-q^{k+i}\right)}{\left(1-q^{i+1}\right)\left(1-q^{\sigma+k+i-1}\right)} y(\mathfrak{t}q^k), \tag{2.10}$$

The Caputo fractional q-derivative of the function y is defined by

$$^{C}\mathbb{D}_q^{\sigma}[y](\mathfrak{t}) = \mathbb{I}_q^{[\sigma]-\sigma}\left[\mathbb{D}_q^{[\sigma]}[y]\right](\mathfrak{t}) = \int_0^{\mathfrak{t}} (\mathfrak{t}-\xi)_q^{([\sigma]-\sigma-1)} \frac{\mathbb{D}_q^{[\sigma]}[y](\xi)}{\Gamma_q\left([\sigma]-\sigma\right)}\,\mathrm{d}_q\xi \tag{2.11}$$

for $\mathfrak{t} \in \{0, 1]$ and $\sigma > 0$ [6, 10]. It has been proved that

$$\mathbb{I}_q^{\nu}\left[\mathbb{I}_q^{\sigma}[y]\right](\mathfrak{t}) = \mathbb{I}_q^{\sigma+\nu}[y](\mathfrak{t}),$$

and $^{C}\mathbb{D}_q^{\sigma}\left[\mathbb{I}_q^{\sigma}[y]\right](\mathfrak{t}) = y(\mathfrak{t})$, where $\sigma, \nu \geq 0$ [6]. Also, [6]

$$\mathbb{I}_q^{\sigma}\left[\mathbb{D}_q^{n}[y]\right](\mathfrak{t}) = \mathbb{D}_q^{n}\left[\mathbb{I}_q^{\sigma}[y]\right](\mathfrak{t}) - \sum_{k=0}^{n-1} \frac{\mathfrak{t}^{\sigma+k-n}\mathbb{D}_q^{k}[y](0)}{\Gamma_q(\sigma+k-n+1)}, \qquad \sigma > 0,\, n \geq 1.$$

Remark 2.3. From Eq.(2.3), Remark 2.1 and Eq. (2.10) in Remark 2.2, we obtain

$$\begin{aligned}
&\int_0^{\mathfrak{t}} (\mathfrak{t}-\xi)_q^{([\sigma]-\sigma-1)} \frac{\mathbb{D}_q^{[\sigma]}[y](\xi)}{\Gamma_q\left([\sigma]-\sigma\right)}\,\mathrm{d}_q\xi \\
=&\int_0^{\mathfrak{t}} \frac{\mathfrak{t}^{[\sigma]-\sigma-1}}{\Gamma_q\left([\sigma]-\sigma\right)}\left[\prod_{i=0}^{\infty}\frac{\mathfrak{t}-\xi q^i}{\mathfrak{t}-\xi q^{[\sigma]-\sigma-1+i}}\right] \times \left(\frac{1}{\mathfrak{t}^{[\sigma]}(1-q)^{[\sigma]}}\sum_{k=0}^{[\sigma]}\left[\prod_{i=0}^{k-1}\frac{\left(1-q^{i-[\sigma]}\right)}{\left(1-q^{i+1}\right)}\right] q^k y(\mathfrak{t}q^k)\right)\mathrm{d}_q\xi \\
=&\frac{1}{\mathfrak{t}^{\sigma}(1-q)^{\sigma-[\sigma]}}\sum_{k=0}^{\infty}\left(\left[\prod_{i=0}^{\infty}\frac{(1-q^{[\sigma]-\sigma+i-1})(1-q^{k+i})}{(1-q^{i+1})(1-q^{[\sigma]-\sigma-1+k+i})}\right] \times \left(\sum_{m=0}^{[\sigma]}\left[\prod_{i=0}^{m-1}\frac{\left(1-q^{i-[\sigma]}\right)}{\left(1-q^{i+1}\right)}\right] q^m y\left(\mathfrak{t}q^{k+m}\right)\right)\right).
\end{aligned}$$

Thus, we have

$$^{c}\mathbb{D}_q^{\sigma}[y](\mathfrak{t}) = \frac{1}{\mathfrak{t}^{\sigma}(1-q)^{\sigma-[\sigma]}} \lim_{n\to\infty}\sum_{k=0}^{n}\left(\left[\prod_{i=0}^{n}\frac{(1-q^{[\sigma]-\sigma+i-1})(1-q^{k+i})}{(1-q^{i+1})(1-q^{[\sigma]-\sigma-1+k+i})}\right] \times \left(\sum_{m=0}^{[\sigma]}\left[\prod_{i=0}^{m-1}\frac{\left(1-q^{i-[\sigma]}\right)}{\left(1-q^{i+1}\right)}\right] q^m y\left(\mathfrak{t}q^{k+m}\right)\right)\right). \tag{2.12}$$

Now, we introduce some basic definitions, lemmas and theorems which are used in the subsequent sections.

Lemma 2.4. [9] Let $y \in AC^n[\mathfrak{t}_1, \mathfrak{t}_2]$. Then, one has

$$\mathbb{I}^{\sigma}[^{C}\mathbb{D}_q^{\sigma}[y]](\mathfrak{t}) = y(\mathfrak{t}) + \sum_{i=0}^{n-1} c_i(\mathfrak{t}-\mathfrak{t}_1)^i,$$

$c_0, c_1, \ldots, c_{n-1} \in \mathbb{R}$, for $n-1 < \sigma \leq n,\ n \in \mathbb{N}$.

Lemma 2.5. [9] Let $n-1 < \sigma \leq n,\ n \in \mathbb{N}$ and $y \in C[\mathfrak{t}_1, \mathfrak{t}_1]$. Then for all $\mathfrak{t} \in [\mathfrak{t}_1, \mathfrak{t}_2]$, we have $^{C}\mathbb{D}_{\mathfrak{t}_1}^{\sigma}[\mathbb{I}_{\mathfrak{t}_1}^{\sigma}[y]](\mathfrak{t}) = y(\mathfrak{t})$.

Lemma 2.6. [9] Let $\sigma \in (0,1)$. Then for each $y \in AC[0,1]$, $\mathbb{I}^{\sigma}[\mathbb{D}^{\sigma}[y]](\mathfrak{t}) = y(\mathfrak{t})$ for a.e. $\mathfrak{t} \in [0,1]$, where

$$\mathbb{D}^{\sigma}[y](\mathfrak{t}) = \frac{\mathrm{d}}{\mathrm{d}\mathfrak{t}}\int_0^{\mathfrak{t}} (\mathfrak{t}-\xi)^{-\sigma}\frac{y(\xi)}{\Gamma(1-\sigma)}\,\mathrm{d}\xi.$$

Lemma 2.7. (Banach fixed point theorem, [5]) Let $\mathfrak{B}$ be a non-empty complete metric space and $\mathcal{T} : \mathfrak{B} \to \mathfrak{B}$ is contraction mapping. Then, there exists a unique point $y \in \mathfrak{B}$ such that $\mathcal{T}(y) = y$.

Lemma 2.8. ([5], Krasnoselskii fixed point theorem) Let $\mathfrak{E}$ be bounded, closed and convex subset in a Banach space $\mathfrak{B}$. If $\mathcal{T}_1$, $\mathcal{T}_2 : \mathfrak{E} \to \mathfrak{E}$ are two applications satisfying the following conditions: (A1) $\mathcal{T}_1(y) + \mathcal{T}_2(z) \in \mathfrak{E}$ for every $y, z \in \mathfrak{E}$; (A2) $\mathcal{T}_1$ is a contraction; (A3) $\mathcal{T}_2$ is compact and continuous. Then there exists $\mathfrak{v}^* \in \mathfrak{B}$ such that $\mathcal{T}_1(\mathfrak{v}^*) + \mathcal{T}_2(\mathfrak{v}^*) = \mathfrak{v}^*$.

3 Existence results

Before presenting our main results, we need the following auxiliary lemma.

Lemma 3.1. Let $\sigma+\nu \in \mathbb{I}$ and $\eta\tau^* \neq 1$. Assume that h_1, h_2 and Θ are three continuous functions. If $y \in C(\bar{\mathbb{I}}, \mathfrak{H})$, then y is solution of (1.1) iff y satisfies the IE

$$y(\mathfrak{t}) = \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\xi)_q^{(\sigma+\nu-1)}}{\Gamma_q(\sigma+\nu)} \left[h_1(\xi, y(\xi)) + \int_0^{\xi} \Theta(\xi, s, y(s))\,\mathrm{d}s + \int_0^{\xi} \frac{(\xi-a)_q^{(\sigma-1)}}{\Gamma_q(\sigma)} h_2(s, y(s))\,\mathrm{d}_q s\right] \mathrm{d}_q\xi$$
$$+ \frac{\eta}{1-\eta\tau^*} \int_0^{\tau^*} \frac{(\tau^*-s)_q^{\sigma+\nu}}{\Gamma_q(\sigma+\nu+1)} \left[h_1(s, y(s)) + \int_0^{s} \Theta(s, r, y(r))\,\mathrm{d}r + \int_0^{s} \frac{(s-r)_q^{(\sigma-1)}}{\Gamma_q(\sigma)} h_2(r, y(r))\,\mathrm{d}_q r\right] \mathrm{d}_q s. \quad (3.1)$$

Proof . Let $y \in C(\bar{\mathbb{I}}, \mathfrak{H})$ be a solution of (1.1). Firstly, we show that y is solution of integral equation (3.1). By Lemma 2.4, we obtain

$$\mathbb{I}_q^{\sigma+\nu}\left[{}^C\mathbb{D}_q^{\sigma+\nu}[y](\mathfrak{t})\right] = y(\mathfrak{t}) - y(0). \quad (3.2)$$

From equation (1.1) we have

$$\mathbb{I}_q^{\sigma+\nu}\left[{}^C\mathbb{D}_q^{\sigma+\nu}[y](t)\right] = \mathbb{I}_q^{\sigma+\nu}\left[h_1(\mathfrak{t}, y(\mathfrak{t})) + +\mathbb{I}_q^{\sigma}[h_2](\mathfrak{t}, y(\mathfrak{t})) \int_0^{\mathfrak{t}} \Theta(\mathfrak{t}, \xi, y(\xi))\,\mathrm{d}\xi\right]$$
$$= \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\xi)_q^{(\sigma+\nu-1)}}{\Gamma_q(\sigma+\nu)} \left[h_1(\mathfrak{t}, y(\mathfrak{t})) + \int_0^{\xi} \Theta(\xi, s, y(s))\,\mathrm{d}s + \int_0^{\xi} \frac{(\xi-s)_q^{\sigma-1}}{\Gamma_q(\sigma)} h_2(s, y(s))\,\mathrm{d}_q s\right] \mathrm{d}_q\xi \quad (3.3)$$

By substituting 3.3 in 3.2 with nonlocal condition in problem 3.1, we get

$$y(\mathfrak{t}) = \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\xi)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)} \left[h_1(\xi, y(\xi)) + \int_0^{\xi} \Theta(\xi, s, y(s))\,\mathrm{d}s + \int_0^{\xi} \frac{(\xi-s)_q^{\sigma-1}}{\Gamma_q(\sigma)} h_2(s, y(s))\,\mathrm{d}_q s\right] \mathrm{d}_q\xi + y(0). \quad (3.4)$$

From integral boundary condition of our problem with using Fubini's theorem and after some computations, we get

$$\begin{aligned}
y(0) &= \eta \int_0^{\tau^*} y(\xi)\,\mathrm{d}\xi \\
&= \eta \int_0^{\tau^*} \left[\int_0^{\xi} \frac{(\xi-s)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)} \left(h_1(s, y(s)) + \int_0^{s} \Theta(s, r, y(r))\,\mathrm{d}r + \int_0^{s} \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)} h_2(r, y(r))\,\mathrm{d}_q r\right) \mathrm{d}_q s\right] \mathrm{d}_q\xi + \eta\tau^* y(0) \\
&= \eta \int_0^{\tau^*} \left[\int_0^{\xi} \frac{(\xi-s)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)} h_1(s, y(s))\,\mathrm{d}_q s\right] \mathrm{d}_q\xi + \eta \int_0^{\tau^*} \left[\int_0^{\xi} \frac{(\xi-s)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)} \int_0^{s} \Theta(s, r, y(r))\,\mathrm{d}r\,\mathrm{d}_q s\right] \mathrm{d}_q\xi \\
&\quad + \eta \int_0^{\tau^*} \left[\int_0^{\xi} \frac{(\xi-s)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)} \int_0^{s} \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)} h_2(r, y(r))\,\mathrm{d}_q r\,\mathrm{d}_q s\right] \mathrm{d}_q\xi + \eta\tau^* y(0) \\
&= \eta \int_0^{\tau^*} \left(\int_s^{\tau^*} \frac{(\xi-s)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)}\,\mathrm{d}_q\xi\right) h_1(s, y(s))\,\mathrm{d}_q s + \eta \int_0^{\tau^*} \left(\int_s^{\tau^*} \frac{(\xi-s)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)}\,\mathrm{d}_q\xi\right) \left(\int_0^{s} \Theta(s, r, y(r))\,\mathrm{d}r\right) \mathrm{d}_q s \\
&\quad + \eta \int_0^{\tau^*} \left(\int_s^{\tau^*} \frac{(\xi-s)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)}\,\mathrm{d}_q\xi\right) \left(\int_0^{s} \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)} h_2(r, y(r))\,\mathrm{d}_q r\right) \mathrm{d}_q s + \eta\tau^* y(0),
\end{aligned}$$

that is

$$y(0) = \frac{\eta}{1-\eta\tau^*} \int_0^{\tau^*} \frac{(\tau^*-s)_q^{\sigma+\nu}}{\Gamma_q(\sigma+\nu)} \left[h_1(s, y(s)) + \int_0^{s} \Theta(s, r, y(r))\,\mathrm{d}r + \int_0^{s} \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)} h_2(r, y(r))\,\mathrm{d}_q r\right] \mathrm{d}_q s. \quad (3.5)$$

Finally, by substituting (3.5) in (3.4), we find (3.1). Conversely, from Lemma 3.1 and by applying the operator ${}^C\mathbb{D}_q^{\sigma+\nu}$ on both sides of (3.1), we find

$$\begin{aligned}
{}^C\mathbb{D}_q^{\sigma+\nu}[y](\mathfrak{t}) &= {}^C\mathbb{D}_q^{\sigma+\nu}\left[\mathbb{I}_q^{\sigma+\nu}\left[h_1(\mathfrak{t}, y(\mathfrak{t})) + \int_0^{\mathfrak{t}} \Theta(\mathfrak{t}, \xi, y(\xi))\,\mathrm{d}s + \mathbb{I}_q^{\sigma} h_2(\mathfrak{t}, y(\mathfrak{t}))\right]\right] + {}^C\mathbb{D}_q^{\sigma+\nu} y(0) \\
&= h_q(\mathfrak{t}, y(\mathfrak{t})) + \mathbb{I}_q^{\sigma} h_2(\mathfrak{t}, y(\mathfrak{t})) + \int_0^{\mathfrak{t}} \Theta(\mathfrak{t}, \xi, y(\xi))\,\mathrm{d}\xi.
\end{aligned} \quad (3.6)$$

This means that y satisfies the equation in problem (1.1). Furthermore, by substituting $\mathfrak{t}$ by 0 in integral equation (3.1), we have clearly that the integral boundary condition in (1.1) holds. Therefore, y is solution of problem (1.1), which completes the proof. □

In order to prove the existence and uniqueness of solution for the problem (1.1) in $C(\overline{\mathbb{I}}, \mathfrak{H})$, we use two fixed point theorems. Firstly, we transform the system (1.1) into fixed point problem as $y = \mathfrak{U}y$, where $\mathfrak{U} : (\overline{\mathbb{I}}, \mathfrak{H}) \to (\overline{\mathbb{I}}, \mathfrak{H})$ is an operator defined by following

$$\begin{aligned} \mathfrak{U}y(\mathfrak{t}) = & \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\xi)_q^{\sigma+\nu}}{\Gamma_q(\sigma+\nu)} \left[h_1(\xi, y(\xi)) + \int_0^{\xi} \Theta(\xi, s, y(s))\,\mathrm{d}s + \int_0^{\xi} \frac{(\xi-s)_q^{\sigma-1}}{\Gamma_q(\sigma)} h_2(s, y(s))\,\mathrm{d}_q s \right] \mathrm{d}_q \xi \\ & + \frac{\eta}{1-\eta\tau^*} \int_0^{\tau^*} \frac{(\tau^*-s)_q^{\sigma+\nu}}{\Gamma_q(\sigma+\nu+1)} \left[h_1(s, y(s)) + \int_0^{s} \Theta(s, r, y(r))\,\mathrm{d}r + \int_0^{s} \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)} h_2(r, y(r))\,\mathrm{d}_q r \right] \mathrm{d}_q s. \end{aligned} \tag{3.7}$$

3.1 Existence result by Krasnoselskii's fixed point

Theorem 3.2. Consider continuous functions $h_1, h_2 : \overline{\mathbb{I}} \times \mathfrak{H} \to \mathfrak{H}$ and $\Theta : \overline{\mathbb{I}}^2 \times \mathfrak{H} \to \mathfrak{H}$ such that satisfying: (H_1) The inequalities

$$\|h_j(\mathfrak{t}, y(\mathfrak{t})) - h_j(\mathfrak{t}, z(\mathfrak{t}))\| \leq \mu_j \|y(\mathfrak{t}) - z(\mathfrak{t})\|, \qquad j = 1, 2,$$

and

$$\|\Theta(\mathfrak{t}, \mathfrak{s}, y(\mathfrak{s})) - \Theta(\mathfrak{t}, \mathfrak{s}, z(\mathfrak{s}))\| \leq \mu^* \|y(\mathfrak{s}) - z(\mathfrak{s})\|,$$

where $\mu^*, \mu_j \geq 0,\ (j = 1, 2)$ with $\mu = \max\{\mu_1, \mu_2, \mu^*\}$; (H_2) There exist three functions $\varrho^*, \varrho_j \in L^\infty(\overline{\mathbb{I}}, \mathbb{R}^+)$, $(j = 1, 2)$, such that

$$\|h_j(\mathfrak{t}, y(\mathfrak{t}))\| \leq \varrho_j(\mathfrak{t}) \|y(\mathfrak{t})\|, \qquad j = 1, 2,$$

and

$$\|\Theta(\mathfrak{t}, \mathfrak{s}, y(\mathfrak{s}))\| \leq \varrho^*(\mathfrak{t}) \|y(\mathfrak{s})\|,$$

$\forall \mathfrak{t} \in \overline{\mathbb{I}}$, $y, z \in \mathfrak{H}$ and

$$(\mathfrak{t}, \mathfrak{s}) \in \mathbb{G} := \Big\{ (\mathfrak{t}, \mathfrak{s}) : 0 \leq \mathfrak{s} \leq \mathfrak{t} \leq 1 \Big\}.$$

If $\lambda \leq 1$ and $\mu\lambda^* \leq 1$, then the problem (1.1) has at least one solution on $\overline{\mathbb{I}}$, where

$$\begin{aligned} \lambda = & \frac{\|\varrho_1\|_{L^\infty} + \|\varrho^*\|_{L^\infty}}{\Gamma_q(\sigma+\nu+1)} + \frac{\|\varrho_2\|_{L^\infty} B_q(\sigma+1, \sigma+\nu)}{\Gamma_q(\sigma+1)\Gamma_q(\sigma+\nu)} + \frac{|\eta|\|\varrho_1\|_{L^\infty} \tau^{*\sigma+\nu+1} + |\eta|\|\varrho^*\|_{L^\infty} \tau^{*\sigma+\nu+1}}{|1-\eta\tau^*|\Gamma_q(\sigma+\nu+2)} \\ & + \frac{|\eta|\|\varrho_2\|_{L^\infty} \tau^{*2\sigma+\nu+1} B_q(\sigma+1, \sigma+\nu+1)}{|1-\nu\tau^* \Gamma_q(\sigma+1)\Gamma_q(\sigma+\nu+1)}, \end{aligned} \tag{3.8}$$

and

$$\lambda^* = \frac{|\eta|}{|1-\eta\tau^*|} \left[\frac{2\tau^{*\sigma+\nu+1}}{\Gamma_q(\sigma+\nu+2)} + \frac{\tau^{*2\sigma+\nu+1} B_q(\sigma+1, \sigma+\nu+1)}{\Gamma_q(\sigma+1)\Gamma_q(\sigma+\nu+1)} \right]. \tag{3.9}$$

Proof . For any function $y \in C(\overline{\mathbb{I}}, \mathfrak{H})$, we define the norm

$$\|y\|_* := \max\Big\{ e^{-\mathfrak{t}} \|y(\mathfrak{t})\| \ :\ \mathfrak{t} \in \overline{\mathbb{I}} \Big\},$$

and consider the closed ball

$$\mathbb{B}_\ell := \Big\{ y \in C(\overline{\mathbb{I}}, \mathfrak{H}) \ :\ \|y\|_* \leq \ell \Big\}.$$

Next, let us define the operators $\mathfrak{U}_1, \mathfrak{U}_2$ on $\mathbb{B}_\ell$ as follows

$$\mathfrak{U}_1 y(\mathfrak{t}) = \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\xi)_q^{\sigma+\nu+1}}{\Gamma_q(\sigma+\nu)} \left[h_1(\xi, y(\xi)) + \int_0^{\xi} \Theta(\xi, s, y(s))\,\mathrm{d}s + \int_0^{\xi} \frac{(\xi-s)_q^{\sigma-1}}{\Gamma_q(\sigma)} h_2(s, y(s))\,\mathrm{d}_q s \right] \mathrm{d}_q \xi. \tag{3.10}$$

and

$$\mathfrak{U}_2 y(\mathfrak{t}) = \frac{\eta}{1-\eta\tau^*}\int_0^{\tau}\frac{(\tau-s)_q^{\sigma+\nu}}{\Gamma_q(\sigma+\nu+1)}\left[h_1(s,y(s))+\int_0^s \Theta(s,r,y(r))\,\mathrm{d}r+\int_0^s\frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)}h_2(r,y(r))\,\mathrm{d}_q r\right]\mathrm{d}_q s. \tag{3.11}$$

For $y,z\in\mathbb{B}_\ell$, $\mathfrak{t}\in\bar{\mathbb{I}}$ and by the assumption (H_2), we find

$$\begin{aligned}
\|\mathfrak{U}_1 y(\mathfrak{t})+\mathfrak{U}_2 z(\mathfrak{t})\| \le& \int_0^{\mathfrak{t}}\frac{(\mathfrak{t}-\xi)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)}\left[\|h_1(\xi,y(\xi))\|+\int_0^{\xi}\|\Theta(\xi,s,y(s))\|\,\mathrm{d}s+\int_0^{\xi}\frac{(\xi-s)_q^{\sigma-1}}{\Gamma_q(\sigma)}\|h_2(\xi,y(\xi))\|\,\mathrm{d}_q s\right]\mathrm{d}_q\xi\\
&+\frac{|\eta|}{|1-\eta\tau^*|}\int_0^{\tau^*}\frac{(\tau^*-s)_q^{\sigma+\nu}}{\Gamma_q(\sigma+\nu+1)}\Bigg[\|h_1(s,z(s))\|+\int_0^s\|\Theta(s,r,z(r))\|\,\mathrm{d}r\\
&+\int_0^s\frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)}\|h_2(r,z(r))\|\,\mathrm{d}_q r\Bigg]\mathrm{d}_q s\\
\le& \int_0^{\mathfrak{t}}\frac{(\mathfrak{t}-\xi)_q^{\sigma+\nu+1}}{\Gamma_q(\sigma+\nu)}\left[\varrho_1(\xi)\|y(\xi)\|+\int_0^{\xi}\varrho^*(\xi)\|y(s)\|\,\mathrm{d}s+\int_0^{\xi}\frac{(\xi-s)_q^{\sigma-1}}{\Gamma_q(\sigma)}\varrho^*(s)\|y(s)\|\,\mathrm{d}_q s\right]\mathrm{d}_q\xi\\
&+\frac{|\eta|}{|1-\eta\tau^*|}\int_0^{\tau^*}\frac{(\tau^*-s)^{\sigma+\nu}}{\Gamma_q(\sigma+\nu+1)}\Bigg[\varrho_1(s)\|z(s)\|+\int_0^s\varrho^*(s)\|z(r)\|\,\mathrm{d}r\\
&+\int_0^s\frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)}\varrho_2(r)\|z(r)\|\,\mathrm{d}_q r\Bigg]\mathrm{d}_q s\\
\le& \int_0^{\mathfrak{t}}\frac{(\mathfrak{t}-\xi)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)}\left[\|\varrho_1\|_{\mu^\infty}\|y\|_* e^{\xi}+\|\varrho^*\|_{L^\infty}\|y\|_*(e^{\xi}-1)+\|\varrho_2\|_{L^\infty}\|y\|_*\int_0^{\xi}\frac{(\xi-s)_q^{\sigma-1}}{\Gamma_q(\sigma)}e^s\,\mathrm{d}_q s\right]\mathrm{d}_q\xi\\
&+\frac{|\eta|}{|1-\eta\tau^*|}\int_0^{\tau^*}\frac{(\tau^*-s)^{\sigma+\nu}}{\Gamma_q(\sigma+\nu+1)}\Bigg[\|\varrho_1\|_{L^\infty}\|z\|_* e^s+\|\varrho^*\|_{L^\infty}\|z\|_*(e^{s-1})\\
&+\|\varrho_2\|_{L^\infty}\|z\|_*\int_0^s\frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)}e^r\,\mathrm{d}r\Bigg]\mathrm{d}_q s.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\mathfrak{U}_1 y+\mathfrak{U}_2 z\|_* \le& \int_0^{\mathfrak{t}}\frac{(\mathfrak{t}-\xi)^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)}\left[\|\varrho_1\|_{L^\infty}\|y\|_*\frac{e^{\xi}}{e^{\mathfrak{t}}}+\|\varrho^*\|_{L^\infty}\|y\|_*\frac{(e^{\xi}-1)}{e^{\mathfrak{t}}}+\|\varrho_2\|_{L^\infty}\|y\|_*\int_0^{\xi}\frac{(\xi-s)_q^{\sigma\sigma-1}}{\Gamma_q(\sigma)}\frac{e^s}{e^{\mathfrak{t}}}\,\mathrm{d}_q s\right]d_q\xi\\
&+\frac{|\eta|}{|1-\eta\tau^*|}\int_0^{\tau^*}\frac{(\tau^*-s)^{\sigma+\nu}}{\Gamma_q(\sigma+\nu+1)}\Bigg[\|\varrho_1\|_{L^\infty}\|z\|_*\frac{e^s}{e^{\mathfrak{t}}}+\|\varrho^*\|_{L^\infty}\|z\|_*\frac{(e^s-1)}{e^{\mathfrak{t}}}\\
&+\|\varrho_2\|_{L^\infty}\|z\|_*\int_0^s\frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)}\frac{e^r}{e^{\mathfrak{t}}}\,\mathrm{d}_q r\Bigg]\mathrm{d}_q s\\
\le& \ell\Bigg[\frac{\|\varrho_1\|_{L^\infty}+\|\varrho^*\|_{L^\infty}}{\Gamma_q(\sigma+\nu+1)}+\frac{\|\varrho_2\|_{L^\infty}}{\Gamma_q(\sigma+1)\Gamma_q(\nu+1)}\int_0^1(1-\xi)_q^{\sigma+\nu+1}\xi^{\sigma}\,\mathrm{d}_q\xi\\
&+\frac{|\eta|\|\varrho_1\|_{L^\infty}{\tau^*}^{\sigma+\nu+1}+|\eta|\|\varrho^*\|_{L^\infty}{\tau^*}^{\sigma+\nu+1}}{|1-\eta\tau^*|\Gamma_q(\sigma+\nu+1)}+\frac{|\eta|\|\varrho_2\|_{L^\infty}}{|1-\eta\tau^*|\Gamma_q(\sigma\sigma+1)\Gamma_q(\sigma+\nu+1)}\int_0^{\tau^*}(\tau^*-s)^{\sigma+\nu}s^{\sigma}\,\mathrm{d}_q s\Bigg]\\
=& \ell\Bigg[\frac{\|\varrho_1\|_{L^\infty}+\|\varrho^*\|_{L^\infty}}{\Gamma_q(\sigma+\nu+1)}+\frac{\|\varrho_2\|_{L^\infty}\nu(\sigma+1,\sigma+\nu)}{\Gamma_q(\sigma+1)\Gamma_q(\nu+1)}+\frac{|\eta|}{|1-\eta\tau^*|}\Bigg(\frac{\|\varrho_1\|_{L^\infty}{\tau^*}^{\sigma+\nu+1}+\|\varrho^*\|_{L^\infty}{\tau^*}^{\sigma+\nu+1}}{\Gamma_q(\sigma+\nu+2)}\\
&+\frac{\|\varrho_2\|_{L^\infty}{\tau^*}^{2\sigma+\nu+1}\nu(\sigma+1,\sigma+\nu+1)}{\Gamma_q(\sigma+1)\Gamma_q(\sigma+\nu+1)}\Bigg)\Bigg]=\ell\lambda\le\ell.
\end{aligned}\tag{3.12}$$

This implies that $(\mathfrak{U}_1 y+\mathfrak{U}_2 z)\in\mathbb{B}_\ell$. Here we used the computations

$$\int_0^1(1-\xi)_q^{\sigma+\nu}\xi^{\sigma}\,\mathrm{d}_q\xi=\beta_q(\sigma+1,\sigma+\nu),$$

$$\int_0^{\tau^*}(\tau^*-s)_q^{\sigma+\nu}s^{\sigma}\,\mathrm{d}_q\xi={\tau^*}^{2\sigma+\nu+1}\nu(\sigma+1,\sigma+\nu+1),$$

and the estimations:

$$\frac{e^{\xi}}{e^{\mathfrak{t}}} \leq 1, \quad \frac{e^{s}}{e^{\mathfrak{t}}} \leq 1, \quad \frac{e^{r}}{e^{\mathfrak{t}}} \leq 1.$$

In this step, we show that $\mathfrak{U}_2$ is a contraction mapping. Let $y, z \in \mathfrak{H}$, $\mathfrak{t} \in \overline{\mathbb{I}}$. We have

$$\begin{aligned}
\|\mathfrak{U}_2 y(\mathfrak{t}) - \mathfrak{U}_2 z(\mathfrak{t})\| &\leq \frac{|\eta|}{|1-\eta\tau^*|}\int_0^{\tau^*} \frac{(\tau^*-s)_q^{\sigma+\nu}}{\Gamma_q(\sigma+\nu+1)}\Big[\|h_1(s,y(s)) - h_1(s,\nu(s))\| + \int_0^s \|\Theta(s,r,y(r)) - \Theta(s,r,z(r))\|\,\mathrm{d}r \\
&\quad + \int_0^s \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)}\|h_2(r,y(r)) - h_2(r,z(r))\|\,\mathrm{d}_q r\Big]\,\mathrm{d}_q s \\
&\leq \frac{|\eta|}{|1-\eta\tau^*|}\int_0^{\tau^*} \frac{(\tau^*-s)_q^{\sigma+\nu}}{\Gamma_q(\sigma+\nu+1)}\Big[\mu_1\|y-z\|_* e^s + \int_0^s \mu^*\|y-z\|_* e^r\,\mathrm{d}r \\
&\quad + \int_0^s \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)}\mu_2\|y-z\|_* e^r\,\mathrm{d}r\Big]\,\mathrm{d}_q s \\
&\leq \frac{|\eta|}{|1-\eta\tau^*|}\int_0^{\tau^*} \frac{(\tau^*-s)_q^{\sigma+\nu}}{\Gamma_q(\sigma+\nu+1)}\Big[\mu\|y-z\|_* e^s + \mu\|y-z\|_*(e^{s-1}) \\
&\quad + \int_0^s \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)}\mu\|y-z\|_* e^r\,\mathrm{d}_q r\Big]\,\mathrm{d}_q s
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\mathfrak{U}_2 y - \mathfrak{U}_2 z\|_* &\leq \frac{|\eta|}{|1-\eta\tau^*|}\int_0^{\tau^*} \frac{(\tau^*-s)_q^{\sigma+\nu}}{\Gamma_q(\sigma+\nu+1)}\Big[\mu\|y-z\|_*\frac{e^s}{e^{\mathfrak{t}}} - \mu\|y-z\|_*\frac{(e^s-1)}{e^{\mathfrak{t}}} + \int_0^s \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)}\mu\|y-z\|_*\frac{e^r}{e^{\mathfrak{t}}}\,\mathrm{d}_q r\Big]\,\mathrm{d}_q s \\
&\leq \frac{|\eta|\mu}{|1-\eta\tau^*|}\Big[\frac{2\tau^{*\sigma+\nu+1}}{\Gamma_q(\sigma+\nu+2)} + \frac{\tau^{*2\sigma+\nu+1}\nu(\sigma+1,\sigma+\nu+1)}{\Gamma_q(\sigma+1)\Gamma_q(\sigma+\nu+1)}\Big]\|y-z\|_*.
\end{aligned}$$

Then since $\mu\lambda^* \leq 1$, $\mathfrak{U}_2$ is a contraction mapping. The continuity of the functions h_1, h_2 and Θ implies that $\mathfrak{U}_1$ is continuous and $\mathfrak{U}_1\mathbb{B}_\ell \subset \mathbb{B}_\ell$, for each $y \in \mathbb{B}_\ell$, i.e., $\mathfrak{U}_1$ is uniformly bounded on $\mathbb{B}_\ell$ as

$$\|(\mathfrak{U}_1 y)(\mathfrak{t})\| \leq \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\xi)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)}\Big[\|h_1(\xi,y(\xi))\| + \int_0^{\xi}\|\Theta(\xi,s,y(s))\|\,\mathrm{d}s + \int_0^{\xi}\frac{(\xi-s)_q^{\sigma-1}}{\Gamma_q(\sigma)}\|h_2(s,y(s))\|\,\mathrm{d}_q s\Big]\,\mathrm{d}_q\xi,$$

which implies that

$$\begin{aligned}
\|\mathfrak{U}_1 y\|_* &\leq \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\xi)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)}\Big[\|\varrho_1\|_{L^\infty}\|y\|_*\frac{e^{\xi}}{e^{\mathfrak{t}}} + \varrho^*\|_{L^\infty}\|y\|_*\frac{(e^{\xi}-1)}{e^{\mathfrak{t}}} + \varrho_2\|_{L^\infty}\|y\|_*\int_0^{\xi}\frac{(\xi-s)_q^{\sigma-1}}{\Gamma_q(\sigma)}\frac{e^s}{e^{\mathfrak{t}}}\,\mathrm{d}_q s\Big]\,\mathrm{d}_q\xi \\
&\leq \ell\Big[\frac{\|\varrho_*\|_{L^\infty} + \|\varrho^*\|_{L^\infty}}{\Gamma_q(\sigma+\nu+1)} + \frac{\|\varrho_2\|_{L^\infty}\nu(\sigma+1,\sigma+\nu)}{\Gamma_q(\sigma+1)\Gamma_q(\nu+1)}\Big] \leq \ell\lambda \leq \ell. \qquad (3.13)
\end{aligned}$$

Finally, we will show that $(\mathfrak{U}_1\mathbb{B}_\ell)$ is equi-continuous. For this end, we put

$$\overline{h}_j = \sup_{(\mathfrak{t},y(\mathfrak{t}))\in\overline{\mathbb{I}}\times\mathbb{B}_\ell}\|h_j(\mathfrak{t},y(\mathfrak{t}))\|, \qquad j=1,2$$

$$\overline{\Theta} = \sup_{(\mathfrak{t},\mathfrak{s},y(\mathfrak{s}))\in\mathbb{G}\times\mathbb{B}_\ell}\int_0^{\xi}\|\Theta(\mathfrak{t},\xi,y(\xi))\|\,\mathrm{d}\xi.$$

Let for any $y \in \mathbb{B}_\ell$ and for each $\mathfrak{t}_1, \mathfrak{t}_2 \in \overline{\mathbb{I}}$ with $\mathfrak{t}_1 \leq \mathfrak{t}_2$, we have

$$\begin{aligned}
\|(\mathfrak{U}_1 y)(\mathfrak{t}_2) - (\mathfrak{U}_1 y)(\mathfrak{t}_1)\| \leq & \int_{\mathfrak{t}_1}^{\mathfrak{t}_2} \frac{(\mathfrak{t}_2-\xi)^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)} \left[\|h_1(\xi, y(\xi))\| + \int_0^\xi \|\Theta(\xi, s, y(s))\| \mathrm{d}_q s + \int_0^\xi \frac{(\xi-s)_q^{\sigma-1}}{\Gamma_q(\sigma)} \|h_2(s, y(s))\| \mathrm{d}s \right] \mathrm{d}_q\xi \\
& + \frac{1}{\Gamma_q(\sigma)} \int_0^{\mathfrak{t}_1} \left[(\mathfrak{t}_1-\xi)_q^{\sigma+\nu-1} - (\mathfrak{t}_2-\xi)_q^{\sigma+\nu-1} \right] \times \left[\|h_1(\xi, y(\xi))\| + \int_0^\xi \|\Theta(\xi, s, y(s))\| \, \mathrm{d}s \right. \\
& \left. + \int_0^\xi \frac{(\xi-s)_q^{\sigma-1}}{\Gamma_q(\sigma)} \|h_2(s, y(s))\| \, \mathrm{d}s \right] \mathrm{d}_q\xi \\
\leq & \int_{\mathfrak{t}_1}^{\mathfrak{t}_2} \frac{(\mathfrak{t}_2-\xi)_q^{\sigma+\nu-1}}{\Gamma(\sigma+\nu)} \left[\overline{h}_1 + \overline{\Theta} + \int_0^\xi \frac{\overline{h}_2 (\xi-s)_q^{\sigma-1}}{\Gamma_q(\sigma)} \, \mathrm{d}s \right] \mathrm{d}_q\xi \\
& + \frac{1}{\Gamma_q(\sigma+\nu)} \int_0^{\mathfrak{t}_1} \left[(\mathfrak{t}_1-\xi)_q^{\sigma+\nu-1} - (\mathfrak{t}_2-\xi)_q^{\sigma+\nu-1} \right] \times \left[\overline{h}_1 + \overline{\Theta} + \int_0^\xi \frac{\overline{h}_2 (\xi-s)_q^{\sigma-1}}{\Gamma_q(\sigma)} \mathrm{d}_q s \right] \mathrm{d}_q\xi \\
\leq & \int_{\mathfrak{t}_1}^{\mathfrak{t}_2} \frac{(\mathfrak{t}_2-\xi)^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)} \left[\overline{h}_1 + \overline{\Theta} + \frac{\overline{h}_2}{\Gamma_q(\sigma+1)} \right] \mathrm{d}_q\xi + \frac{1}{\Gamma_q(\sigma+\nu)} \int_0^{\mathfrak{t}_1} \left[(\mathfrak{t}_1-\xi)_q^{\sigma+\nu-1} - (\mathfrak{t}_2-\xi)_q^{\sigma+\nu-1} \right] \\
& \times \left[\overline{h}_1 + \overline{\Theta} + \frac{\overline{h}_1}{\Gamma_q(\sigma+1)} \right] \mathrm{d}_q\xi + \frac{1}{\Gamma_q(\sigma+\nu+1)} \left[\overline{h}_1 + \overline{\Theta} \right. \\
& \left. + \frac{\overline{h}_2}{\Gamma_q(\sigma+1)} \right] \left[2(\mathfrak{t}_2-\mathfrak{t}_1)_q^{\sigma+\nu} + \mathfrak{t}_1^{\sigma+\nu} - \mathfrak{t}_2^{\sigma+\nu} \right].
\end{aligned}$$

The RHS of the last inequality is independent of y and tends to zero when $|\mathfrak{t}_2 - \mathfrak{t}_1| \to 0$, this means that

$$|\mathfrak{U}_1 y(\mathfrak{t}_2) - \mathfrak{U}_1 y,$$

which implies that $\mathfrak{U}_1 \mathbb{B}_\ell$ is equi-continuous, then $\mathfrak{U}_1$ is relatively compact on $\mathbb{B}_\ell$. Hence by Arzelá-Ascoli theorem, $\mathfrak{U}_1$ is compact on $\mathbb{B}_\ell$. Now, all hypothesis of Theorem 3.2 hold, therefore the operator $\mathfrak{U}_1 + \mathfrak{U}_2$ has a fixed point on $\mathbb{B}_\ell$. So the problem (1.1) has at least one solution on $\overline{\mathbb{I}}$. This proves the theorem. □

3.2 Existence and uniqueness result

Theorem 3.3. Assume that (H_1) holds. If $\mu\lambda < 1$, then the BVP (1.1) has a unique solution on $\overline{\mathbb{I}}$.

Proof . Define $m = \max\{m_1, m_2, m^*\}$, where m_j and m^* are positive numbers such that

$$m_j = \sup_{\mathfrak{t} \in \overline{\mathbb{I}}} \|h_j(\mathfrak{t}, 0)\|, (j = 1, 2), \qquad m^* = \sup_{(\mathfrak{t}, \mathfrak{s}) \in \mathbb{G}} \|\Theta(\mathfrak{t}, \mathfrak{s}, 0)\|.$$

We fix

$$\ell \geq \frac{m^* \lambda}{1 - \mu\lambda},$$

and we consider

$$\mathbb{N}_\ell = \left\{ y \in C(\overline{\mathbb{I}}, \mathfrak{H}) \, : \, \|y\|_* \leq \ell \right\}.$$

Then, in view of the assumption (H_1), we have

$$\begin{aligned}
\|h_q(\mathfrak{t}, y(\mathfrak{t}))\| &= \|h_1(\mathfrak{t}, y(\mathfrak{t})) - h_1(\mathfrak{t}, 0) + h_1(\mathfrak{t}, 0)\| \\
&\leq \|h_q(\mathfrak{t}, y(\mathfrak{t})) - h_q(\mathfrak{t}, 0)\| + \|h_1(\mathfrak{t}, 0)\| \\
&\leq \mu_1 \|y\|_* + m_1,
\end{aligned}$$

$$\|h_2(\mathfrak{t}, y(\mathfrak{t}))\| \leq \mu_2 \|y\|_* + m_2,$$

and

$$\|\Theta(\mathfrak{t}, \mathfrak{s}, y(\mathfrak{s}))\| \leq \mu^* \|y\|_* + m^*.$$

In the first step, we show that $\mathfrak{U}\mathbb{N}_\ell \subset \mathbb{N}_\ell$. For each $\mathfrak{t} \in \bar{\mathbb{I}}$ and for any $y \in \mathbb{N}_\ell$,

$$\begin{aligned}\|\mathfrak{U}y(\mathfrak{t})\| \leq & \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\xi)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)} \left[\|h_1(\xi,y(\xi))\| + \int_0^{\xi} \|\Theta(\xi,s,y(s))\|\,\mathrm{d}s + \int_0^{\xi} \frac{(\xi-s)_q^{\sigma-1}}{\Gamma_q(\sigma)} \|h_2(s,y(s))\|\,\mathrm{d}_q s\right] \mathrm{d}_q\xi \\ & + \frac{|\eta|}{|1-\eta\tau^*|} \int_0^{\tau^*} \frac{(\tau^*-s)_q^{\sigma+\nu}}{\Gamma_q(\sigma+\nu+1)} \Big[\|h_1(s,z(s))\| \\ & + \int_0^{\xi} \|\Theta(s,r,z(r))\|\,\mathrm{d}r + \int_0^{\xi} \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)} \|h_2(r,z(r))\|\,\mathrm{d}r\Big] \mathrm{d}_q s \\ \leq & (\mu\ell+m)\lambda \leq \ell.\end{aligned}$$

Hence, $\mathfrak{U}\mathbb{N}_\ell \subset \mathbb{N}_\ell$. Now, in the second step, we shall show that $\mathfrak{U}: \mathbb{N}_\ell \to \mathbb{N}_\ell$ is a contraction. From the assumption (H_1) we have for any $y, z \in \mathbb{N}_\ell$ and for each $\mathfrak{t} \in \bar{\mathbb{I}}$

$$\begin{aligned}\|\mathfrak{U}y(\mathfrak{t}) - \mathfrak{U}z(\mathfrak{t})\| \leq & \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\xi)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)} \Big[\|h_1(\xi,y(\xi)) - h_1(\xi,z(\xi))\| + \int_0^{\xi} \|\Theta(\xi,s,y(s)) - \Theta(\xi,s,z(s))\|\,\mathrm{d}s \\ & + \int_0^{\xi} \frac{(\xi-s)_q^{\sigma-1}}{\Gamma_q(\sigma)} \|h_2(s,y(s)) - h_2(s,z(s))\|\,\mathrm{d}_q s\Big] \mathrm{d}_q\xi \\ & + \frac{|\eta|}{|1-\eta\tau^*|} \int_0^{\tau^*} \frac{(\tau^*-s)_q^{\sigma+\nu}}{\Gamma-q(\sigma+\nu+1)} \Big[\|h_1(s,y(s)) - h_1(s,z(s))\| + \int_0^{s} \|\Theta(s,r,y(r)) - \Theta(s,r,z(r))\|\,\mathrm{d}r \\ & + \int_0^{s} \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)} \|h_2(r,y(r)) - h_2(r,z(r))\|\,\mathrm{d}_q r\Big] \mathrm{d}_q s \\ \leq & \mu\lambda\|y-z\|_*. \end{aligned} \tag{3.14}$$

Since $\mu\lambda < 1$, it follows that $\mathfrak{U}$ is a contraction. All assumptions of Lemma 2.2 are satisfied, then there exists $y \in C(\bar{\mathbb{I}}, \mathfrak{H})$ such that $\mathfrak{U}y = y$, which is the unique solution of the problem (1.1) in $C(\bar{\mathbb{I}}, \mathfrak{H})$. □

4 Generalized Ulam stabilitiestle

The aim is to discuss the Ulam stability for problem (1.1), by using the integration

$$\begin{aligned} z(\mathfrak{t}) = & \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\xi)_q^{\sigma+\nu-1}}{\Gamma_q(\sigma+\nu)} \left[\|h_1(\xi,z(\xi))\| + \int_0^{\xi} \|\Theta(\xi,s,z(s))\|\,\mathrm{d}s + \int_0^{\xi} \frac{(\xi-s)_q^{\sigma-1}}{\Gamma_q(\sigma)} \|h_2(s,z(s))\|\,\mathrm{d}_q s\right] \mathrm{d}_q\xi \\ & + \frac{\eta}{1-\eta\tau^*} \int_0^{\tau^*} \frac{(\tau^*-s)_q^{\sigma+\nu}}{\Gamma_q(\sigma+\nu+1)} \left[h_1(s,z(s)) + \int_0^{s} \Theta(s,r,z(r))\,\mathrm{d}r + \int_0^{s} \frac{(s-r)_q^{\sigma-1}}{\Gamma_q(\sigma)} h_2(r,z(r))\,\mathrm{d}_q r\right] \mathrm{d}_q s. \end{aligned}$$

Here $z \in C(\bar{\mathbb{I}}, \mathfrak{H})$ possess a fractional derivative of order $\sigma+\nu$, where $0 < \sigma+\nu < 1$ and $h_j : \bar{\mathbb{I}} \times \mathfrak{H} \to \mathfrak{H}$ and $\Theta : \bar{\mathbb{I}}^2 \times \mathfrak{H} \to \mathfrak{H}$, are continuous functions. Then we define the nonlinear continuous operator $\mathfrak{P} : C(\bar{\mathbb{I}}, \mathfrak{H}) \to C(\bar{\mathbb{I}}, \mathfrak{H})$, as follows

$$\mathfrak{P}z(\mathfrak{t}) = {}^{C}\mathbb{D}_q^{\sigma+\nu} z(\mathfrak{t}) - h_1(\mathfrak{t}, v(\mathfrak{t})) - \mathbb{I}_q^{\sigma} h_2(\mathfrak{t}, v(\mathfrak{t})) - \int_0^{\mathfrak{t}} \Theta(\mathfrak{t}, \xi, z(\xi))\,\mathrm{d}\xi.$$

For each $\epsilon > 0$ and for each solution z of problem (1.1), such that

$$\|\mathfrak{P}z\|_* \leq \epsilon, \tag{4.1}$$

the problem (1.1), is said to be Ulam–Hyers stable if we can find a solution $y \in C(\bar{\mathbb{I}}, \mathfrak{H}\mathfrak{U})$ of problem (1.1) and $\gamma \in \mathbb{R}^{\geq 0}$, satisfying the inequality $\|y - z\|_* \leq \gamma\epsilon^*$, is a positive real number depending on ϵ. Consider function $\wp$ in $C(\mathbb{R}^+, \mathbb{R}^+)$ such that for each solution z of problem (1.1), we can find a solution $u \in C(\bar{\mathbb{I}}, \mathfrak{H})$ of the problem (1.1) such that

$$\|y(\mathfrak{t}) - z(\mathfrak{t})\|_* \leq \wp(\epsilon), \qquad \mathfrak{t} \in \bar{\mathbb{I}}.$$

Then the problem (1.1), is said to be generalized Ulam–Hyers stable. For each $\epsilon > 0$ and for each solution z of problem (1.1), the problem (1.1) is called Ulam-Hyers-Rassias stable with respect to $\varrho \in C(\bar{\mathbb{I}}, \mathbb{R}^+)$ if

$$\|\mathfrak{P}z(\mathfrak{t})\|_* \leq \epsilon\varrho(t), \qquad \mathfrak{t} \in \bar{\mathbb{I}}, \tag{4.2}$$

and there exist a real number $\gamma > 0$ and a solution $z \in C(\bar{\mathbb{I}}, \mathfrak{H})$ of problem (1.1) such that

$$\|y(\mathfrak{t}) - z(\mathfrak{t})\| \leq \gamma \epsilon_* \varrho(\mathfrak{t}), \qquad \forall \mathfrak{t} \in \bar{\mathbb{I}},$$

where ϵ_* is a positive real number depending on ϵ.

Theorem 4.1. Under assumption (H_1) in Theorem 3.1, with $\mu\lambda < 1$. The problem (1.1) is both Ulam–Hyers and generalized Ulam–Hyers stable.

Proof . Let $y \in C(\bar{\mathbb{I}}, \mathfrak{H})$ be a solution of problem (1.1), satisfying (3.1) in the sense of Theorem 3.2. Let z be any solution satisfying (4.1). Lemma 2.4 implies the equivalence between the operators $\mathfrak{P}$ and $\mathcal{T} - \mathfrak{I}_d$ (where $\mathfrak{I}_d$ is the identity operator) for every solution $z \in C(\bar{\mathbb{I}}, \mathfrak{H})$ of problem (1.1) satisfying $\mu\lambda < 1$. Therefore, we deduce by the fixed-point property of the operator $\mathcal{T}$ that

$$\begin{aligned}
\|z(\mathfrak{t}) - y(\mathfrak{t})\|_* &= \|z(\mathfrak{t}) - \mathcal{T}z(\mathfrak{t}) + \mathcal{T}z(\mathfrak{t}) - y(\mathfrak{t})\|_* \\
&= \|z(t) - \mathcal{T}z(\mathfrak{t}) + \mathcal{T}z(\mathfrak{t}) - \mathcal{T}y(\mathfrak{t})\|_* \\
&\leq \|\mathcal{T}z(\mathfrak{t}) - \mathcal{T}y(\mathfrak{t})\| + \|\mathcal{T}z(\mathfrak{t}) - z(\mathfrak{t})\|_* \\
&\leq \mu\lambda\|y - z\|_* + \epsilon,
\end{aligned}$$

because $\mu\lambda < 1$ and $\epsilon > 0$, we find

$$\|u - v\|_* \leq \frac{\epsilon}{1 - \mu\lambda}.$$

Fixing $\epsilon_* = \frac{\epsilon}{1-\mu\lambda}$ and $\gamma = 1$, we obtain the Ulam–Hyers stability condition. In addition, the generalized Ulam-Hyers stability follows by taking $\wp(\epsilon) = \frac{\epsilon}{1-\mu\lambda}$. □

Theorem 4.2. Assume that (H_1) holds with $\mu < \lambda - 1$, and there exists a function $\varrho \in C(\bar{\mathbb{I}}, \mathbb{R}^+)$ satisfying the condition 4.2. Then the problem (1.1) is Ulam-Hyers-Rassias stable with respect to ϱ.

Proof . We have from the proof of Theorem 4.1, $\|y(\mathfrak{t}) - z(\mathfrak{t})\|_* \leq \epsilon_* \varrho(\mathfrak{t}),\ \forall \mathfrak{t} \in \bar{\mathbb{I}}$, where $\epsilon_* = \frac{\epsilon}{1-\mu\lambda}$, and so the proof is cimpleted. □

5 Illustrative of our outcome

First we present Example 5.1, for illustrative our main result.

Example 5.1. Consider the following fractional integro-differential problem

$$^{C}\mathbb{D}_q^{\frac{68}{77}}[y](\mathfrak{t}) = \frac{(15 - 2\mathfrak{t})y(\mathfrak{t})}{25} + \mathbb{I}_q^{\frac{5}{11}}\left[\frac{(5 - \mathfrak{t})\sin(y(\mathfrak{t}))}{43}\right] + \int_0^{\mathfrak{t}} \frac{y(\xi)\exp(-(\mathfrak{t} + \xi))}{20}\, \mathrm{d}\xi, \tag{5.1}$$

with boundary condition

$$y(0) = -\frac{15}{2}\int_0^{0.6} y(\xi)\, \mathrm{d}\xi, \qquad \forall \mathfrak{t} \in \mathbb{I}.$$

Clearly $\sigma + \nu = \frac{68}{77}$, $\sigma = \frac{5}{11}$, $\tau^* = 0.6$ and $\eta = -\frac{15}{2}$. To illustrate our results in Theorem 3.2 and Theorem 4.1, we take for $y, z \in \mathfrak{H} = \mathbb{R}^+$ and $\mathfrak{t} \in [0, 1]$ the following continuous functions:

$$h_1(\mathfrak{t}, y(\mathfrak{t})) = \frac{(15 - 2\mathfrak{t})y(\mathfrak{t})}{25}, \qquad h_2(\mathfrak{t}, y(\mathfrak{t})) = \frac{(5 - \mathfrak{t})\sin(y(\mathfrak{t}))}{43},$$

and

$$\Theta(\mathfrak{t}, \mathfrak{s}, y(\mathfrak{s})) = \frac{y(\mathfrak{s})\exp(-(\mathfrak{t} + \mathfrak{s}))}{20}.$$

Now, for $y, z \in \mathfrak{H}$, we have

$$\|h_1(\mathfrak{t}, y(\mathfrak{t})) - h_1(\mathfrak{t}, z(\mathfrak{t}))\| \leq \frac{3}{5}\|y(\mathfrak{t}) - z(\mathfrak{t})\|,$$

$$\|h_2(\mathfrak{t}, y(\mathfrak{t})) - h_2(\mathfrak{t}, z(\mathfrak{t}))\| \leq \frac{5}{43}\|y(\mathfrak{t}) - z(\mathfrak{t})\|,$$

Table 1: Numerical results of λ and λ^* for $q=\frac{3}{8},\frac{1}{2},\frac{8}{9}$ in Example 5.1.

n	$q=\frac{3}{8}$		$q=\frac{1}{2}$		$q=\frac{8}{9}$	
	λ	λ^*	λ	λ^*	λ	λ^*
1	0.93177	1.34571	0.71630	0.99360	0.11402	0.07701
2	0.94654	1.39205	0.73885	1.06376	0.11895	0.09638
3	0.95212	1.40943	0.75025	1.09885	0.12377	0.11354
4	0.95422	1.41595	0.75598	1.11640	0.12828	0.12878
5	0.95500	1.41840	0.75885	1.12518	0.13242	0.14232
6	0.95530	1.41931	0.76029	1.12957	0.13618	0.15436
7	0.95541	1.41966	0.76101	1.13176	0.13957	0.16506
8	0.95545	1.41978	0.76137	1.13286	0.14262	0.17458
9	0.95546	1.41983	0.76155	1.13341	0.14536	0.18304
10	0.95547	1.41985	0.76164	1.13368	0.14781	0.19057
11	0.95547	1.41986	0.76168	1.13382	0.15001	0.19727
12	0.95547	1.41986	0.76170	1.13389	0.15197	0.20322
13	0.95547	1.41986	0.76172	1.13392	0.15372	0.20852
14	0.95547	1.41986	0.76172	1.13394	0.15528	0.21323
15	0.95547	1.41986	0.76172	1.13395	0.15667	0.21741
16	0.95547	1.41986	0.76173	1.13395	0.15791	0.22114
17	0.95547	1.41986	0.76173	1.13396	0.15901	0.22445
18	0.95547	1.41986	0.76173	1.13396	0.16000	0.22739
⋮	⋮	⋮	⋮	⋮	⋮	⋮
76	0.95547	1.41986	0.76173	1.13396	0.16792	0.25095
77	0.95547	1.41986	0.76173	1.13396	0.16792	0.25096
78	0.95547	1.41986	0.76173	1.13396	0.16793	0.25096
79	0.95547	1.41986	0.76173	1.13396	0.16793	0.25096
80	0.95547	1.41986	0.76173	1.13396	0.16793	0.25096

and

$$\begin{aligned}\|\Theta(\mathfrak{t},\mathfrak{s},y(\mathfrak{s}))-\Theta(\mathfrak{t},\mathfrak{s},z(\mathfrak{s}))\| &= \left\|\frac{y(\mathfrak{s})\exp(-(\mathfrak{t}+\mathfrak{s}))}{20}-\frac{y(\mathfrak{s})\exp(-(\mathfrak{t}+\mathfrak{s}))}{20}\right\| \\ &\leq \frac{1}{20}\|y(\mathfrak{s})-z(\mathfrak{s})\|,\end{aligned}$$

for each $\mathfrak{t},\mathfrak{s}\in\mathbb{I}$ and $(\mathfrak{t},\mathfrak{s})\in\mathbb{G}$. Hence, $\mu_1=\frac{17}{25}$, $\mu_2=\frac{7}{43}$, $\mu^*=\frac{1}{20}$ and so

$$\mu=\max\left\{\mu_1,\mu_2,\mu^*\right\}=\frac{17}{25}.$$

Also, we obtain

$$\begin{aligned}\|h_1(\mathfrak{t},y(\mathfrak{t}))\| &= \left\|\frac{(15-2\mathfrak{t})y(\mathfrak{t})}{25}\right\|\leq\left|\frac{15-2\mathfrak{t}}{25}\right|\|y(\mathfrak{t})\|,\\ \|h_2(\mathfrak{t},y(\mathfrak{t}))\| &= \left\|\frac{(5-2\mathfrak{t})\sin(y(\mathfrak{t}))}{43}\right\|\leq\left|\frac{5-2\mathfrak{t}}{43}\right|\|y(\mathfrak{t})\|,\\ \|\Theta(\mathfrak{t},\mathfrak{s},y(\mathfrak{s}))\| &\leq \left\|\frac{y(\mathfrak{s})\exp(-(\mathfrak{t}+\mathfrak{s}))}{20}\right\|\leq\left\|\frac{\exp(-(\mathfrak{t}+\mathfrak{s}))}{20}\right\|\|y(\mathfrak{s})\|,\end{aligned}$$

for each $\mathfrak{t},\mathfrak{s}\in\mathbb{I}$. Hence,

$$\varrho_1(\mathfrak{t})=\frac{15-2\mathfrak{t}}{25},\quad \varrho_2(\mathfrak{t})=\frac{5-2\mathfrak{t}}{43},\quad \varrho^*(\mathfrak{t})=\frac{\exp(-\mathfrak{t})}{20},$$

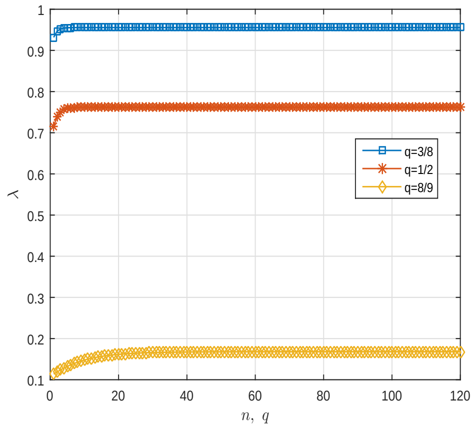
for all $\mathfrak{t}\in\overline{\mathbb{I}}$, $y,z\in\mathfrak{H}$ and $(\mathfrak{t},\mathfrak{s})\in\mathbb{G}$. By the above, we find that

$$\begin{aligned}\lambda=&\frac{\frac{3}{5}+\frac{1}{20}}{\Gamma_q\left(\frac{5}{11}+\frac{3}{7}+1\right)}+\frac{\frac{5}{43}B_q\left(\frac{5}{11}+1,\frac{5}{11}+\frac{3}{7}\right)}{\Gamma_q\left(\frac{5}{11}+1\right)\Gamma_q\left(\frac{5}{11}+\frac{3}{7}\right)}+\frac{\left|-\frac{15}{2}\right|\times\frac{3}{5}\times0.6^{\frac{5}{11}+\frac{3}{7}+1}+\left|-\frac{15}{2}\right|\times\frac{1}{20}\times0.6^{\frac{5}{11}+\frac{3}{7}+1}}{\left|1-0.6\left(-\frac{15}{2}\right)\right|\Gamma_q\left(\frac{5}{11}+\frac{3}{7}+2\right)}\\ &+\frac{\left|-\frac{15}{2}\right|\times\frac{5}{43}\times0.6^{\frac{10}{11}+\frac{3}{7}+1}B_q\left(\frac{5}{11}+1,\frac{5}{11}+\frac{3}{7}+1\right)}{\left|1-\frac{5}{11}\times0.6\right|\Gamma_q\left(\frac{5}{11}+1\right)\Gamma_q\left(\frac{5}{11}+\frac{3}{7}+1\right)},\end{aligned}\tag{5.2}$$

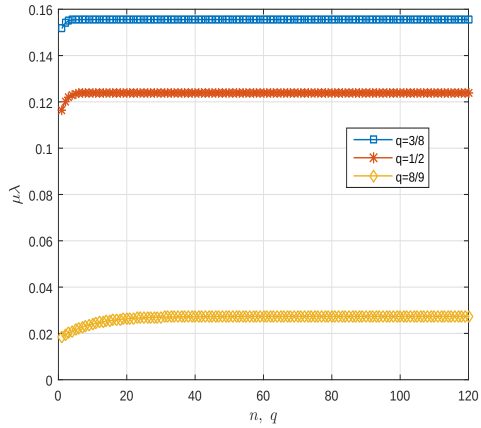
and

$$\lambda^* = \frac{\left|-\frac{15}{2}\right|}{\left|1-0.6\left(-\frac{15}{2}\right)\right|}\left[\frac{2\times 0.6^{\frac{5}{11}+\frac{3}{7}+1}}{\Gamma_q\left(\frac{5}{11}+\frac{3}{7}+2\right)}+\frac{0.6^{2\frac{5}{11}+\frac{3}{7}+1}B_q\left(\frac{5}{11}+1,\frac{5}{11}+\frac{3}{7}+1\right)}{\Gamma_q\left(\frac{5}{11}+1\right)\Gamma_q\left(\frac{5}{11}+\frac{3}{7}+1\right)}\right]. \tag{5.3}$$

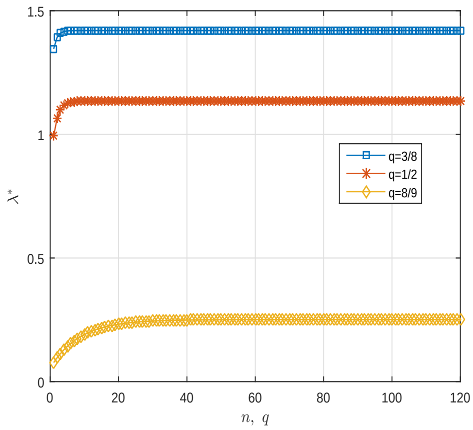
With consider $q=\frac{3}{8},\frac{1}{2},\frac{8}{9}$, we can see the results of λ and λ^* in Table 1. These results are plotted in Fig. 1. Then, we



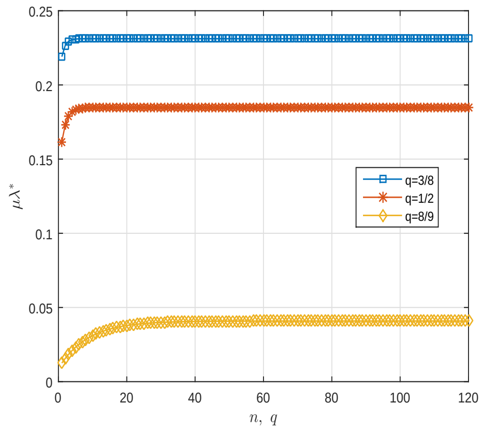
(a) λ, Eq.(3.8)



(b) $\mu\lambda$



(c) λ^*, Eq.(3.9)



(d) $\mu\lambda^*$

Figure 1: Graphical representation of λ, λ^* and $\mu\lambda$, $\mu\lambda^*$ for $q=\frac{3}{8},\frac{1}{2},\frac{8}{9}$ in Example 5.1.

get

$$\begin{aligned}\lambda_j &= 0.95547<1, \quad 0.76172<1, \quad 0.16793<1,\\ \lambda_j^* &= 1.41986, \quad 1.13395, \quad 0.25096,\\ \mu\lambda_j^* &= 0.9655<1, \quad 0.7711<1, \quad 0.1707<1,\end{aligned}$$

for $q_j=\frac{3}{8},\frac{1}{2},\frac{8}{9}$ respectively. All assumptions of Theorem 3.2 are satisfied. Hence, there exists at least one solution for the problem (5.1) on $\mathbb{I}$. By take the same functions, we result the assumption

$$\mu\lambda_j=\left\{\begin{array}{ll}0.6497, & q_j=\frac{3}{8},\\ 0.5180, & q_j=\frac{1}{2},\\ 0.1142, & q_j=\frac{8}{9},\end{array}\right\}<1,$$

then the system (5.1) is Ulam–Hyers stable, then it is generalized Ulam–Hyers stable. It is Ulam-Hyers-Rassias stable if there exists a continuous and positive function $\varrho_j\in C(\mathbb{I},\mathbb{R}^+)$ such that

$$\|y(\mathrm{t})-z(\mathrm{t})\|\le \epsilon_{j*}\varrho(\mathrm{t})=\frac{\epsilon_j\varrho(\mathrm{t})}{1-\mu\lambda_j},$$

which it satisfies in assumption of the Theorem 4.2.

In the next example, we review and check Theorem 3.3 numerically.

Example 5.2. Consider the following fractional integro-differential problem

$$ {}^{C}\mathbb{D}_q^{\frac{29}{45}}[y](\mathfrak{t}) = \frac{(16-\sqrt{\mathfrak{t}})\tan^{-1}(y(\mathfrak{t}))}{75} + \mathbb{I}_q^{\frac{4}{9}}\left[\frac{2\mathfrak{t}\sin^{-1}(y(\mathfrak{t}))}{21}\right] + \int_0^{\mathfrak{t}} \frac{y(\xi)\exp(-(3\mathfrak{t}+\xi))}{10}\,\mathrm{d}\xi, \tag{5.4} $$

with boundary condition

$$ y(0) = -\frac{5}{2}\int_0^{0.95} y(\xi)\,\mathrm{d}\xi, \qquad \mathfrak{t}\in\mathbb{I}. $$

Clearly $\sigma+\nu = \frac{29}{45}$, $\sigma = \frac{4}{9}$, $\tau^* = 0.95$ and $\eta = \frac{5}{2}$. To illustrate our results in Theorem 3.3, we take for $y, z \in \mathfrak{H} = \mathbb{R}^+$ and $\mathfrak{t} \in \bar{\mathbb{I}}$ the following continuous functions:

$$ h_1(\mathfrak{t}, y(\mathfrak{t})) = \frac{(16-\sqrt{\mathfrak{t}})\tan^{-1}(y(\mathfrak{t}))}{75}, \qquad h_2(\mathfrak{t}, y(\mathfrak{t})) = \frac{2\mathfrak{t}\sin^{-1}(y(\mathfrak{t}))}{21}, $$

and

$$ \varTheta(\mathfrak{t},\mathfrak{s},y(\mathfrak{s})) = \frac{y(\mathfrak{s})\exp(-(3\mathfrak{t}+\mathfrak{s}))}{10}. $$

Now, for $y, z \in \mathfrak{H}$, we have

$$ \begin{aligned} \|h_1(\mathfrak{t},y(\mathfrak{t})) - h_1(\mathfrak{t},z(\mathfrak{t}))\| &= \left\| \frac{(16-\sqrt{\mathfrak{t}})\tan^{-1}(y(\mathfrak{t}))}{75} - \frac{(16-\sqrt{\mathfrak{t}})\tan^{-1}(z(\mathfrak{t}))}{75} \right\| \\ &\le \frac{17}{75}\|y(\mathfrak{t}) - z(\mathfrak{t})\|, \end{aligned} $$

$$ \begin{aligned} \|h_2(\mathfrak{t},y(\mathfrak{t})) - h_2(\mathfrak{t},z(\mathfrak{t}))\| &= \left\| \frac{2\mathfrak{t}\sin^{-1}(y(\mathfrak{t}))}{21} - \frac{2\mathfrak{t}\sin^{-1}(z(\mathfrak{t}))}{43} \right\| \\ &\le \frac{2}{21}\|y(\mathfrak{t}) - z(\mathfrak{t})\|, \end{aligned} $$

and

$$ \begin{aligned} \|\varTheta(\mathfrak{t},\mathfrak{s},y(\mathfrak{s})) - \varTheta(\mathfrak{t},\mathfrak{s},z(\mathfrak{s}))\| &= \left\| \frac{y(\mathfrak{s})\exp(-(3\mathfrak{t}+\mathfrak{s}))}{10} - \frac{y(\mathfrak{s})\exp(-(3\mathfrak{t}+\mathfrak{s}))}{10} \right\| \\ &\le \frac{1}{10}\|y(\mathfrak{s}) - z(\mathfrak{s})\|, \end{aligned} $$

for each $\mathfrak{t}, \mathfrak{s} \in \mathbb{I}$ and $(\mathfrak{t},\mathfrak{s}) \in \mathbb{G}$. Hence, $\mu_1 = \frac{17}{75}$, $\mu_2 = \frac{2}{21}$, $\mu^* = \frac{1}{10}$ and so

$$ \mu = \max\left\{\mu_1, \mu_2, \mu^*\right\} = \frac{17}{25}. $$

Also, we obtain

$$ \begin{aligned} \|h_1(\mathfrak{t},y(\mathfrak{t}))\| &= \left\| \frac{(16-\sqrt{\mathfrak{t}})\tan^{-1}(y(\mathfrak{t}))}{75} \right\| \le \left| \frac{16-\sqrt{\mathfrak{t}}}{75} \right| \|y(\mathfrak{t})\|, \\ \|h_2(\mathfrak{t},y(\mathfrak{t}))\| &= \left\| \frac{2\mathfrak{t}\sin^{-1}(y(\mathfrak{t}))}{21} \right\| \le \left| \frac{2\mathfrak{t}}{21} \right| \|y(\mathfrak{t})\|, \\ \|\varTheta(\mathfrak{t},\mathfrak{s},y(\mathfrak{s}))\| &\le \left\| \frac{y(\mathfrak{s})\exp(-(3\mathfrak{t}+\mathfrak{s}))}{10} \right\| \le \left\| \frac{\exp(-(3\mathfrak{t}+\mathfrak{s}))}{10} \right\| \|y(\mathfrak{s})\|, \end{aligned} $$

for each $\mathfrak{t}, \mathfrak{s} \in \mathbb{I}$. Hence,

$$ \varrho_1(\mathfrak{t}) = \frac{16-\sqrt{\mathfrak{t}}}{75}, \qquad \varrho_2(\mathfrak{t}) = \frac{2\mathfrak{t}}{21}, $$

Table 2: Numerical results of λ and $\mu\lambda$ for $q = \frac{2}{7}, \frac{1}{2}, \frac{9}{11}$ in Example 5.2.

n	$q=\frac{2}{7}$		$q=\frac{1}{2}$		$q=\frac{9}{11}$	
	λ	$\mu\lambda$	λ	$\mu\lambda$	λ	$\mu\lambda$
1	0.81214	0.55225	0.54150	0.36822	0.15811	0.10752
2	0.81764	0.55600	0.55700	0.37876	0.16610	0.11295
3	0.81923	0.55708	0.56491	0.38414	0.17332	0.11785
4	0.81969	0.55739	0.56890	0.38685	0.17947	0.12204
5	0.81982	0.55748	0.57090	0.38821	0.18462	0.12554
6	0.81986	0.55750	0.57190	0.38889	0.18887	0.12843
7	0.81987	0.55751	0.57240	0.38923	0.19238	0.13082
8	0.81987	0.55751	0.57265	0.38940	0.19526	0.13278
9	0.81987	0.55751	0.57278	0.38949	0.19763	0.13439
10	0.81987	0.55751	0.57284	0.38953	0.19956	0.13570
11	0.81987	0.55751	0.57287	0.38955	0.20115	0.13678
12	0.81987	0.55751	0.57289	0.38956	0.20245	0.13767
13	0.81987	0.55751	0.57290	0.38957	0.20352	0.13839
14	0.81987	0.55751	0.57290	0.38957	0.20439	0.13898
15	0.81987	0.55751	0.57290	0.38957	0.20510	0.13947
⋮	⋮	⋮	⋮	⋮	⋮	⋮
43	0.81987	0.55751	0.57290	0.38957	0.20830	0.14164
44	0.81987	0.55751	0.57290	0.38957	0.20830	0.14165
45	0.81987	0.55751	0.57290	0.38957	0.20830	0.14165
46	0.81987	0.55751	0.57290	0.38957	0.20831	0.14165
47	0.81987	0.55751	0.57290	0.38957	0.20831	0.14165

and so

$$\varrho^*(\mathfrak{t}) = \frac{\exp(-3\mathfrak{t})}{10}, \qquad \forall \mathfrak{t} \in \bar{\mathbb{I}},$$

$y, z \in \mathfrak{H}$ and $(\mathfrak{t}, \mathfrak{s}) \in \mathbb{G}$. By the above, we find that

$$\begin{aligned}
\lambda &= \frac{\|\varrho_1\|_{L^\infty} + \|\varrho^*\|_{L^\infty}}{\Gamma_q(\sigma+\nu+1)} + \frac{\|\varrho_2\|_{L^\infty} B_q(\sigma+1,\sigma+\nu)}{\Gamma_q(\sigma+1)\Gamma_q(\sigma+\nu)} + \frac{|\eta|\|\varrho_1\|_{L^\infty}\tau^{*\sigma+\nu+1} + |\eta|\|\varrho^*\|_{L^\infty}\tau^{*\sigma+\nu+1}}{|1-\eta\tau^*|\Gamma_q(\sigma+\nu+2)} \\
&\quad + \frac{|\eta|\|\varrho_2\|_{L^\infty}\tau^{*2\sigma+\nu+1} B_q(\sigma+1,\sigma+\nu+1)}{|1-\eta\tau^*|\Gamma_q(\sigma+1)\Gamma_q(\sigma+\nu+1)} \\
&= \frac{\frac{16}{75}+\frac{1}{10}}{\Gamma_q(\frac{4}{9}+\frac{1}{5}+1)} + \frac{\frac{2}{21}B_q(\frac{4}{9}+1,\frac{4}{9}+\frac{1}{5})}{\Gamma_q(\frac{4}{9}+1)\Gamma_q(\frac{4}{9}+\frac{1}{5})} + \frac{|2.5|\times\frac{16}{75}0.95^{\frac{4}{9}+\frac{1}{5}+1} + |2.5|\frac{1}{10}0.95^{\frac{4}{9}+\frac{1}{5}+1}}{|1-2.5\times 0.95|\Gamma_q(\frac{4}{9}+\frac{1}{5}+2)} \\
&\quad + \frac{|2.5|\frac{4}{9}\times\frac{2}{21}\times 0.95^{\frac{8}{9}+\frac{1}{5}+1}B_q(\frac{4}{9}+1,\frac{4}{9}+\frac{1}{5}+1)}{|1-2.5\times 0.95|\Gamma_q(\frac{4}{9}+1)\Gamma_q(\frac{4}{9}+\frac{1}{5}+1)}.
\end{aligned} \tag{5.5}$$

With consider $q = \frac{2}{7}, \frac{1}{2}, \frac{9}{11}$, we can see the results of λ and λ^* in Table 2. These results are plotted in Fig. 2. Then, we get

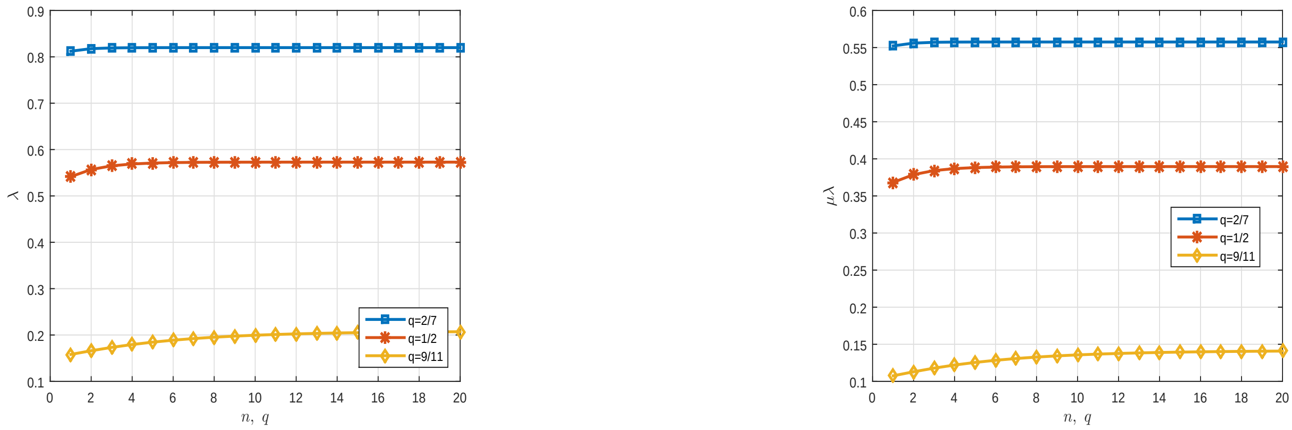
$$\lambda_j = \begin{cases} 0.81987, & q_j = \frac{2}{7}, \\ 0.57290, & q_j = \frac{1}{2}, \\ 0.20831, & q_j = \frac{9}{11}, \end{cases}$$

$$\mu\lambda_j == \left\{ \begin{array}{ll} 0.55751, & q_j = \frac{2}{7}, \\ 0.38957, & q_j = \frac{1}{2}, \\ 0.14165, & q_j = \frac{9}{11}, \end{array} \right\} < 1.$$

All assumptions of Theorem 3.3 are satisfied. Hence, there exists at least one solution for the problem (5.4) on $\bar{\mathbb{I}}$.

6 Conclusion

The q-integro-differential boundary equations and their applications represent a matter of high interest in the area of fractional q-calculus and its applications in various areas of science and technology. q-integro-differential boundary



(a) λ, Eq.(3.8) (b) $\mu\lambda$

Figure 2: Graphical representation of λ and $\mu\lambda$ for $q = \frac{2}{7}, \frac{1}{2}, \frac{9}{11}$ in Example 5.2.

value problems occur in the mathematical modeling of a variety of physical operations. Using the Krasnoselskii's, Banach fixed point theorems, we proof the existence and uniqueness results. Based on the results obtained, conditions are provided that ensure the generalized Ulam stability of the original system. The results for investigating Eq. (1.1) on a time scale, are illustrated by two examples.

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